MALLIAVIN CALCULUS APPROACH TO PRICING AND HEDGING OF OPTIONS WITH MORE THAN ONE UNDERLYING ASSETS

BY

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Abstract

The problems of pricing and hedging in financial market are fundamental because of uncertainties in the market which are measured by the sensitivities of the underlying assets. Ito calculus has been used to develop several models that deal with the problems of pricing and hedging of options with smooth payoff functions. However, Ito calculus becomes ineffective when dealing with options with multiple underlying assets, whose payoffs are non-smooth functions. Therefore, this study was designed to consider the sensitivities of options with multiple underlying assets whose payoff are non-smooth function.

The Malliavin integral calculus given by the Skorohod integral and the integration by part technique for stochastic variation were used to derive weight functions of the Greeks for Best of Asset Option (BAO) and Asian Option (AO). The Clark-Ocone formula was used to derive an extension of the Malliavin derivative chain rule to finite dimensional vector form. This, together with the weight functions were used to derive expressions for the Greeks which represent the sensitivities of the options with respect to the parameters; price of the underlying asset at initial time S_0, second derivative of the option with respect to S_0, volatility σ, expiration time T, interest rate μ, namely: δ, γ, ρ, θ and ν respectively. Randomly generated data was used to compute the sensitivities.

The weight functions obtained were $\omega^{\Delta} = \frac{W_t}{S_0 \sigma T}$, $\omega^{\Gamma} = \frac{1}{(\sigma T)^2}\frac{1}{2S_0^2}(W_T^2 - T - \frac{W_T}{\sigma T})$, $\omega^{\rho} = \frac{W_T}{\sigma}$, $\omega^{\Theta} = (\frac{\mu - \frac{\sigma^2}{2}}{)}\sigma T)W_T$ and $\omega^{\nu} = \frac{W_T^2 - T - 2W_T}{2\sigma T}$. The Malliavin derivative chain rule obtained was $D(g(F_k^j)) = \sum_{j=1}^{n} g^{'}(F_k^j)DF_k^j, k \geq 1$ and the Greek expression were obtained as:

$$\Delta^{BAO} = \frac{e^{-rT}}{S_0 \sigma T}\mathbb{E}_Q(max(S_i - K)I_{S_i > S_j}, i \neq j, i, j = 1, 2...nW_T),$$

$$\Gamma^{BAO} = \frac{-e^{-rT}}{S_0^2}\mathbb{E}_Q[max(S_i - K)I_{S_i > S_j}, i \neq j, i, j = 1, 2...n\frac{1}{(\sigma T)^2}\frac{W_T^2 - T}{2} - \frac{W_T}{\sigma T}],$$

$$\Theta^{BAO} = -e^{-rT}\mathbb{E}_Q[max(S_i - K)I_{S_i > S_j}, i \neq j, i, j = 1, 2...n(\frac{\mu - \frac{\sigma^2}{2}}{\sigma T})W_T],$$

$$\rho^{BAO} = \frac{e^{-rT}}{\sigma}\mathbb{E}_Q[max(S_i - K)I_{S_i > S_j}, i \neq j, i, j = 1, 2...n]W_T],$$

$$\nu^{BAO} = \frac{e^{-rT}}{2\sigma T}\mathbb{E}_Q[max(S_i - K)I_{S_i > S_j}, i \neq j, i, j = 1, 2...n(W_T^2 - T - 2W_T)],$$

and

$$\Delta^{AO} = e^{-rT}\mathbb{E}_Q[(\frac{1}{T}\int_0^T S_t dt - k)(\frac{W_t}{S_0 \sigma T})],$$

$$\Gamma^{AO} = \frac{e^{-rT}}{S_0^2}\mathbb{E}_Q[(\frac{1}{T}\int_0^T S_t dt - k)\frac{1}{(\sigma T)^2}\frac{W_T^2 - T}{2} - \frac{W_T}{\sigma T}],$$

$$\rho^{AO} = \frac{e^{-rT}}{\sigma}\mathbb{E}_Q[(\frac{1}{T}\int_0^T S_t dt - k)W_T],$$

$$\Theta^{AO} = -e^{rT}\mathbb{E}_Q[(\frac{1}{T}\int_0^T S_t dt - k)(\frac{\mu - \frac{\sigma^2}{2}}{\sigma T})W_T],$$

$$\nu^{AO} = \frac{e^{-rT}}{2\sigma T}\mathbb{E}_Q[(\frac{1}{T}\int_0^T S_t dt - k)(W_T^2 - T - 2W_T)]$$

where $\mathbb{E}_Q$ represent the expectation with respect to the equivalent martingale measure, W_T is the standard Brownian motion at time T, S_T is the price of the underlying asset at time T and K is the strike price. The computed sensitivities showed that the risk associated with the model was minimal when there were more than one underlying asset.

The sensitivities of options with multiple underlying assets with non-smooth payoffs was obtained, and these can be applied in financial market to monitor and minimise risk.

Keywords: Multiple underlying assets, Best of asset options, Asian options, Greek expectation, Brownian motion.

Word count: 496

Dedication

I dedicate this work to the glory of God, to my wife, Akeju Oluwakemi Ayoola, my children; Favour and Daniel for the their understanding, prayers and love and to my mother, Akeju Christianah Oluwatola for her prayers and support all through.

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Certification

I certify that this work was carried out by Adeyemi Olu AKEJU with matriculation number 136977 in the Department of Mathematics, Faculty of Science, University of Ibadan under my supervision.

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Chapter 1
INTRODUCTION

1.1 Introduction

We examine in this thesis, the pricing and hedging of two types of Rainbow options namely; Asian option and Best of assets options. These types of option are options whose payoffs are defined with respect to multiple underlying assets.

Rainbow Option is a particular kind of exotic option whose closed form formula are not smooth. Rainbow options involves portfolio with more than one underlying assets. This is most suitable for our study since our interest is pricing and hedging in a multi- dimensional framework using Malliavin calculus.

This calculus involves the integration by part technique of the stochastic of variation. We use this calculus to derive the expectation of the payoff function of Rainbow Options. The study of Malliavin calculus and the applications in finance involve the use of integration by part formula to give a mathematical approach to the computation of the price sensitivities.

Options whose formulas can be computed explicitly can be derived in the Ito framework, but it is challenging to work in the Ito framework when the payoff function are not regular. This type of payoff can be computed in Malliavin sense. This is because one of the original ideas behind the development of Malliavin calculus is the study of smoothness of solution of stochastic differential equations with discontinuous coefficients.
The real advantage of using Malliavin calculus by means of the integration by part is that, it is applicable when dealing with random variables with unknown density functions and when there are options with non smooth payoffs.

Malliavin, P (1978), introduced the theory of Malliavin calculus being an integration by part procedure that has infinite dimension with the purpose of proving results concerning the smoothness of solution densities of stochastic differential equations that are driven by Brownian motion. These solution densities were shown by Oksendal, B (2003), using probability distribution of random variables defined in $\mathbb{D}^{1,2}$ (the space of Malliavin differentiable random variables).

Options are derivative contracts which permit its holder to buy or to sell a

given number of derivatives (which can be a financial stock, a currency e.t.c) at a given and agreed price and at a particular time $\tau < T$ which are fixed in the contract. Options are generally classified into two main classes. They are either Call or Put option. An option is known as a Call if the person holding the option has the right to purchase it while we refer to the option as a Put if the person holding it has right to dispose it by way of selling the option. If the person holding the option decides to exercise the right, the other party, who is refered to as a writer is expected to buy the asset(s) underlying the option at a specified price which is refered to as Strike price. The option holder, that is the buyer is expected to pay a certain amount known as the premium fee to the other party who is known as the writer, in exchange for holding the option.
The conditions and time to exercise differ, it is a function of the style of option in view.

- Options style which can be exercised at the end of the contract (maturity time) is known as European option.
- An option style that can be exercised before or at maturity time T is known as American option.

In what follows, we shall state relevant notations that relate to the definition of Call options and Put options as follows;

Let $\mathtt{S}_\tau$ represent the market price of the underlying asset at any time τ, $\mathbf{K}$ is the agreed strike price of the option, $\mathtt{C}_\tau$ represent the Call option value at time τ and P_τ represent the Put option value at any time τ, where τ satisfies the condition $0 \leq \tau \leq T$, then the values of the Call and Put options can be defined respectively at the time of exercise as

$$\mathtt{C}_T = max((\mathtt{S}_T - \mathbf{K}), 0), \tag{1.1.1}$$

and

$$\mathtt{P}_T = max((\mathbf{K} - \mathtt{S}_T), 0). \tag{1.1.2}$$

These types of options that is, Call and Put options are known as Vanilla options. Apart from the Vanilla type of Options, there are other complicated types which are generally called exotic options.
This type of option is completely different from the main Vanilla in terms of the contract payment plans, strike price and the nature of the underlying assets. Due to the variations associated with the underlying assets, investors have opportunity to several investment plans and strategies. One important feature of this type of

option contract is the possibility to customize it to meet up with the investor risk tolerance. This will enable the investor to achieve a set desired profit. Exotic option is a mixture of European and American options in terms of the time to exercise.

There are several examples of exotic option. This include but not limited to the following;
Compound option: This option allows the holder the right to buy another option at a specific time and at a specific price.
Barrier option: This type is exercised when the underlying assets attained a predetermine price.
Binary option: This is also referred to as digital option. e.t.c.
In this work, we specifically considered Asian option and Best of assets options. These two options are also examples of exotic option. We considered these two because, they are suitable for our study since they are option styles with more than one underlying assets.

Asian option considered the average of the assets underlying the contract over a certain period of time to determine if there is profit when compared with the strike price.
Best of Asset option is the type that considered the maximum of the underlying assets prices in comparison with the strike price to determine the profitability of the contract.

There are two questions that often arise when dealing with options:
(1) How do we find a premium price at initial time $\tau = 0$ for the option, that is, the contract price that is acceptable to the buyer and that is acceptable to the writer?.
(2) How do we determine, at maturity time, the option value given that a premium has been paid at initial time?. This is known as hedging problem. In other to deal with these problems, we assume the absence of arbitrage opportunities that is, it is impossible to obtain benefit without taking risk.
The dynamics of pricing and hedging of options is such that at maturity time, a flow of the payoff $h(\mathtt{S}_T)$ can be guaranteed by the option owner. Then the option owner can purchase with the premium, a portfolio that has equal flow of price with one of the options. This process is known as the portfolio hedging or dynamic strategy of buying and selling of options.
We shall denote, at any time τ the value of the hedged portfolio simply as Υ_τ, $0 \leq \tau \leq T$ and the possibility of not having arbitrage is such that

$$P(\Upsilon_T > 0) > 0 \quad \Upsilon_0 = 0$$

This means that, the possibility that the portfolio will always be replicated is positive at every time τ.

1.2 Research Question / Statement of The Problem

The problem of pricing and hedging is fundamental because of uncertainties in the financial market. These uncertainties are measured by the sensitivities of the underlying assets which can also be referred to as the derivative security.
Derivatives securities are important assets in financial markets. However the prices of derivative securities are subject to fluctuation, this fluctuation is the reason decision to invest in financial market becomes uncertain and highly volatile.
Hence, the question is, if there is a portfolio or a contract that has more than one underlying assets, is it possible to use this portfolio to hedge and mitigate the risk associated with the market uncertainties?.

1.3 Motivation of Study

This study is focused on options with multiple underlying assets geared towards the formulation and development of effective hedging strategy that mitigate risks in financial market. However, Ito calculus has been used to developed several models that deal with the problem of pricing and hedging of options with smooth payoff functions. This becomes ineffective when dealing with options with multiple underlying assets whose payoff are non smooth. Hence the reason for considering Malliavin calculus, since it can handle non smooth payoff functions.

1.4 The Theoretical Framework

This study will rely on the theory of Malliavin calculus which is essential in dealing with non smooth payoff functions.
In this regard, we shall use the fundamental theories of Skorohod integral, integration by part formula for handling Malliavin derivative of Clark Ocone formula, divergence operator and some of the features of stochastic differential equations.

1.5 Research Objectives

The objectives of this research are;

i To form an expression for pricing and hedging of Rainbow Option using the integration by part technique of Malliavin Calculus,

ii To compute the numerical approximate results of the greeks by means of Excel and Matlab softwares,

iii To compare the results obtained in (ii) above with the result obtained with Black-Schole model.

1.6 Definitions and Basic Results

In this section, we shall state some basic concepts and fundamental definitions that are used in this work.

Definition 1.1 (Stochastic Process):
A random variable X is said to be a stochastic process if $X = \{X(t), t \in [0, T]\}$ is a collection of random variables on a common probability space indexed by parameter $t \in T \subset \mathbb{R}_+$. Stochastic process can be formulated as a function that is, $X : T \times \Omega \longrightarrow \mathbb{R}$, such that $X(t, .)$ is $\mathcal{A}$- measurable for each $t \in T$ where Ω is a non empty set, $\mathcal{A}$ is σ-algebra generated by Ω. $X(t)$ can be written also as X_t.

Definition 1.2 (Brownian Motion):
A stochastic process $B(\tau)_{\tau\in[0,T]}$ is said to be a Brownian motion if the following properties are satisfy;

- $B(0) = 0$ almost surely
- $(B(\tau) - B(s)), \quad \tau > s$ is independent of the past (Independent Increment)
- $(B(\tau) - B(s))$ has normal distribution with mean 0 and variance $\tau - s$. This implied that, for $s = 0$, $(B(\tau) - B(0))$ has normal distribution with mean 0 and variance t, that is $(B(\tau) - B(s)) \sim N(0, t)$ (Normal increment).
- $B(\tau), \tau > s$ is a continuous function of τ (Continuity of path)

Remark:
Brownian motion can be described in the setting of isonormal Gaussian processes as we shall discuss in section 3.2

Definition 1.3 (Measurable Space):
Let Ω be a non empty set, and let $\mathcal{A}$, a σ-algebra, be the collection of subsets of Ω, then the pair $(\Omega, \mathcal{A})$ is called a measurable space.

Definition 1.4 (Probabiltity Space):
Let Ω be a non empty set, let $\mathcal{A}$, a σ-algebra, be the collection of subsets of Ω, and let P be probability measure such that $P(\Omega) = 1$ and $0 \leq P(A) \leq 1$ for every $A \in \mathcal{A}$, then the triple $(\Omega, \mathcal{A}, P)$ is refered to as a probability space.

Definition 1.5 (Filtered Probabiltity Space):
Let Ω be a non empty set, let $\mathcal{A}$, a σ-algebra, be the collection of subsets of Ω, let P be a probability measure, if there exists $(\mathcal{A}_t, t \in [0, T])$, a family of sub σ-algebra of $\mathcal{A}$, then $(\Omega, \mathcal{A}, P, \mathcal{A}_t)$ is refered to as a filtered probability space.

Remark:
1. A sequence $(f_n, n \in \mathbb{N})$ of σ-algebra is called filtration if $f_n \subset f_{n-1} \subset \mathcal{A}$ for every $n \in \mathbb{N}$ where $\mathcal{A} \subset \Omega$
2. $(\mathcal{F}_t, t \in [0, T])$ is called filtration of the probability space $(\Omega, \mathcal{F}, P)$ if and only if

(i) $\mathcal{F}_0$ contains all subsets of any P- null set.

(ii) $\mathcal{F}_s$ is a sub σ-algebra of $\mathcal{F}_t, t \geq s$

Filtration can always be used with the property $P(\Omega)$ which represents the power set of Ω such that;

(1) $\mathcal{F}_0 = (\emptyset, \Omega)$: At the beginning, there is no information.

(2) $\mathcal{F}_T = P(\Omega)$: At the end , there is full information.

(3) $\mathcal{F}_0 \subset \mathcal{F}_1 \subset ... \subset \mathcal{F}_T$: The information available increases over time.

Filtration are used to model the flow of information over time. At time t, we can decide if the event $A \in \mathcal{F}_t$ has occurred or not.

Definition 1.6 (Adapted Processes):
A sequence $(X_t, t \geq 0)$ of random variables is said to be adapted to a filtration $\mathcal{F}_t$ if for each t, the random variable $(X_t$ is $\mathcal{F}_t$- measurable, that is, for any t, $\mathcal{F}_t$ contains all the information about X_t.

Definition 1.7 (Martingale):
A stochastic process $M(t)$ (if t is continuous, then, $0 \leq t \leq T$, or if t is discrete, then $t = 0, 1, ..., T$), adapted to a filtration $\mathcal{F}_t$, is a martingale if for any t, $M(t)$ is integrable, that is $\mathbb{E}|M(t)| < \infty$ and for any t and s, such that $0 \leq s \leq t \leq T$, then $\mathbb{E}(M(t)/\mathcal{F}_s) = M(s)$

Definition 1.8 (Super Martingale):
A stochastic process $M(t), t \geq 0$, adapted to a filtration $\mathcal{F}_t$, is a super martingale if for any t, $M(t)$ is integrable, that is $\mathbb{E}|M(t)| < \infty$ and for any t and s, such that $0 \leq s \leq t \leq T$, $\mathbb{E}(M(t)/\mathcal{F}_s) \leq M(s)$

Definition 1.9 (Sub Martingale):
A stochastic process $M(t), t \geq 0$, adapted to a filtration $\mathcal{F}_t$, is a sub martingale if for any t, $M(t)$ is integrable, that is $\mathbb{E}|M(t)| < \infty$ and for any t and s, such that $0 \leq s \leq t \leq T$, $\mathbb{E}(M(t)/\mathcal{F}_s) \geq M(s)$

Remark:
A stochastic process that is both super martingale and sub martingale is a martingale.

Definition 1.10 (Black Schole Financial Market):
A market, in the Black-Schole sense is made up of an asset that is risk free A and an asset that is risky S.
The price of the risk free asset A is expected to satisfy the differential equation

$$dA(\tau) = rA(\tau)d\tau \qquad A(0) = 1 \tag{1.6.1}$$

which is an ordinary differential equation, provided the interest rate r is constant. The solution of equation (1.6.1) is

$$A(\tau) = A_\tau = e^{r\tau}$$

satisfies the price process of the risk free asset.
If the interest rate r is a non-negative adapted process, then r will satisfy the condition that

$$\int_0^T r_\tau d\tau < \infty$$

The price of the asset that is risky $\mathtt{S}$ is expected to have the dynamics

$$d\mathtt{S}(\tau) = \kappa\mathtt{S}(\tau)d\tau + \sigma\mathtt{S}(\tau)dB(\tau) \qquad \mathtt{S}(0) = \mathtt{S}_0, \quad \mathtt{S}(\tau) = \mathtt{S}_\tau, \qquad \tau \in [0, T] \quad (1.6.2)$$

a stochastic differential equation (SDE).
The solution

$$\mathtt{S}(\tau) = \mathtt{S}_0 \exp((\kappa - \frac{\sigma^2}{2})\tau + \sigma B(\tau))$$

of the stochastic differential equation (1.6.2) shown by Kloeden P.E and Platen.E (1999) satisfy the price process of the risky asset $\mathtt{S}$ where $\mathtt{S}_0$ represents the initial price of the asset $\mathtt{S}$, κ is the drift term which is taken to be constant, σ represents the volatility of the process which is also known as the noise term, this volatility is also assumed to be constant, $B = \{B(\tau), \tau \in [0, T]\}$ represents a Brownian motion defined on a filtered probability space $(\Omega, \mathcal{A}, P, \mathcal{A}_\tau)$, and $\{\mathcal{A}_\tau, \tau \in [0, T]\}$ is a filtration, that is the flow of available information determined by the Brownian motion.

If an investor invested the sum $\chi > 0$ in an asset described in line with Black-scholes market, assumed $N(\tau)$ represents the quantity of the risk-free assets while $\mathcal{N}(\tau)$ represents the quantity of risky assets that an investor owned at time τ, then we can define the following terms;

Definition 1.11 (Trading Strategy):
Trading strategy is also known as dynamic portfolio. A strategy described the investment of an investor in each asset at any time $\tau \in [0, T]$, that is, the ratio of amount of money invested in each asset in a portfolio. Meanwhile, a trading strategy or dynamic portfolio process $\varrho(\tau)$ described how the investment were combined and its defined as

$$\varrho(\tau) = (N(\tau), \mathcal{N}(\tau)), \tau \in [0, T]$$

so

$$\int_0^T |\mathcal{N}_\tau \kappa_\tau| d\tau < \infty, \qquad \int_0^T N_\tau r_\tau d\tau < \infty$$

and $x = N_0 + \mathcal{N}_0\mathtt{S}_0$ a.s

Definition 1.12 (Self Financing Portfolio):
A self financing portfolio is also known as a self financing stategy. A portfolio or a stategy is said to be self financing if all the changes in the portfolio are due to gains realized on investment, that is no fund are borrowed or withdrawn from the portfolio at any time.

Definition 1.13 (Wealth Process):
The wealth at time τ which represents the portfolio value is given by

$$\begin{aligned}\mathcal{W}(\tau) &= \mathcal{W}_\tau(\varrho)\\ &= N_\tau A_\tau + \mathcal{N}_\tau \mathtt{S}_\tau\\ &= N_\tau e^{r\tau} + \mathcal{N}_\tau \mathtt{S}_\tau\end{aligned}$$

The investor gain(the gain process) $\mathcal{G}_\tau(\varrho)$ will satisfy

$$\mathcal{G}_\tau(\varrho) = \int_0^\tau N_s dA_s + \int_0^\tau \mathcal{N}_s d\mathtt{S}_s$$

The process ϱ is self-financing provided that we cannot have an inward and outward movement of money into the market so that the wealth process satisfies,

$$\begin{aligned}\mathcal{W}_\tau(\varrho) &= \mathcal{W}_0(\varrho) + \mathcal{G}_\tau(\varrho),\ \tau \in [0,T]\\ &= x + \int_0^\tau N_s dA_s + \int_0^\tau \mathcal{N}_s d\mathtt{S}_s\end{aligned}$$

Let the discounted process be given by

$$\begin{aligned}\tilde{\mathtt{S}}_\tau &= A_\tau^{-1}\mathtt{S}_\tau\\ &= e^{-r\tau}\mathtt{S}_\tau\end{aligned}$$

$$\tilde{\mathtt{S}}_\tau = \mathtt{S}_0 \exp\left(\int_0^\tau \left(\kappa_s - r_s - \frac{\sigma_s^2}{2}\right) ds + \int_0^\tau \sigma_s dB_s\right)$$

then we can write the discounted portfolio as

$$\begin{aligned}\tilde{\mathcal{W}}_\tau(\varrho) &= A_\tau^{-1}\mathcal{W}_\tau(\varrho)\\ &= e^{-r\tau}(N_\tau e^{r\tau} + \mathcal{N}_\tau \mathtt{S}\tau)\\ &= N_\tau + \mathcal{N}_\tau e^{-r\tau}\mathtt{S}\tau\\ &= N_\tau + \mathcal{N}_\tau \tilde{\mathtt{S}}\tau\end{aligned}$$

Differentiating $\tilde{\mathcal{W}}_\tau$ we get

$$d\tilde{\mathcal{W}}_\tau(\varrho) = \mathcal{N}_\tau d\tilde{\mathtt{S}}_\tau$$

Integrating, we get

$$\begin{aligned}\tilde{\mathcal{W}}_\tau(\varrho) &= x + \int_0^\tau \mathcal{N}_s d\tilde{\mathtt{S}}_s \\ &= x + \int_0^\tau (\kappa_s - r_s)\mathcal{N}_s \tilde{S}_s ds + \int_0^\tau \sigma_s \mathcal{N}_s \tilde{S}_s dB_s \end{aligned} \tag{1.6.3}$$

Therefore, for a self financing portfolio,

$$\begin{aligned}\alpha &= \tilde{\mathcal{W}}_\tau(\varrho) - \mathcal{N}_\tau \mathtt{S}_\tau \\ &= x + \int_0^\tau \mathcal{N}_s d\tilde{S}_s - \mathcal{N}_\tau \mathtt{S}_\tau\end{aligned}$$

Note: (1.6.3) becomes a local martingale if $\kappa_s = r_\tau$.

Definition 1.14 (Tamed Trading Strategy):
A trading strategy denoted as ϱ is said to be tamed if its associated wealth process is always non-negative i.e $\mathcal{W}_\tau(\varrho) \geq 0, \tau \in [0, T]$.

Definition 1.15 (Arbitrage):
Arbitrage is defined as a stategy that gives opportunity to make a profit out of nothing without taking any risk.
A self-financing strategy which satisfies the conditions
(1) $\mathcal{W}_0(\varrho) = 0$
(2) $P(\mathcal{W}_T(\varrho) \geq 0) = 1$
(3) $P(\mathcal{W}_T(\varrho) > 0) > 0$
is called an arbitrage.
If we have a self financing portfolio, and the manager fail to consider in his decision when the value of the portfolio is renegotiated, with respect to the underlying asset value, then the difference at time $\Delta\tau$ in the portfolio value is subject to the difference in the option value and in the interest on the inverted cash at hand given as,

$$\mathcal{W}_\tau - \mathcal{N}_\tau \mathtt{S}_\tau = N_\tau e^{r\tau}$$

$$\mathcal{W}_\tau = \mathcal{N}_\tau \mathtt{S}_\tau + N_\tau e^{r\tau}$$

$$\begin{aligned}d\mathcal{W}_\tau &= \mathcal{N}_\tau d\mathtt{S}_\tau + (\mathcal{W}_\tau - \mathcal{N}_\tau \mathtt{S}_\tau) r d\tau \\ &= r\mathcal{W}_\tau d\tau + \mathcal{N}_\tau (d\mathtt{S}_\tau - r\mathtt{S}_\tau d\tau\end{aligned}$$

The problem of pricing and hedging involves looking for a portfolio strategy which is self financing and that can replicate the terminal flow $h(\mathtt{S}_T)$, that is $\mathcal{W}(T, \mathtt{S}_T) = h(\mathtt{S}_T)$

This problem can be interpreted as finding two kinds of sufficiently regular functions, that is functions that are continuously differentiable along it sample path, denoted as $v(\tau, x)$ and $\mathcal{N}(\tau, x)$ which are described as

$$dv(\tau, \mathtt{S}_\tau) = v(\tau \mathtt{S}_\tau) r d\tau + \mathcal{N}(\tau, \mathtt{S}_\tau)(d\mathtt{S}_\tau - r\mathtt{S}_\tau d\tau)$$

$$\mathcal{W}(T, \mathtt{S}_T) = h(\mathtt{S}_T)$$

$\mathcal{N}(\tau, \mathtt{S}_\tau)$ is the hedging portfolio of the derivative with payoff function $h(\mathtt{S}_T)$

The investors that engage in the trading of derivative securities are of three types; they are refered to as Hedgers, Speculators and the Arbitrageurs. These are defined as follows;

Definition 1.16 (Hedgers):
This group uses options and other derivatives to reduce the risk that they face from potential future movement in market variables such as underlying asset price, interest rate, volatility e.t.c. Hedgers prefer to forgo the chance to make exceptional profits, even if future uncertainty appears to work to their advantage by protecting themselves against exceptional loss.

Definition 1.17 (Speculators):
This group uses options and other derivatives to bet on the future direction of a market. They take the opposite position to hedgers in the sense that, they are always out to make opportunistically high profits. Speculators are needed in financial markets to make hedging possible, since a hedger wishing to lay off risk cannot do so unless someone is willing to take it on.

Definition 1.18 (Arbitrageurs):
This group like to lock in riskless profit by simultaneously entering into transactions in two or more markets. An arbitrage opportunity exists if for example, a security can be bought in south at one price and sold at a slightly higher price in the north at the same time.

Remark:
In this work, we shall assume that there are no arbitrage opportunities. This eliminates the presence of arbitrageurs.

Definition 1.19 (Predictable Process):
A stochastic process $X(t), t \in [0,T]$ is said to be predictable if it is measurable with respect to the σ-field on $(\Omega \times \mathbb{R}_+)$ generated by an adapted processes.

Definition 1.20 (Local Martingale):
A local martingale $\mathcal{M}(t), t \geq 0$ is an adapted process such that there exists a sequence of stopping time T_n satisfying the condition that

$$T_n \leq T_{n+1}; T_n \longrightarrow +\infty$$

as

$$n \longrightarrow +\infty,$$

and for any $n \in \mathbb{N}$, $(\mathcal{M}_t \vee T_n)_{t \geq 0}$ is a martingale.
A stopping time is a random variable $T : \Omega \longrightarrow \mathbb{R} - +$ such that $(T \leq t) \in \mathcal{F}_t, t \in \mathbb{R}_+$. When working with local martingale, we can revert to the study of martingale by introducing the sequence T_n. Rose-Anne. D and Monique. J (2007)

Definition 1.21:
Assume P and Q are equivalent probability measure defined on $(\Omega, \mathcal{A})$, the measurable space, then Q is a risk-less measure that is, an equivalent martingale measure (EMM) provided the process

$$\begin{aligned}\tilde{\mathtt{S}}_\tau &= A_\tau^{-1}\mathtt{S}_\tau \\ &= e^{-r\tau}\mathtt{S}_\tau\end{aligned}$$

is a discounted process and it is a local martingale with respect to the probability measure Q.
Remark:
A stochastic process χ_τ is a sub-martingale respectively(a super-martingale) if and only if $\chi_\tau = \mathcal{M}_\tau + \tilde{A}_\tau$ respectively $(\chi_\tau = \mathcal{M}_\tau - \tilde{A}_\tau)$. $\tilde{A}$ represent an increasing predictable process and $\mathcal{M}$ represent the local martingale.

If we let $\sigma_\tau > 0$ $\forall \tau \in [0,T]$ and $\int_0^T \|\vartheta_s\|^2 ds < \infty$ a.s where

$$\vartheta = \frac{\kappa_\tau - r_\tau}{\sigma_\tau}$$

then a local (positive) martingale process is defined as

$$\mathtt{J}_\tau = \exp\Big(- \int_0^\tau \vartheta_s dB_s - \frac{1}{2}\int_0^\tau \|\vartheta_s\|^2 ds\Big)$$

provided

$$\mathbb{E}\Big(\exp\Big(- \int_0^T \vartheta_s dB_s - \frac{1}{2}\int_0^T \|\vartheta_s\|^2 d\tau\Big)\Big) = 1$$

then the process $\mathtt{J}_\tau$ is refered to as martingale where the measure Q and it equivalent measure P are related as $\frac{dQ}{dP} = \mathtt{J}_T$ such that

$$\tilde{\mathcal{W}}_\tau = \mathcal{W}_\tau + \int_0^\tau \vartheta_s ds$$

under Q represents a Brownian motion, Eric Fournie et al (1999).
Therefore under probability measure Q, the price process will be defined as

$$\mathtt{S}_\tau = \mathtt{S}_0 \exp\Big(\int_0^\tau (r_s - \frac{\sigma^2}{2}) ds + \int_0^\tau \sigma_s d\tilde{B}_s \Big)$$

and the discounted price process forms a local martingale, Steven,(2004)

$$\begin{aligned}\tilde{\mathtt{S}}_\tau &= A_\tau^{-1}\mathtt{S}_\tau \\ &= \mathtt{S}_0 \exp\Big(\int_0^\tau \sigma_s d\tilde{B}_s - \frac{1}{2}\int_0^\tau \sigma_s^2 ds\Big)\end{aligned}$$

The discounted wealth process of any self-financing strategy is also a local martingale, therefore,

$$\begin{aligned}\tilde{\mathcal{W}}_t(\varrho) &= x + \int_0^\tau \mathcal{N}_s d\tilde{\mathtt{S}}_s \\ &= x + \int_0^\tau \sigma_s \mathcal{N}_s \tilde{S}_s d\tilde{\mathcal{W}}_s\end{aligned}$$

If there are no opportunities for arbitrage, then

$$\mathbb{E}^Q\Big(\int_0^T (\sigma_s \mathcal{N}_s \tilde{\mathtt{S}}_s)^2 ds\Big) < \infty$$

This implies that $\tilde{\mathcal{W}}_\tau$ is a martingale under measure Q. Using the property of martingale, Rose-Anne.D and Monique. J (2007), we have

$$\mathbb{E}^Q(\tilde{\Upsilon}_T(\varrho)) = \Upsilon_0(\varrho) = 0$$

Remark:
Subsequently, to reduce ambiguity in our notations, we shall write the expectation of any process with respect to probability measure Q, $\mathbb{E}^Q(.)$ simply as $\mathbb{E}(.)$

Definition 1.22 (Admissible):
If $\mathcal{W}_\tau$ is bounded from below by some fixed real numbers, then the strategy is said to be admissible. If the value process of a portfolio ϱ satisfies $\mathcal{W}_\tau(\varrho) \geq 0$ for a pre-investment $x > 0$, that is, the initial amount invested in the risk free asset, then the portfolio is refered to as admissible.
Remarks:
1) The class of admissible portfolio do not permit arbitrage opportunity. This mean that the condition

$$\mathbb{E}(\tilde{\mathcal{W}}_T(\varrho)) \leq \mathcal{W}_0(\varrho) = 0$$

is satisfied. Hence, $\mathcal{W}_T(\varrho) = 0$ with respect to measure Q. This contradict the assumption $P(\Upsilon_T(\varrho) > 0) > 0$.
2) Suppose σ_τ is a uniformly bounded process, then $\{\tilde{\mathrm{S}}_\tau, 0 \leq \tau \leq T\}$, a discounted price process is a martingale with respect to measure Q.Steven. E. S (2004).

Definition 1.23 (Replicating Portfolio):
A portfolio is said to be a replicating portfolio if the portfolio consists of cash deposit and a certain unit of assets that can re-generate themself over time t. The idea is to keep this unit of assets constant over a small time δt.
The changes that occured in the portfolio has two sources;
1) Asset price fluctuation and
2) The interest accrued on the cash deposit over time.

Definition: 1.24 (Complete Market):
A complete market is a financial market where every contingent claim which is also known as financial derivative is replicable, otherwise, it is incomplete.

Remarks:

(1) By financial derivative, it means that the value of financial instuments, for example, option contract are derived from the underlying assets and not derivative, that is differentiation.

(2) If $\mathcal{C}$ is a contingent claim whose price $x \in \mathbb{R}$ is arbitrage free, then there is an admissible strategy ϱ such that $\mathcal{C} = \mathcal{W}_T^{x,\varrho}$ a.s.

(3) If a financial market is completely free from having arbitrage opportunities, then any claim $\mathcal{C}$ has a unique arbitrage free price, Rose-Anne. D and Monique. J (2007)

$$x = \mathbb{E}(e^{-rT}C)$$

(4) In incomplete market, there is generally no possibility for portfolio to replicate.

Definition 1.25:
Suppose that an investor holds a Call option with strike price K. If $\tau = 0$ is the time when the Call option was acquired and $S(\tau)$ is the price of the underlying asset at time τ, then, if at maturity time T,

- $S(T) > K$, then, the option is in the money.
- $S(T) = K$, then, the option is at the money.
- $S(T) < K$, then, the option is out of the money.

1.6.1 Change of Probability Measure

In this section, we consider the relationship between the probability measure P and the risk neutral measure Q. The price process $S(t)$ is defined on the probability space $(\Omega, \mathcal{A}, P)$ with probability measure P. When a model is neutral with respect to risk, that is when an investment in the riskless assets could yield the same return as the investment in the risky assets, then a no arbitrage opportunity position is attained. To attain this position, there is a need to change from probability measure P to a risk-neutral measure Q.
The connection between the two probability measures shall be discuss in what follows

Let Ω be a non empty set, and let $\mathsf{A} \subset \Omega$. If $\varpi_1, \varpi_2 \in \Omega$ then $P(\varpi_1) = p$, and $P(\varpi_2) = 1 - p$, implies that ϖ_1, ϖ_2 are compliments where $0 < p < 1$

Definition 1.26:
The probability measure P is said to be equivalent to the probability measure Q expressed as $(P \sim Q)$, if P and Q have equal null sets such that

$$Q(\varpi) = 0$$

if and only if

$$P(\varpi) = 0$$

$\varpi \in \Omega$.

If $Q(\varpi_1) = q$ and $Q(\varpi_2) = 1 - q$, where $0 < q < 1$ then we define the relation

$$\wedge(\mathsf{A}) = \frac{Q(\mathsf{A})}{P(\mathsf{A})}$$

as the ratio of the two probability measures P and Q. Steven. E. S (2004). This implies that

$$\wedge(\varpi_1) = \frac{Q(\varpi_1)}{P(\varpi_1)} = \frac{q}{p}$$

and

$$\wedge(\varpi_2) = \frac{Q(\varpi_2)}{P(\varpi_2)} = \frac{1-q}{1-p}.$$

So by definition of $\wedge(\mathsf{A})$, $\forall\ \mathsf{A} \subset \Omega$,

$$Q(\mathsf{A}) = P(\mathsf{A}) \wedge (\mathsf{A})$$

If υ is a random variable, then it expectation with respect to the probability measure P is defined as

$$\begin{aligned} E_P(\upsilon) &= \upsilon(\varpi_1)P(\varpi_1) + \upsilon(\varpi_2)P(\varpi_2) \\ &= p\upsilon(\varpi_1) + (1-p)\upsilon(\varpi_2) \end{aligned}$$

and with respect to the probability measure Q, it is defined as

$$\begin{aligned} E_Q(\upsilon) &= \upsilon(\varpi_1)Q(\varpi_1) + \upsilon(\varpi_2)Q(\varpi_2) \\ &= q\upsilon(\varpi_1) + (1-q)\upsilon(\varpi_2) \\ &= \upsilon(\varpi_1)\wedge(\varpi_1)P(\varpi_1) + \upsilon(_2)\wedge(\varpi_2)P(\varpi_2) \\ &= E_P(\wedge\upsilon) \end{aligned}$$

let $H = \wedge\upsilon$, then

$$\begin{aligned} E_P(\upsilon) &= H(\varpi_1)P(\varpi_1) + H(\varpi_2)P(\varpi_2) \\ &= E_P(H) \end{aligned}$$

If $\upsilon = 1$, then $E_Q(\upsilon) = E_P(\wedge) = 1$
If we consider a random variable $\wedge > 0$ where $E_P(\wedge) = 1$ and

$$Q(\varpi_k) = \wedge(\varpi_k)P(\varpi_k) > 0, \qquad k = 1, 2$$

then

$$\begin{aligned} Q(\Omega) &= Q(\varpi_1) + Q(\varpi_2) \\ &= \wedge(\varpi_1)P(\varpi_1) + \wedge(\varpi_2)P(\varpi_2) \\ &= E_P(\wedge) = 1 \end{aligned}$$

So $\wedge$ is strictly positive random variable. This implies $E_P(\wedge) = 1$ and $Q(\varpi) = \wedge(\varpi)P(\varpi)$, $\quad E_Q(\upsilon) = E_P(\wedge\upsilon)$, for any equivalent change of measure.

Change of Measure for Normal Random Variables

Here, we consider a change of measure with respect to the normal random variable distributed normally with mean κ and variance 1.
Let $f_\kappa(x)$ be the probability density function of a random variable x normally distribution such that $x \sim N(\kappa, 1)$, with mean κ, a real number and variance 1 and let P_κ be the probability measure of $N(\kappa, 1)$ on $\mathbb{R}, (B, (\mathbb{R}))$. then

$$\begin{aligned} f_\kappa(x) &= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\kappa)^2} \\ &= f_0(x) e^{\kappa x - \frac{\kappa^2}{2}} \\ &= f_0(x)\wedge(x) \end{aligned}$$

so

$$\begin{aligned} P(\mathsf{A}) &= \int_{\mathsf{A}} f(x)dx \\ &= \int_{\mathsf{A}} dP \end{aligned}$$

so that

$$dP = P(dx) = f(x)dx$$

and
if a random variable $\chi \geq 0$ then

$$E(\chi) = 0$$

if and only if

$$P(\chi = 0) = 1$$

by the property of expectation.
So

$$P_\kappa(\mathsf{A}) = \int_{\mathsf{A}} \wedge(x) P_0(dx)$$
$$= E_p(I_{\mathsf{A}} \wedge) = 0$$

It therefore means that $P_0(I_{\mathsf{A}} \wedge = 0) = 1$, where I_{A} represents the Indicator function.
Since $\wedge(x) > 0 \; \forall \; x$, then $P_0(\mathsf{A}) = 0$. This means that $\wedge$ can be expressed as

$$\wedge = \frac{dP_\kappa}{dP_0}, \quad \wedge = \frac{dP_\kappa}{dP_0}(x) = e^{\kappa x - \frac{\kappa^2}{2}}$$

This implies that by an equivalent change of probability measure, any $N(\kappa, 1)$ probability can be determined from the $N(0, 1)$ distribution.

Theorem 1.1(Removal of the mean):
Let $\gamma = \chi + \kappa$ where χ and γ are distributed normally as $N(0, 1)$, then there exists an equivalent probability $Q \sim P$ such that

$$\frac{dQ}{dP}(x) = \wedge(x) = e^{-\kappa x - \frac{\kappa^2}{2}}$$

Change of Measure on a General Space
Assume on the same space, we define P and Q representing two probability measures, then we have the following;

Definition 1.27:
Let $Q(\mathsf{A}) = 0$ whenever $P(\mathsf{A}) = 0$, then P is said to be equivalent to Q if P and Q are absolutely continuous with respect to each other expressed as $Q \ll P$ and $P \ll Q$.

Theorem 1.2(Radon Nikodyn):
Randon Nikodyn derivative helps to determine all the equivalent martingale measure (EMM). Assume $Q \ll P$, then there is a random variable $\wedge$ with $\wedge \geq 0$, $E_P(\wedge) = 1$ and $Q(\mathsf{A}) = E_P(\wedge I(\mathsf{A})) = \int_{\mathsf{A}} \wedge dP$, where A is any measurable set. Conversely, if the random variable $\wedge$ and Q are as defined above, then $\wedge$ is a

probability measure and $Q \ll P$. The random variable $\wedge$ is refered to as the Radon-Nikodyn derivative or simply as the Q density with respect to P which is represented as $\frac{dQ}{dP} = \wedge$.
If $Q \ll P$, it means that the expectation of any integrable random variable χ with respect to Q can be related by $E_Q(\chi) = E_P(\wedge\chi)$ under P and Q.

Definition 1.28:
Assume we define on the same space P and Q representing two probabilty measures, if we have A, a set where $P(\mathsf{A}) = 0$, and $Q(\mathsf{A}) = 1$, then P and Q are singular.
By singularity, it is possible to decide on the probability model with level of certainty simply by observing the outcome of the model.

Chapter 2
REVIEW OF LITERATURE

2.1 Introduction

In this work, we consider some studies where Malliavin calculus has been applied to finance especially in pricing and hedging of options.
Options whose formulas can be computed explicitly can be derived with the Ito framework, but it is challenging to work in the Ito framework when the payoff function are not regular. These types of payoff can be handled in Malliavin sense. This is because the original idea of Malliavin calculus is to study how smooth the densities of stochastic differential equations solutions are especially when they have discontinuous coefficients
The real advantage of using Malliavin calculus by means of the integration by part is that, it is applicable when dealing with random variables with unknown density functions and when we have options with non smooth payoffs.
In this chapter, we summarize some of the findings that have been studied over time about the applications of Malliavin calculus in finance

2.2 Review of Relevant Literature

A lot of work and publications have appeared in recent years about the Malliavin calculus and its applications in pricing and hedging of options. Here, we consider some of these literature.

Ocone, D. (1984) discovered an explicit representation of the Clark representation formula using the Malliavin derivative. This formula is refer to as Clark Ocone formula. This formula has become famous among the users of Malliavin calculus. In 1991, Ocone together with Karatzas applied the representation formula to finance,(and since then, different authors have applied the theory to the study of finance). They shows that an explicit formula that can replicate the contingent claims portfolio can be obtained through the representation formula especially when dealing with complete market.

Eric Fournie et al,(1999) presented a probabilistic approach to the numerical computations of Greeks in Finance using the Malliavin Calculus principles. The Greeks formulae they obtained were for path-dependent discontinuous pay off functional. Their result was applied to the study of European options in the Black and Scholes model framework. When compared with the approach of Monte Carlo finite difference, the method was found to be more efficient especially when the payoff functional is discontinuous. Where as, the Monte Carlo finite difference approximation(FDA) has a convergence rate of $n^{-\frac{1}{4}}$ against Eric Fournie method which have a convergence rate of $n^{-\frac{1}{2}}$

Broadie and Glassermann(1996) obtained a convergence rate of $n^{-\frac{1}{3}}$ with the central finite difference approximation.

Ivanenko and Kulik (2003), did a study of Integral representation of the likelihood function and the derivative of the log- likelihood function using Malliavin Calculus for a model that is centred on discrete time observations of the solution to the equation of the form

$$dx_t = a_\theta(x_t)dt + dz_t \tag{2.2.1}$$

where z represent a levy process, $a : \theta \times \mathbb{R} \to \mathbb{R}$ represents a measurable function, $\theta \subset \mathbb{R}$ is a parametric set.

Due to the implicit nature of the likelihood function of (2.2.1), the authors used the Malliavin Calculus to control the properties of the likelihood and log-likelihood functions with respect to the objects involved in the model. The Malliavin Calculus becomes a tool used for showing both existence and smoothness of distribution densities. This is crucial when studying the sensitivities of expectations with respect to the parameters. Their approach follows from K. Bichteler , et al (1987) and was used by Bally, V and Clement, E (2011), followed by Bouleau, N and Denis, L. (2011). The integral representation for the likelihood functions together with the differential of the log-likelihood function in terms of the parameter were used for proving the regularity of the experiment generated via set of discrete time observations of the solution of equation (2.2.1). The representations also provide a basis for asymptotical analysis of the behaviours of the model when sample size increase to infinity.

Wanyang Dai (2013) consider the numerical schemes, the adapted solution, and the corresponding convergence analysis in the study of unified backward stochastic partial differential equation (BSPDE) described as a vector valued function.

$$U(s,y) = G(y) + \int_t^T \mathcal{K}(v,y,U) + \int_t^T (\mathcal{R}(v,y,U) - \tilde{U}(v,y))dW(v), \qquad (2.2.2)$$

where $\mathcal{K}$ and $\mathcal{R}$ are non linear partial differential operators that depend on U, $\tilde{U}$ and their associated high order partial derivatives.
So

$$\mathcal{K}(v,y,U) = \mathcal{K}(v,y,U(v,y),...,\tilde{U}^{(k)}(v,y),\tilde{U}(v,y),...,\tilde{U}^{(m)}(v,y))$$

$$\mathcal{R}(v,y,U) = \mathcal{R}(u,y,U(u,y),...,U^{(n)}(v,y))$$

(2,2.2) becomes a BSDE if the value of $\mathcal{J} = 0$ and $\mathcal{L}$ does not depend on their associated parial defferentials but on x, V, and $\tilde{V}$ which was study by Peng (1990). Also, (2.2.2) reduces to a non linear BSPDE derived by Zariphopoulou and Musiela in the study of optimal-utilty based portfolio chose the value of $\mathcal{J}$ to be zero and allow $\mathcal{L}$ to depends the derivatives of V and $\tilde{V}$. The BSPDE in (2.2.2) was developed in line with the BSPDE studied by Becherer, Zariphopoulou and Musiela, Dai . In order to solve (2.2.2), two numerical algorithms were proposed. The first is an iterative scheme while the second is not exactly iterative because it require to solve equations that is either non linear or linear at every point.
The error estimation and the error analysis or rate of convergence of the scheme was conducted with respect to a completely discrete criterion. The analysis was based on the theory of random field developed to show both the uniqueness and the existence of adapted solutions of the Malliavin derivative of first and second order with randomness environments.

Yuzuru Inahama (2014) studied rough differential equations driven by Gaussian rough paths using Malliavian Calculus under mild assumptions on co-efficient vector fields and underlying Gaussian processes. It was proved that solutions at a fixed time is smooth in the Malliavin calculus sense.
Dahl,Mohammed,and Oksendel (2015) worked on optimal stochastic controlled process $\chi(\tau)$, whose state dynamics represent a controlled stochastic differential equations which has jumps, delay and noisy memory. The dynamic of $\chi(\tau)$ is defined on $\int_{t-\tau}^{t} \chi(s)dW(s)$, where $W(t)$ is a Brownian motion, τ is the memory span, and it involves memory due to the influence from the previous values of the state process.

They derived in two different ways, the necessary and sufficient maximum principles for the process $\chi(\tau)$ which resulted in two set of maximum principle. The first set was deduce by using Malliavin derivative techniques while the second set was deduce by reducing the problem to a discrete delay optimal control problem.

Clyin (1989) worked on finite difference approximation where he use Monte-Carlo simulation method to approximate the derivatives of payoff of certain exotic option. Though his approach has error because expectation of the derivatives were approximated numerically especially when the pay off is discontinuous.This was first observed by Curran (1994) when he determine the greeks by using the expectation of the payoff derivatives.

Broadie and Glasserman (1996) came up with the process of differentiating the density function of the pay off function using the likelihood ratio to determine the greek delta. For instance, the delta obtained is represented as

$$\Delta = \frac{\partial}{\partial x} E^X[\varphi(X(T))]$$

$$= E[\varphi(X^X(T))\frac{\partial}{\partial x}\ln P(X^X(T))]$$

The density function in their approach require an explicit expression even though the approach was adjudge to be efficient.

Avellanda et al (2000) motivated by the work of Kullback-Leibler (1998) on relative entropy maximization, developed yet another way by which the weight function can be obtained. They worked on the inclusion of a weight functional by taking the derivative of the pay off function.

Benhamon (2003) studied how to characterized and choose the weights by;

(i) expressing the weights function as skorohod integrals which allow the introduction of the idea of weighting function generator.

(ii) choosing the weights, he focuses on those random variable that provide a minimum variance described as $\varphi(X(T))$W

(iii) The weight with minimum variance is described as the conditional expectation of the weight given $X(T)$ as the process iv The link in the density method with the likelihood ratio was provided by the result.

Arturo Kohatsu-Higa and Miquel Montero (2003) discussed the significance of Malliavin calculus in finance and applied the ideas to the simulation and computation of greeks using Monte Carlo simulation. Their work focussed on European -type option whose formula are computed explictly.Their approach shows that it is not possible to get the integration by part formula which guarantee a small variance,because they are of the opinion that,for a minimal variance to be attained,the probability density of the random variable must be known.

Ali, S. U (2008) studied the existence and uniqueness probability solutions of the variational inequalities for American style of option using the main tool of the Malliavin calculus,which was the extension of the Ito calculus. It was shown that the American option possess a unique solution when the calculus moved from the Ito type to the Malliavin type. This study follows from the idea of Kusuoka (1987).

Youssef El-Khatib (2009) did a study of stochastic volatility model using the theories of Malliavin calculus in calculating the sensitivities of the price of certain underlying. This was first considered by Fournie et al (1999) for deterministic volatility models,and this became the tool for studying the case of the stochastic volatility model which this author studied. The author computed the sensitivities of the price of underlying assets driven by Brownian motion which takes into consideration the noise effect. In doing this,the theory of Malliavin calculus was engaged as in the case of Fournie et al (1999)

Nicola, C. P and Piergiacomo, S (2013) studied Asian basket option's problem of hedging and pricing using the method of Quasi-Monte carlo simulation in a Black-Schole market associated with a time-dependent volatilities.This method as highlighted by the authoronly generated result for the delta of the price. This Quasi-Monte Carlo simulation method was observed not independent and not sufficient for evaluating the delta without the concept of the Malliavin derivative as discussed by Sabino (2008).

Abbas-Turki, L. A and Lpeyre, B (2011) was concerned with pricing of American option with the aid of Monte-Carlo method and Malliavin calculus.The aim is to use these technique to reduce the variance of the computation.This was carried out by using the Monte-Carlo non parametric variance reduction method rather than using the localization function reduction method of Bally et al (2005) and Tsitsiklis and Roy (2001) . Their method require writing the conditional expectation of the stochastic process without localization by using the Malliavin calculus to estimate the variance based on high number of simulated path contrary to the assumption of Abbas-Turki (2009).

Deya, A. and Tindel, S. (2013) highlighted in their study of a class of finite dimensional generated stochastic heat equation some results about its smoothness and existence of solution. These results was obtained using the theories of Malliavin calculus and the pathwise estimates for integrals generated by rough signals.

Yaozhong, H., Nualart, D. and Xiaoming, S (2011) did a study of backward stochastic differential equation(BSDE) which has a general terminal value and a general random generator both of which are not particularly from a forward equation.The authors obtained by Malliavin calculus,the convergence scheme for the L^p-Holder continuity solution of the BSDE and several numerical approximation was obtained for the scheme.The study did not specifically assumed any terminal value, which means that the terminal value could be any random variable and that the generator can also be any random variable that is $\mathcal{F}_t$ - measurable. Due to the problem in constucting a numerical scheme for the BSDE with adapted process,and the approximation of the adapted process,Z_t, Malliavin calculus becomes the appropriate tool since the random variable (the adapted process) is writen as $Z_t = D_t Y_t$ as shown by Karoui et al(1997) and used by Zhang, J (2004) and Ma, J (2002), where Y_t represent trace of Malliavin derivative

Samy, J. and Saporito, Y. F. (2018) developed an approach that is centered around the theories of Malliavin calculus to compute the sensitivities of path-dependent derivative security. They considered in the Ito calculus framework, a measure of path-dependence of functionals and time functional derivatives which are use for the classification of functionals with respect to the degree of path-dependence. They use the Malliavin calculus integration by part technique for the computation of the sensitivities for path- dependence derivative securities.Through this technique, the weighted expectation formula for the greeks were obtained.

Federico, D. O and Ernesto, M (2014) use the theory of integration by part technique of the Malliavin calculus and the method of likelihood ratio and finite difference to compute the greeks for exponential Levy model. Exact formula for greeks of European option were obtained via the likelihood ratio method and the Malliavin calculus. The authors also worked using the method of fast Fourier transform in finding an approximation and the associated error which shows a considerable improvement when compare with the Black-Schole model. An approximation was also obtained for the variance gamma model associated with the levy process and the error was minimal because the error was generated by approximation of the integral.

Christian, B and Peter, P (2016) provides some necessary and sufficient condition for weak and strong L^2-convergenceof a discretized Malliavin derivative, skorohod integral,the discrete form of Clark-Ocone formula and the continuous form. They showed that there is a connection between the Malliavin calculus on Bernoulli and Wiener space.

Anselm, H and Ludger, R (2018) use the principle of Malliavin calculus to determine an explicit representation for sensitivities of Asian and European derivatives where the underlying assets are driven by an exponential levy process through the Monte-carlo procedure of the Malliavin calculus.This method takes care of the jump in the process.

Viktor, B., Luca, P. D. and Yuliya, M. (2016) considered the pricing of derivatives that has payoff with discontinuous polynomial growth.They consider underlying asset whose dynamics are defined in the Black-Scholes setting associated with a stochastic volatility. Three different methods were considered in solving this problem. One, they consider a process by which they can tranform the initial asset price so that the discontinuity can be eliminated. This makes the fractional Brownian motion and the Wiener process discretization possible and consequently the estimate of the rate of convergence of the discretized prices. Secondly, they considered on the fractional Brownian motion trajectory the conditional expectation of the process. Then, a closed expression was obtained for the fractional Brownian motion, which was used to evaluate the price. Lastly, the density of the integral functional was calculated using Malliavin calculus as it rely on the trajectory of the fractional Brownian motion

Kuchuk-Iatsenko, S., Mishura, Y. and Munchak, Y. (2016) considered a problem of exact price of European option in a financial market with stochastic volatility defined by a functional of Cox-Ingersoll-Ross process or Ornstein-Uhlenbeck process. The random variable density function that described the mean of the volatility over time to expiration was obtained using the Malliavin calculus. With this, the option price can be calculated with respect to minimum martingale measure especially when the Wiener process driving the dynamic of the asset price and the

Wiener process that defines the volatility are uncorrelated.

Nacira, A and Oksendal, B (2018) considered an alternative method for determining the optimal stochastic control of stochastic process with jumps contrary to Peng, S. (1990) by ensuring that the coefficient of jump and diffusion depends on the control without considering the BSDE with second order derivative as in the case of Oksendal. B (2017)

Youssef El-Khatib, Abdulnasser, H. J (2019) considered the general form of the dynamics of asset price volatility as a stochastic volatility. The objective of the study is to calculate the price sensitivities for the stochastic volatility models using the Malliavin calculus. Their result shows that each of the price sensitivities represent a source of financial risk and the result provide an improvement on the hedging of the underlying risk.

Caroline Hillairet,Ying Jiao and Anthony, R. (2018) provides a valuation formula for various kind of contracts in actuarial, using the Malliavin calculus when the contract is generally on loss process. The expected cash flow, according to the authors was expressed in term of a building block in line with the Black-Schole formula. The loss process depend on the jump and the intensity time of the counting process. The building block represent the cumulated loss in line with stop-loss contract, considered when the expected shortfall risk measure is been computed.

Julien, H., Philip Ngare and Antonis, P. (2018) works on the formula for pricing European quanto options written on LIBOR rate. They use domestic forward measure to derived the system dynamics and then consider the price of the quanto option.The author consider the local volatility model for the LIBOR rate and the FX rate so that smile effect in the fixed income and FX market might be taken into consideration.They observed that,due to the structure of the local volatility function, a closed form solution for quanto option does not exist.

Bilgr Yilmaz (2018) consider computing option sensitivities problems under the condition that the underlying asset and the interest rate emanated from a stochastic volatility model and a stochastic interest rate respectively using the theory of the Malliavin calculus which leads to effective numerical implementation of a running Monte-Carlo algorithm. This algorithm ,the author implied can be used for different types of option even if their payoff functions are not differentiable. This is similar to our work except that the author consider a stochastic volatility model.

Takuji, A and Ryoichi, S (2019) consider the explicit martingale representation for random variable which are described as a functional of a levy process.The integrands that appear in this martingale representation described by the theorem of Clark-Ocone are expressed by the conditional expectation of the Malliavin derivatives. The author extend this to random variable that are not Malliavin differentiable using the Ito formula rather than the Malliavin calculus. This extension was applied to an explicit representation of locally risk-minimizing strategy of digital option for exponential levy models. The author also discussed the Malliavin differentiabilty in terms of the levy process of digital option whose payoff is described by an indicator function.

Chapter 3
METHODOLOGY

3.1 Introduction

In this section, we shall discuss the theory of Malliavin calculus and its properties. This calulus is a tool used to develop our formulations in this study.
The formulation of the Wiener process (Wiener.N,1923) as a mathematical model of Brownian motion leads to the development in the theory of integration on a function space and to the study of stochastics analysis.
Malliavin calculus, was introduced by Malliavin, P in 1978. This calculus is also known as the calculus of variation with a theory which extends the calculus of variation to the study of stochastic calculus. One of the benefits is that, the theory gives a probabilistic proof of the Hormander criterion (Hormander .L,1967) of hypoellipticity by relating the smoothness of the solution of a second order partial differential equation with the smoothness of the law of the solution of a stochastic differential equation.

Malliavin, P. (1978) studied the solution of stochastic differential equation generated by Brownian noise by considering the regularity of the law of functionals of the Brownian motion. The calculus can be adapted to both finite dimensional space, like $\mathbb{R}^n$ and infinite dimensional space like the Wiener space.
Malliavin Calculus helps us to obtain the derivative of the functions of Brownian motion and this derivative is referred to as Malliavin derivative.

3.2 Malliavin Calculus for Gaussian Processes

The study of Mallivian Calculus started with the concept of Gaussian Calculus, that is, a Calculus with respect to a Gaussian field, and in the abstract setting with respect to abstract Wiener Space. Mallivian Calculus is an element of stochastic analysis that is valid for a general class of Gaussian objects namely the Isonormal

Gaussian processes.

Definition 3.1:
Let $\mathcal{R}$ represents a real separable Hilbert space (i.e $\mathcal{R}$ admits a countable orthonormal basis) with $\langle \cdot, \cdot \rangle_{\mathcal{R}}$ representing the inner product for $r \in \mathcal{R}$, we let $||r||_{\mathcal{R}} := \sqrt{\langle r, r \rangle_{\mathcal{R}}}$.
If $Z := \{Z(r), r \in \mathcal{R}\}$ is a stochastic process defined on the complete probability space $(\Omega, \mathcal{A}, P)$ then Z is an isonormal Gaussian process provided

(i) the random variable $Z(r)$ is a centered Gaussian random variable $\mathbb{E}(Z(r)) = 0$ and variances $||r||^2_{\mathcal{R}}$ $\forall$ $r \in \mathcal{R}$

(ii) $\mathbb{E}[Z(g)Z(r)] = \langle g, r \rangle_{\mathcal{R}}$ $\forall$ $(g, r) \in \mathcal{R}^2$

(iii) The map $r \to Z(r)$ is linear.

We refer to the pair $(Z, \mathcal{R})$ as Isonormal Gaussian process but for convenience of notation we simply call it Z on $\mathcal{R}$, a real seperable Hilbert space

Z, by the definition is a Gaussian process indexed by functions in some Hilbert space which describes the covariance of Z.

Definition 3.2: $B := (B_\tau)_{\tau \in [0,T]}$ is a standard Brownian motion with respect to a right - continuous filtration $(\mathcal{A}_\tau)_{\tau \in [0,T]}$ if

(i) B is adapted with respect to $(\mathcal{A}_\tau)_{\tau \in [0,T]}$

(ii) $B_0 = 0$

(iii) B possess a stationary Independent increments

(iv) B is a Gaussian process that has Variance τ $\forall$ $0 \leq \tau_0 \leq \tau_1 \leq \cdots \leq \tau_n \leq T$, the random vector $(B_{\tau_1} - B_{\tau_0}, \ldots, B_{\tau_n} - B_{\tau_{n-1}})$ is Centered Gaussian with Covariance matrix $Diag(\tau_1 - \tau_0, \ldots, \tau_n - \tau_{n-1})$.The Brownian motion can be described in the setting of isonormal Gaussian process.

Let $\mathcal{R} := L^2([0, T], d\tau)$ be the space of deterministic functions $h : [0, \tau] \to \mathbb{R}$ such that $\int_0^\tau h(s)^2 ds < \infty$. then define $Z(r) := \int_0^T r(s) dB_s$, $r \in \mathcal{R}$ where the stochastic integral is defined in the sense of Ito calculus. By linearity of the Ito stochastic integral,we have that

- Z is a linear map

- $\mathbb{E}[\int_0^T h(s)dB_s] = 0 \; \forall \; h \in \mathcal{R}$

- Z is Centered Gaussian random variable with variance $\int_0^T h(s)^2 ds \; \forall \; h \in \mathcal{R}$

- $\mathbb{E}[\int_0^T g(s)dB_s \int_0^T h(s)dB_s] = \int_0^T g(s)h(s)ds = \langle g, h\rangle_{\mathcal{R}} \; \forall \; (g, h) \in \mathcal{R}^2$.

3.3 Decomposition of Wiener Chaos

Malliavin calculus on abstract Wiener space represents an infinite dimensional space. This space can be decomposed into orthogonal sum of subspace $\mathcal{R}_n$. Giulia Di Nunno (2009). This decomposition is obtained via the hermite polynomial. This is because, the family of hermite polynomials constitutes an orthonomal basis for $L^2(\mathbb{R}, \mu(dx))$, where $\mu d(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{x^2}{2}} d(x)$, Schoutens. W (2000). The review of the hermite polynomial shall be discuss in this section.

Let $r \in \mathcal{R}$ with an inner product defined as $\langle ., . \rangle_{\mathcal{R}}$ where $\mathcal{R}$ represents a real separable Hilbert space, then we denoted by $||r||_{\mathcal{R}}$ the norm of r.

Definition 3.3:

If Z is as defined in definition (3.1) above and it satisfies condition (2), then Z is a centred Gaussian family of random variables

This means that $\{Z(r)\}$ is classified as a Gaussian family

If we have the Hilbert space $\mathcal{R}$, we can always form a probability space and a Gaussian process $\{Z(r)\}$. The mapping $r \to Z(r)$ gives a linear isometry between $\mathcal{R}$ and $\mathcal{R}_1$ where $\mathcal{R}_1$ is a closed subspace of $L^2(\Omega, \mathcal{A}, P)$. The members of $\mathcal{R}_1$ are zero- mean Gaussian random variables.

So,

$$||Z(r)||^2_{L^2} = E(Z(r)^2) = ||r||^2_{\mathcal{R}}$$

If the σ-field $\mathcal{G}$ is formed by $\{Z(r), r \in \mathcal{R}\}$ some random variables then we consider by the Hermite polynomial, the decompsition of $L^2(\Omega, \mathcal{G}, P)$.

Let $H_m(y)$ denote the mth Hermite polynomial, then

$$H_m(y) = \frac{(-1)^m}{m!} \exp^{\frac{y^2}{2}} \frac{d^m}{dy^m} (\exp^{\frac{-y^2}{2}})$$

$m \geq 1$, such that $H_0(y) = 1$

In the expansion of

$$G(\tau, y) = \exp(\tau y - \frac{\tau^2}{2})$$

in powers of τ, the coefficients of the expansion represent the hermite polynomials which can be expressed as

$$G(\tau, y) = \exp(\frac{y^2}{2} - \frac{1}{2}(y - \tau^2))$$

This function G has some particular properties i.e.

$$\frac{\partial G}{\partial y} = \tau \exp(\tau y - \frac{\tau^2}{2}) = \tau G(\tau, y)$$

$$\frac{\partial G}{\partial \tau} = (y - \tau) \exp(\tau y - \frac{\tau^2}{2})$$
$$= (y - \tau) G(\tau, y)$$
$$G(-y, \tau) = \exp(-\tau y - \frac{\tau^2}{2})$$
$$= G(y, -\tau)$$

These can be compared for $m \geq 1$ with the Hermite polynomials properties i.e.

$$H'_m(y) = H_{m-1}(y)$$
$$(m+1)H_{m+1}(y) = yH_m(y) - H_{m-1}(y)$$
$$H_m(-y) = (-1)^m H_m(y)$$

These can be shown by induction as follows ;

Let m=1, from

$$H_m(y) = \frac{(-1)^m}{m!} e^{\frac{y^2}{2}} \frac{d^m}{dy^m}(e^{\frac{-y^2}{2}}),$$

we have

$$H'_1(y) = \left(-e^{\frac{y^2}{2}} \frac{d}{dy} e^{\frac{-y^2}{2}}\right)' = \left(-e^{\frac{y^2}{2}}(-y)e^{\frac{-y^2}{2}}\right)' = y' = 1 = H_{m-1} = H_0(y)$$

Let m=2, then

$$H'_2(y) = \left(\frac{1}{2} e^{\frac{y^2}{2}} \frac{d^2}{dy^2} e^{\frac{-y^2}{2}}\right)' = \left(\frac{1}{2} e^{\frac{y^2}{2}} \frac{d}{dy}(-ye^{\frac{-y^2}{2}})\right)' = \left(\frac{1}{2} e^{\frac{y^2}{2}}(-e^{\frac{-y^2}{2}} + y^2 e^{\frac{-y^2}{2}})\right)'$$

$$= \frac{1}{2}\left(-1+y^2\right)' = y = H_1(y)$$

Let m=3, then

$$H_3'(y) = \left(\frac{-1}{6} e^{\frac{y^2}{2}} \frac{d^3}{dy^3} e^{\frac{-y^2}{2}}\right)' = \frac{-1}{6}\left(y+2y-y^3\right)' = \frac{1}{2}\left(y^2-1\right) = H_2(y)$$

So,

$$H_1'(y) = H_{1-1}(y) = H_0(y)$$

$$H_2'(y) = H_{2-1}(y) = H_1(y)$$

$$H_3'(y) = H_{3-1}(y) = H_2(y)$$

showing that $H_m'(y) = H_{m-1}(y)$

Also,

$$(1+1)H_{1+1}(y) = 2H_2(y) = yH_1(y) - H_0(y)$$

$$\implies 2\left(\frac{1}{2}(y^2-1)\right) = y(y) - 1$$

$$\implies y^2 - 1 = y^2 - 1$$

$$(2+1)H_{2+1}(y) = 3H_3(y) = yH_2(y) - H_1(y)$$

$$\implies 3\left(\frac{-1}{6}(y+2y-y^3)\right) = y\left(\frac{1}{2}(y^2-1)\right) - y$$

$$\implies \frac{-1}{2}\left(y+2y-y^3\right) = \frac{1}{2}\left(y^3-y\right) - y$$

$$\implies \frac{1}{2}\left(y^3-3y)\right) = \frac{1}{2}\left(y^3-y-2y\right)$$

$$\implies \frac{1}{2}\left(y^3-3y\right) = \frac{1}{2}\left(y^3-3y\right)$$

Showing that $(m+1)H_{m+1}(y) = yH_m(y) - H_{m-1}(y)$

Lastly,

$$H_2(-y) = \frac{1}{2}\left(-(y^2)-1\right) = -(1)^2 H_2(y)$$

$$\implies \frac{1}{2}\left((-y^2)-1\right) = H_2(y)$$

$$\implies \frac{1}{2}\left(y^2-1\right) = \frac{1}{2}\left(y^2-1\right)$$

$$H_3(-y) = -\frac{1}{6}\left((-y) + 2(-y) - (-y)^3\right) = (-1)^3 H_3(y)$$

$$\Longrightarrow -\frac{1}{6}\left(-y - 2y + y)^3\right) = +\frac{1}{6}\left(y + 2y - y^3\right)$$

$$\Longrightarrow \frac{1}{6}\left(y + 2y - y^3\right) = \frac{1}{6}\left(y + 2y - y^3\right)$$

Showing that $H_m(-y) = (-1)^m H_m(y)$

Suppose $B(\tau) = \left(B^1(\tau), ..., B^d(\tau)\right), \tau \geq 0$ is a d-dimensional Brownian motion defined on its canonical probability space $(\Omega, \mathcal{A}, P)$ i.e $\Omega = C_0(\mathbb{R}_+; \mathbb{R}^d)$, P is the d-dimesional Wiener measure and $\mathcal{A}$ is the completion of the Borel σ-field of Ω with respect to P, so that the underlying Hilbert space $\mathcal{R} = L^2(\mathbb{R}_+; \mathbb{R}^d)$ and for any $r \in \mathcal{R}$, $Z(r) = \sum_{i=1}^d \int_{\mathbb{R}_+} r_i(s) dB^i(s)$ (the wiener integral)

The next lemma shows that $\mathbb{E}(H_n$ and $\mathbb{E}(H_m$ are orthogonal if $n \neq m$

Lemma 3.1: [Giulia Di Nunno (2009)]
Let χ,υ represent two random variables with joint Gaussian distribution where $\mathbb{E}(\chi) = \mathbb{E}(\upsilon) = 0$ and $\mathbb{E}(\chi^2) = \mathbb{E}(\upsilon^2) = 1$, then $\forall\ m, n \geq 0$, we have

$$\begin{aligned} \mathbb{E}(H_n(\chi)H_m(\upsilon)) &= 0 \quad if \quad n \neq m \\ &= \frac{1}{n!}(E(\chi\upsilon)^n) \quad if \quad n = m \end{aligned}$$

Lemma 3.2:[Schoutens. W (2000)]
The random variable $\{e^{Z(r)}, r \in \mathcal{R}\}$ forms a total subset of

$$L^2(\mathcal{G}) = L^2(\Omega, \mathcal{G}, P)$$

Theorem 3.1 [Giulia Di Nunno (2009)]
$L^2(\Omega, \mathcal{G}, P)$ can be decomposed infinitely into orthogonal sum of subspace $\mathcal{R}_n$ represented as

$$L^2(\Omega, \mathcal{G}, P) = \bigoplus_{n=0}^{\infty} \mathcal{R}_n$$

Since the total subset of $L^2(\Omega, \mathcal{G}, P)$ was formed by $\{e^{Z(r)}, \quad r \in \mathcal{R}\}$, then

$\mathbb{E}(\chi \exp(\tau Z(r))) = 0$ implies that $\chi = 0$. If $\mathcal{R}$ is one-dimensional, then $(\Omega, \mathcal{A}, P) = (\mathbb{R}, \mathcal{B}(\mathbb{R}), \mathcal{U})$ where $\mathcal{U}$ represents the standard normal law with mean 0 and variance 1. We set $W(h)(x) = hx$ for each $h \in \mathbb{R}$, where $\mathcal{H} = \mathbb{R}$. So in $\mathcal{R}$, there are two members of norm one (i.e. 1 and -1), and we can relate them respectively with the random variables x and $-x$. From $H_n(-x) = (-1)^n H_n(x)$, $n \geq 1$, we have that x_n has one dimension generated by $H_n(x)$. The above theorem implies that a complete orthonormal system is formed by the Hermite polynomial in$L^2(\mathbb{R}, \mathcal{V})$

Assume an orthonormal basis of $\mathcal{R}$ is represented by $\{e_i \quad i \geq 1\}$ and that $\mathcal{R}$ is infinite-dimensional. If $a = (a_1, a_2, ...)$ $a_i \in \mathbb{N}$, the set of all sequences is represented by Λ so that except for a finite number of them, all the terms varnish. We represent a!, for each $a \in \Lambda$, by

$$a! = \prod_{i=1}^{\infty} a_i! \quad , |a| = \sum_{i=1}^{\infty} a_i$$

So that $H_a(y) \quad y \in \mathbb{R}^N$, the generalized Hermite polynomial is defined as

$$H_a(y) = \prod_{i=1}^{\infty} H_{a_i}(y) \qquad where \quad H_0(y) = 1$$

An orthonormal system is the family of random variables φ_a described as

$$\varphi_a = \sqrt{a}! \prod_{i=1}^{\infty} H_{a_i}(Z(e_i))$$

For any $a \in \Lambda$.
Let $a, b \in \Lambda$, we have

$$\begin{aligned}
\mathbb{E}\left(\prod_{i=1}^{\infty} H_{a_i}(Z(e_i)) H_{b_i}(Z(e_i))\right) &= \prod_{i=1}^{\infty} \mathbb{E}\left(H_{a_i}(Z(e_i)) H_{b_i}(Z(e_i))\right) \\
&= \frac{1}{a!} \qquad if \quad a = b \\
&= 0 \qquad if \quad a \neq b
\end{aligned}$$

3.3.1 Wiener Chaos Expansion

In the study of Stochastic analysis especially Malliavin Calculus, the Wiener-Ito chaos expansion is important. It was shown by Ito (1951) that the expansion can be expressed as iterated Ito integrals.

Here, we shall consider a one-dimensional Wiener process $B(\tau) = B(\tau, \varpi) : \tau \geq 0, \varpi \in \Omega$ where $B(0, \omega) = 0$, defined on $(\Omega, \mathcal{A}, P)$. A real function S is such that

$$S(\varsigma_{\sigma_1}, ..., \varsigma_{\sigma_n}) = S(\varsigma_1, ..., \varsigma_n) \tag{3.3.1}$$

is called symmetric given that $S : [0, T]^n \to \mathbb{R}$

If together with (3.3.1),

$$||S||^2_{L^2([0,T]^n)} : \int_{[0,T]^n} S^2(\varsigma_1, ..., \varsigma_n) d\varsigma_1, ..., d\varsigma_n < \infty$$

then $S \in \widehat{L}^2([0, T]^n)$, where $\widehat{L}^2([0, T]^n)$

If $S \in \widehat{L}^2([0, T]^n)$ and the set $\mathcal{S}_n$ is defined such that

$$\mathcal{S}_n = (\varsigma_1, ..., \varsigma_n) \in [0, T]^n; 0 \leq \varsigma_1 \leq \varsigma_2 \leq ... \leq \varsigma_n \leq T$$

then we have

$$||S||^2_{L^2([0,T]^n)} = n! \int_{\mathcal{S}_n} S^2(\varsigma_1, ..., \varsigma_n) d\varsigma_1, ..., d\varsigma_m = n! ||s||^2_{L^2(\mathcal{S}_n)}$$

The symmetrization of S denoted as $\widehat{S}$ is defined over all permutations σ of (1,...,k) by

$$\widehat{S}(\varsigma_1, ..., \varsigma_k) = \frac{1}{k!} \sum_{\sigma} S(\varsigma_{\sigma_1}, ..., \varsigma_{\sigma_k})$$

If S is symmetric, then $\widehat{S} = S$

For example, suppose

$$S(\varsigma_1, \varsigma_2) = \varsigma_1^2 + \varsigma_2 \sin \varsigma_1$$

then

$$\widehat{S}(\varsigma_1, \varsigma_2) = \frac{1}{2!} \left[\varsigma_1^2 + \varsigma_2^2 + \varsigma_2 \sin \varsigma_1 + \varsigma_1 \sin \varsigma_2 \right]$$

A k-fold iterated Ito integral of the form

$$I_k(h) = \int_0^T \int_0^{\tau_k} ... \int_0^{\tau_3} \left(\int_0^{\tau_2} h(\tau_1 ... \tau_n) dB(\tau_1) dB(\tau_2) ... dB(\tau_{k-1}) dB(\tau_k) \right)$$

can be formed given h such that

$$||h||^2_{L^2(\mathcal{S}_k)} := \int_{\mathcal{S}_k} h^2(\tau_1, ..., \tau_k) d\tau_1 ... d\tau_k < \infty$$

This is because the integrand is square integrable with respect to $dB(\tau_i)$ at each Ito integration and its $\mathcal{A}_\tau$-adapted.
Iteratively, by Ito isometry properties,

$$E[I_k^2(r)] = E\left[\{ \int_0^T (\int_0^{\tau_k} ... \int_0^{\tau_2} r(\tau_1 ... \tau_k) dB(\tau_1) ...) dB(\tau_k) \}^2 \right]$$

$$= \int_0^T E\left[(\int_0^{\tau_k} ... \int_0^{\tau_2} r(\tau_1 ... \tau_k) dB(\tau_1) ... dB(\tau_{k-1}))^2 \right] d\tau_k$$

$$= ... = \int_0^T \int_0^{\tau_n} ... \int_0^{\tau_2} r^2(\tau_1 ... \tau_k) d\tau_1 ... d\tau_k = ||r||^2_{L^2(\mathcal{S}_n)}$$

Similarly, applying iteratively the Ito isometry, where $r \in L^2(\mathcal{S}_k)$ and $g \in L^2(\mathcal{S}_m)$ with $k > m$, then we have that

$$E[I_k(r) I_m(g)]$$

$$= E[\{ \int_0^T \int_0^{s_m} ... \int_0^{s_2} r(\tau_1 ... \tau_{k-m}, s_1 ... s_m) dB(\tau_1) ... dB(s_m) \}$$

$$\{ \int_0^T \int_0^{s_m} ... \int_0^{\tau_2} g(s_1 ... s_m) dB(s_1) ... dB(s_m) \}]$$

$$= \int_0^T E[\{ \int_0^{s_m} ... \int_0^{\tau_2} r(\tau_1 ... s_{m-1} s_m) dB(\tau_1) ... dB(S_{m-1}) \}] ds_m$$

$$\int_0^{s_m} ... \int_0^{s_2} g(s_1 ... s_{m-1} s_m) dB(s_1) ... dB(s_{m-1})$$

$$= \int_0^T \int_0^{s_m} ... \int_0^{s_2} E\left[g(s_1 s_2 ... s_m) \int_0^{s_1} ... \int_0^{\tau_2} r(\tau_1 ... \tau_{k-m}, s_1 ... s_m) dB(\tau_1) ... dB(t\tau_{k-m}) \right] ds_1 ... ds_m$$

$$= \int_0^T \int_0^{s_m} ... \int_0^{s_2} E\left[g(s_1 s_2 ... s_m) \int_0^{s_1} ... \int_0^{\tau_2} r(\tau_1 ... \tau_{k-m}, s_1 ... s_m) dB(\tau_1) ... dB(\tau_{k-m})\right] ds_1 ... ds_m$$

$$= 0$$

Since an Ito integral has its expectation as zero, then, these results can be summarized as follows

$$E[I_m(g) I_k(r)] = \{(g, r)_{L^2(\mathcal{S}_k)} \quad if \quad k = m$$

$$= \{0 \quad if \quad k \neq m$$

where the inner product of $L^2(\mathcal{S}_k)$ is repesented as

$$< g, r >_{L^2(\mathcal{S}_k)} = \int_{\mathcal{S}_k} g(\varsigma_1 ... \varsigma_k) r(\varsigma_1 ... \varsigma_n) d\varsigma_1 ... d\varsigma_k$$

Theorem 3.2: (The Wiener-Ito Chaos expansion)[Giulia Di Nunno (2009)] If we have $\{h_n\}_{n=0}^{\infty}$, a sequence of deterministic functions such that

$$\varrho(\varpi) = \sum_{n=0}^{\infty} I_n(h_n)$$

(where $h_n \in \widehat{L}^2([0,T]^n)$) Converges in $L^2(P)$ then for an $\mathcal{A}_T$- measurable random variable ϱ we have

$$||\varrho||^2_{L^2(\Omega)} := ||\varrho||^2_{L^2(P)} := E_P[\varrho^2] < \infty$$

Moreover, the isometry

$$||\varrho||^2_{L^2(P)} = \sum_{n=0}^{\infty} n! ||h_n||^2_{L^2([0,T]^n)}. \qquad (3.3.2)$$

Given the process $\varrho_1(t_1, \varphi)$ such that

$$\left[\int_0^T \varrho_1^2(t_1, \varpi) dt_1\right] \leq ||\varrho||^2_{L^2(P)} \qquad (3.3.3)$$

and

$$\varrho(\varpi) = E[\varrho] + \int_0^T \varrho_1(t_1, \varpi) dB(t_1) \tag{3.3.4}$$

where $E[\varrho] = s_0$ (constant)
Applying Ito representation theorem to $\varrho_1(t_1, \varpi)$, $t_1 \leq T$, we have

$$E\left[\int_0^{S_1} \varrho_2^2(t_2, t_1, \varpi) dt_2\right] \leq E\left[\varrho_1^2(t_1)\right] < \infty$$

where $\varrho_2(t_2, t_1, \varpi), 0 \leq t_2 \leq t_1$ is an $\mathcal{A}_\tau$ -adapted process and

$$\varrho_1(t_1, \varpi) = E[\varrho_1(t_1)] + \int_0^{t_1} \varrho_2(t_2, t_1, \varpi) dB(t_2) \tag{3.3.5}$$

Substituting (3.3.5) in (3.3.4), we have

$$\varrho(\varpi) = s_0 + \int_0^T s_1(t_1) dB(t_1) + \int_0^T \int_0^{t_1} \varrho_2(t_2, t_1, \varpi) dW(t_2) dB(t_1) \tag{3.3.6}$$

here, we use

$$E[\varrho_1(t_1)] = s_1(t_1) \tag{3.3.7}$$

By (3.3.4) and (3.3.7),

$$E\left[\{\int_0^T (\int_0^{t_1} \varrho_2(t_1, t_2, \varpi) dB(t_2)) dB(t_1)\}^2\right] = \int_0^T (\int_0^{t_1} E\left[\varrho_2^2(t_1, t_2, \varpi)\right] dt_2) dt_1 \leq ||\varrho||^2_{L^2(P)}$$

Likewise, $\varrho_3(t_3, t_2, t_1, \varpi)$ an $\mathcal{A}_\tau$-adapted process $(0 \leq t_3 \leq t_2)$ was obtained by applying the Ito representation theorem $(t_2 \leq t_1 \leq T)$, to $\varrho_2(t_2, t_1, \varpi)$ such that

$$E\left[\int_0^{t_2} \varrho_3^2(t_3, t_2, t_1, \varpi) dt_3\right] \leq E\left[\varrho_2^2(t_2, t_1)\right] < \infty$$

and

$$\varrho_2(t_2, t_1, \varpi) = E[\varrho_2(t_2, t_1, \varpi)] + \int_0^{t_2} \varrho_3(t_3, t_2, t_1, \varpi) dB(t_3) \tag{3.3.8}$$

Substitute (3.3.8) in (3.3.6), we get

$$\varrho(\varpi) = s_0 + \int_0^T s_1(t_1)dB(t_1) + \int_0^T \int_0^{t_1} s_2(t_2,t_1)dB(t_2))dB(t_1) + \int_0^T \int_0^{t_1} \int_0^{t_2} \varrho_3(t_3,t_2,t_1,\varpi)dB(t_3)d$$

where

$$E\left[\varrho_2(t_2,t_1)\right] = s_2(t_2,t_1); \quad 0 \leq t_2 \leq t_1 \leq T \tag{3.3.9}$$

Using (3.3.4), (3.3.7), (3.3.9),

$$E\left[\{\int_0^T \int_0^{t_1} \int_0^{t_2} \varrho_3(t_3,t_2,t_1,\varpi)dB(t_3)dB(t_2)dB(t_1)\}^2\right] \leq ||\varrho||^2_{L^2(P)}$$

If we follow this procedure iteratively by induction after n steps, we have $\varrho_{n+1}(\tau_1,\tau_2,...\tau_{n+1},\varpi)$ and $s_0,s_1,...,s_n$ and after n steps a process where s_0 is constant and s_k is defined such that

$$\varrho(\varpi) = \sum_{k=0}^{n} I_k(s_k) + \int_{\mathcal{S}_{n+1}} \varrho_{n+1} dB^{\otimes(n+1)}$$

with $0 \leq \tau_1 \leq \tau_2 \leq ... \leq \tau_{n+1} \leq T$ where ϱ_{n+1} an (n+1)-fold iterated integral is given by

$$\int_{\mathcal{S}_{n+1}} \varrho_{n+1} dB^{\otimes(n+1)} = \int_0^T \int_0^{\tau_{n+1}} ... \int_0^{\tau_2} \varrho_{n+1}(\tau_1...\tau_{n+1},\varpi)dB(\tau_1)...dB(\tau_{n+1})$$

and

$$E\left[\left\{\int_{\mathcal{S}_{n+1}} \varrho_{n+1} dB^{\otimes(n+1)}\right\}^2\right] \leq ||\varrho||^2_{L^2(\Omega)}$$

If $\varphi_{n+1} := \int_{\mathcal{S}_{n+1}} \varrho_{n+1} dB^{\otimes(n+1)}$ $n = 1,2,...$ is bounded in $L^2(P)$ and

$$(\varphi_{n+1}, I_k(h_k))_{L^2(\Omega)} = 0 \quad \forall k \leq n, s_k \in L^2([0,T]^k)$$

Then,

$$||\varrho||^2_{L^2(\Omega)} = \sum_{k=0}^{n} ||I_k(s_k)||^2_{L^2(\Omega)} + ||\varphi_{n+1}||^2_{L^2(\Omega)}$$

by the Pythagorean theorem where,

$$\sum_{k=0}^{n} ||I_k(s_k)||^2_{L^2(\Omega)} < \infty$$

and so, in $L^2(\Omega)$

$$\sum_{k=0}^{\infty} I_k(s_k)$$

is strongly convergent and

$$\lim_{n\to\infty} \varphi_{n+1} := \varphi \qquad exist \quad \in \quad L^2(\Omega)$$

$I_k(h_k), \varphi)_{L^2(\Omega)} = 0$, $h_k \in L^2([0,T]^k)$
Hence,

$$\varrho(\varpi) = \sum_{k=0}^{\infty} I_k(s_k) \quad convergence \quad in \quad L^2(\Omega)$$

and

$$||\varrho||^2_{L^2(\Omega)} = \sum_{k=0}^{n} ||I_k(s_k)||^2_{L^2(\Omega)} \tag{3.3.10}$$

Finally, we proceed to obtain (3.3.2)-(3.3.3), as follows:
Taking

$$s_n(\tau_1, ..., \tau_n) = 0 \quad if \quad (\tau_1...\tau_n) \in [0,T]^n/t_n \tag{3.3.11}$$

the function s_n is defined on $\mathcal{S}_n$ and can be extended to $[0,T]^n$.
Define the symmetrization $h_n = \widehat{s}_n$ of s, then

$$J_n(h_n) = n! I_n(h_n) = n! I_n(\widehat{s}_n)$$

$$= I_n(s_n)$$

So, (3.3.2)-(3.3.3) follows from (3.3.10) and (3.3.11).

We define as

$$h \otimes s(\varsigma_1, \varsigma_2) = h(\varsigma_1)s(\varsigma_2)$$

for two functions h and s the tensor product $h \otimes s$ and the symmetriztion of $h \otimes s$ as the symmetrized tensor product $h \widehat{\otimes} s$

3.3.2 Multiple Wiener-Ito Integrals

In this section, we define the Malliavin derivative via the wiener-Ito decomposition.[Kuo. H (2005)].
Suppose the Hilbert space $\mathcal{H}$ be represented as $L^2(B, \mathcal{B}, \mu)$ such that $(B, \mathcal{B})$ represent a measurable space and a σ - finite measure μ, i.e the Gaussian process Z is characterized by the family of random variables $\{Z(A), A \in \mathcal{B}, \mu(A) < \infty\}$ where $Z(A) = Z(1_A)$. We assume Z(A) to be an $L^2(\Omega, \mathcal{A}, P)$-valued measure on the measurable space $(B, \mathcal{B})$, which takes independent values on any family of disjoint subsets of B such that any random variable W(A) has the distribution $N(0, \mu(A))$, where $\mu(A) < \infty$.
This measure is also known as the white noise. To this end, for any function $h \in L^2(B)$, we shall define the stochastic integral $W(h)$ as

$$W(h) = \int_B h dW$$

It is possible to expressed as multiple stochastic integral the nth Wiener chaos $\mathcal{H}_n$ with respect to W. Next, the multiple stochastic integral $I_n(f)$ is define in what follows;
For a function $f \in L^2(B^k, \mathcal{B}^k, \mu^k)$, $k \geq 1$, a stochastic integral is defined where B^k is the k-times product of space B and μ^k is the corresponding product measure. Let E_k represent the set of simple functions defined as

$$f(\tau_1 ... \tau_k) = \sum_{i_1 ... i_k}^{n} a_{i_1 ... i_k} 1_{A_{i_1} \times ... \times A_{i_k}}(t_1, ..., t_k)$$

such that whenever we have any two equal indices,the coeffifient $a_{i_1...i_k}$ vanish and the set $A_1, ..., A_k$ are pairwise disjoint in $\mathcal{B}_0$. So,

$$I_k(f) = \sum_{i_1...i_k=1}^{n} a_{i_1...i_k} W(A_{i_1})...W(A_{i_k})$$

defined the multiple-stochastic integral

Remarks:

The multiple stochastic integral $I_k(f)$ has the following properties

(1) $I_k(f)$ is linear.

(2) Let

$$\widehat{f}(\tau_1...\tau_k) = \frac{1}{k!}\sum_{\sigma} f(\tau_{\sigma(1)}...\tau_{\sigma(k)})$$

be the symmetrization of f and σ run over all permutation of $\{1, ..., k\}$ then $I_k(f) = I_k(\widehat{f})$

(3)

$$\mathbb{E}I_k(f)I_n(f) = \{0 \quad if n \neq k$$

$$= k! \quad if n = k$$

Definition 3.5:

Let

$$I_n(f) := \int_0^T \int_0^{\tau_n} ... \int_0^{\tau_3} \int_0^{\tau_2} f(\tau_1...\tau_n) dB(\tau_1) d(\tau_2)...dB(\tau_n)$$

represent n-fold iterated Ito integrals where $f = J_0(f) \quad ; f \in \mathbb{R}$

We have by Ito integrals properties that

- $I_n(f) \in L^2(P)$ and by Ito isometry, $||I_n(f)||^2_{L^2(P)} = ||f||^2_{L^2(S_N)}$
- $f \in L^2(s_n)$ and $g \in L^2(s_m)$ such that $n > m$, then $\mathbb{E}[I_n(f)I_m(g)] = 0$.

$f \in \widehat{L}^2$ implies that the function f is a symmetric square integrable.

3.4 Skorohod Integral

In this section, we present the theory of Skorohod integral. This integral will be used to formulate the Malliavin weight function. This is very important in the calculation of the Greeks.

Consider a Hilbert space $\mathcal{H}$ defined as $\mathcal{H} = L^2(D, \mathcal{A}, \kappa)$, an L^2-space where κ is define on a measurable space $(D, \mathcal{A})$. Here, the square integrable processes are members of $Dom\delta \subset L^2(T\times\Omega)$, and the Skorohod stochastic integral is represented as $\delta(v)$ of the process $v = v(\tau, \varpi) \quad \tau \in T, \varpi \in \Omega$.

Definition 3.7

Suppose the stochastic process $u(\tau)$ is measurable such that $\tau \in [0, T]$. If

$$\mathbb{E}\left[\int_0^T v^2(\tau)d\tau\right] < \infty$$

then $v(\tau)$ is $\mathcal{A}_\tau$-measurable.

Suppose for $f_n(\cdot, \tau) \in \widehat{L}([0, T]^n)$, we define Wiener Ito expansion of the stochastics process $v(\tau)$ as

$$v(\tau) = \sum_{n=0}^{\infty} J_n(f_n(\cdot, \tau))$$

then,

$$\delta(v) := \int_0^T v(\tau)dB(t) := \sum_{n=0}^{\infty} I_{n+1}(\widehat{f}_n)$$

defined the Skorohod integral of $\sqsubseteq$ where the symmetrization of $f_n(., t)$ is represented as $\widehat{f}_n$

Moreso,

$$||\delta(v)||^2_{L^2(P)} = \sum_{n=0}^{\infty}(n+1)!||\widehat{f}_n||_{L^2([0,T])^{n+1})} < \infty$$

We can write $f_{n,\tau}(\tau_1...\tau_n) = f_n(\tau_1, ..., \tau_n, \tau)$ since $f_n(\cdot, \tau) = f_{n,\tau}(.)$ is a function of the parameter τ.

Since the function f_n is symmetric with respect to its first n variables then f_n and the symmetrization $\widehat{f}_n$ are function of n+1 variables $\tau_1, ..., \tau_n, \tau$ where the symmetrization with $\tau_{n+1} = \tau$ is given by,

$$\widehat{f}_n(t_1, ..., t_{n+1}) = \frac{1}{n+1}\left[f_n(t_1...t_{n+1}) + ... + f_n(t_1...t_{i-1}, t_i, t_{i+1}...t_{n+1}) + ... + f_n(t_2...t_{n+1}, t_1)\right]$$

where the sum is taken over those permutations σ of the indices $(1, ..., n+1)$ which inter- change the last component with one of the others and leave the rest in place.

The Skorohod integral satisfies the following properties

- it is a linear operator
- its expectation is zero i.e $E[\delta(v)] = 0$
- If $v, Xv, \in Dom(\delta)$ then,

$$\int_0^T Xv(\tau)\delta B(\tau) \neq X \int_0^\infty v(\tau)\delta B(\tau)$$

provided the random variable X is an $\mathcal{A}_\tau$-measurable.

Theorem 3.3[Giulia Di Nunno, (2009)]:
The Ito-integral can be extended to the Skorohod integral i.e
Let $E\left[\int_0^T v^2(t)d\tau\right] < \infty$ where the stochastic process $v(\tau), \tau \in [0,T]$ is a $\mathcal{A}$-adapted measurable process then

$$\int_0^T v(\tau)\delta B(\tau) = \int_0^T v(\tau)dB(\tau)$$

i.e v is Skorohod integrable and it is also Ito integrable.

Proposition (3.2) (Nualart, D.(2006)): If in $L^2(\Omega)$, the series

$$\delta(v) = \sum_{n=0}^{\infty} I_{n+1}\widehat{f}_n$$

converges and v can be expanded as

$$v(\tau) = \sum_{n=0}^{\infty} J_n(f_n(\cdot,\tau))$$

where $v \in L^2(T \times \Omega)$, then v is in $Dom\delta$.

Proposition (3.3): (Nualart D (2006))
Assume $u \in L^{1,2}$, and if $\{\int_T D_t u_s dB_s, \tau \in T\}$ exist in $L^2(T \times \Omega)$ and there is a

Skorohod integrable process $\{D_\tau v_s, s \in T\}$, then $\delta(v) \in \mathbb{D}^{1,2}$ and

$$D_\tau(\delta(v)) = v_\tau + \int_s^T D_\tau v_s dB_s$$

Proposition (3.4)Nualart D (2006): If $X, Y \in \mathbb{D}^{1,2}$ such that $\mathbb{E}(\langle DX, v^0\rangle_{\mathcal{H}}) = 0$ and $\mathbb{E}(Y) = 0$, then v has a unique orthogonal decomposition $v = DY + v^0$ where $v \in L^2(T \times \Omega)$. In addition, v^0 is Skorohod integrable and $\delta(v^0) = 0$

Lemma(3.4): [Ocone. D (1984)]
Let

$$v(\tau, \varpi) = \sum_{n=0}^{\infty} J_n(f_n(\cdot, \tau))$$

represent the Wiener Ito expansion of the stochastic process $v(\tau, \varpi)$ where $\tau \in [0, T]$, then the stochastic process v is $\mathcal{A}_\tau$-adapted iff $f_n(\tau_1 ... \tau_n, \tau) = 0$ and

$$\tau < \max_{1 \leq i \leq n} \tau_i$$

Theorem 3.4: [Da Prato G, (2007)]
Suppose $v(\tau, \varpi)$ is a $\mathcal{A}_\tau$-adapted stochastic process and
$E\left[\int_0^T v^2(\tau, \varpi) d\tau\right] < \infty$ where $\tau \in [0, T]$ then

$$\int_0^T v(\tau, \varpi)\delta B(\tau) = \int_0^T v(\tau, \varpi) dB(\tau)$$

and $v \in Dom(\delta)$

3.5 Malliavin Derivative/Derivative Operator

In this section, we define the Mallivian derivative and its adjoint, the divergence operator. The derivative operator is a derivative with respect to the inverse operator of the stochastic integral.
Let $\mathcal{A}$ represent a σ-field generated by B and let $(A, \mathcal{A}, P)$ represent a complete

probability space on which a Hilbert space $\mathcal{R}$ is defined, then we can represent by $Z = \{Z(r), r \in \mathcal{R}\}$ an Isonormal Gaussian process.
The space of infinitely continuously differentiable functions $f : \mathbb{R}^n \to \mathbb{R}$ is represented as $C_b^\infty(\mathbb{R}^n)$ (respectively $C_p^\infty(\mathbb{R}^n)$) such that its partial derivatives are bounded (respectively have polynomial growth). We represent also $C_0^\infty(\mathbb{R}^n)$ as the space of all infinitely continuously differentiable functions with compact support.

Definition 3.8:

(1) Let $Y : \Omega \to \mathbb{R}$ and let denote by S the set of smooth random variables, if there is a function y in $C_p^\infty(\mathbb{R}^n)$, then

$$Y = y(Z(r_1) \dots Z(r_n)) \tag{3.3.12}$$

for $n \geq 1$ and elements $r_1, \dots, r_n \in \mathcal{R}$

(2) The set $\mathcal{P}$ denotes the set of random variables of the form (3.3.12) where y is a polynomial

(3) S_b(respectively S_0) denotes the space of random variables of the form (3.3.12) with y in $C_b^\infty(\mathbb{R}^n)$ (respectively) $C_0^\infty(\mathbb{R}^n)$)

Definition 3.9:
Assume Y is a member of S with expression (3.3.12), then DY, the Mallivian derivative of Y is defined as

$$DY = \sum_{i=1}^{n} \frac{\delta y(Z(r_1), \dots, Z(r_n)) r_i}{\delta \varsigma_i} \tag{3.3.13}$$

The derivative is a mapping $DY : \Omega \to \mathcal{R}$
By iteration for $m \geq 2$ we define $D^m Y$ in $L^2(\Omega, \mathcal{R}^{\otimes m})$ as

$$DY = \sum_{i=1 \dots i_m = 1} \frac{\delta^m y(Z(r_1) \dots Z(r_n)) r_{i_1} \otimes \dots \otimes r_{i_m}}{\delta \varsigma_1 \dots \delta \varsigma_m}$$

This represent the mth order Malliavim derivative.

Proposition 3.6:[Malliavin. P (2005)]
If a smooth random variable Y admit two different representation of the form (3.3.12)

$$Y = y(Z(r_1)...Z(r_n))$$

$$= g(Z(g_1)...Z(g_m))$$

then

$$\sum_{i=1}^{n} \frac{\delta y}{\delta \varsigma_i}(Z(r_1), Z(r_2), ..., Z(r_n-1), Z(r_n))r_i = \sum_{i=1}^{m} \frac{\delta g}{\delta \varsigma_i}(Z(g_1), Z(g_2), ..., Z(g_m-1), Z(g_m))g_i$$

In other words,the Mallivian derivative DF of F is well defined by (3.3.13)

Remark:
By the definition of the gradient operator for smooth random variables,

$$D(YX) = YDX + XDY$$

for every smooth random variables of the form (3.3.12).

3.6 Integration by Part Formula

We use the Malliavin derivative and the relation between it and Skorohod integral to obtain an integration by part formula which play an important role in the calculation of the Greeks.
The integration by part formula is very essential in the study of smoothness of random variables and the absolutely continuity of the Malliavin calculus. This is fundamental in its application to finance.

Proposition 3.7: [Nualart. D (2006)]
Let $r \in \mathcal{R}$ and let Y be a smooth random variable of the form (3.3.12). then

$$\mathbb{E}[\langle DY, r\rangle_{\mathcal{R}}] = \mathbb{E}[YZ(r)]$$

the integration by parts formula holds.

Proposition 3.8:[Oksendal. B (2003)]
Suppose that $(DY_n)_n$ converges to η, a stochastic process in $L^p(\Omega, \mathcal{R})$ such that the

sequence $\{Y_n\}_{n\in\mathbb{N}}$ of smooth random variables, $n \to \infty$ converges to zero in $L^p(\Omega)$. Then, $\eta = 0$ and D, the Malliavin derivative operator is closable from $L^p(\Omega)$ to $L^p(\Omega, \mathcal{R})$

Proposition 3.9: [Oksendal. B (2003)]
Suppose $Y = (Y^1, \ldots, Y^m)$ where $Y^i \in \mathbb{D}^{1,P}$, $P \geq 1$, $\varrho(Y) \in \mathbb{D}^{1,P}$ and $\varrho\mathbb{R}^m \to \mathbb{R}$ then

$$D(\varrho(Y)) = \sum_{i=1}^{m} \frac{d\varrho}{d\varsigma_i}(Y)DY^i$$

Proposition 3.10 (D. Nulart 2006): Suppose $\varrho\mathbb{R}^m \to \mathbb{R}$ is a function, where $x, y \in \mathbb{R}^m$ and $k > 0$ then ϱ is a Lipschitz function provided $|\varrho(x) - \varrho(y)| \leq k\|x - y\|$. Given a random vector $Y = (Y', \ldots, Y^m)$ such that $Y^i \in \mathbb{D}^{1,P}$, $P \geq 1$, if there exist random variables X^i and $\varrho(Y)$ belongs to $\mathbb{D}^{1,P}$ then

$$D(\varphi(Y)) = \sum_{i=1}^{m} X^i DY^i$$

In addition, if Y is an absolutely continuous random variable on $\mathbb{R}^m$ then $\mathcal{G}^i = \dfrac{d\varphi}{dx_i}(Y)$. Note that since ϱ is Lipschitz, $\dfrac{d\varrho}{dx_i}(x)$ exist for almost all x in $\mathbb{R}^m$.

Theorem 3.5: :
Suppose $Y_k \in \mathbb{D}^{1,2}$ for every $Y \in L^2(P)$ where $k = 1, 2, ...$ then
(1.) $Y_k \to Y$, in $L^2(P)$ as $k \to \infty$
(2.) Given that $D_tY_k \to D_tY$ in $L^2(P \times \lambda)$ where $Y \in \mathbb{D}^{1,2}$ then $\{D_tY_k\}_{k=1}^{\infty}$ converges in $L^2(P \times \lambda)$ as $k \to \infty$

Proposition 3.11: [Malliavin P and Thalmaier, (2005)]
Let $Y \in \mathbb{D}^{1,2}$ be a square integrable random variable with a decomposition given above, then

$$D_tY = \sum_{n=1}^{\infty} nJ_{n-1}(y_n(\cdot, t))$$

Proposition 3.12: [Malliavin P and Thalmaier, (2005)]
Suppose

$$E(Y|\mathcal{A}_A) = \sum_{n=0}^{\infty} J_n(y_n 1_A^{\otimes n})$$

represent the conditional expectation of Y where $A \in \mathcal{B}$ then

$$Y = \sum_{n=0}^{\infty} J_n(y_n)$$

is a square integrable random variable.

Let G be a Borel set in [0,T]. Let $\mathcal{A}_G \subseteq \mathcal{A}_T$ be defined as the completed σ-alegbra generated by $\int_0^T 1_A(\tau)dB(\tau)$ for all Borel sets $A \subseteq G$. If $Y \in \mathbb{D}^{1,2}$, then $E[Y|\mathcal{A}_G] \in \mathbb{D}^{1,2}$ and

$$D_\tau E[Y|\mathcal{A}_G] = E[D_\tau Y|\mathcal{A}_G] \cdot 1_G(\tau)$$

If v is a $\mathcal{A}$-adapted stochastic process such that $v(s) \in \mathbb{D}^{1,2}$ for all s. In particular

$$D_\tau v(s) = D_\tau E[v(s)|\mathcal{A}_s] = E[D_\tau v(s)|\mathcal{A}_s] \cdot 1_{[0,s]}(\tau)$$

Let $Y = (Y^1, ..., Y^m)$ with $Y^l \in \mathbb{D}^{1,2}$, the Maliavin covariance matrix of Y is defined as the symmetric positive definite matrix given by

$$\sigma_Y^{ij} = \langle DY^i, DY^j \rangle = \int_0^1 D_s Y^i D_s Y^j ds$$

Then σ_Y satisfies the non-degeneracy assumption provided $\mathbb{E}((det\sigma_Y)^{-P}) < \infty$ for all $P \in \mathbb{N}$
If this is true, then, σ_Y is almost surely invertible.

Let $\mathbb{P}$ represent the family of all random variable $Y : \Omega \to \mathbb{R}$ of the form $Y(\varpi) = \xi(\theta_{..}\theta_n)$ where $\xi(\varsigma_1 ... \varsigma_n)$ is a polynomial in n variables $\varsigma_1, ..., \varsigma_n$ and

$$\theta_i = \int_0^T y_i(\tau)dB(t\tau) \qquad y_i \in L^2([0,T])$$

such random variables are called Wiener polynomials.
$\mathbb{D}^{1,2}$ is the closure (with respect to the norm $||\cdot||_{1,2}$ of $\mathbb{P}$ which represent family of random variable of the form $Y(\varpi) = \xi(\theta_{.}..\theta_n)$ where $Y : \Omega \to \mathbb{R}$
A function $y : \mathbb{R}^m \to \mathbb{R}$ is said to be Lipschitz continuous provided

$$|f(x) - f(y)| \leq L|x - y|$$

for all $x, y \in \mathbb{R}^m$, and L is the Lipschitz constant.

Proposition:(Integration by Part formula)[Oksendal. P (2000, 2003), Nualart. D (2006)]
Given the function $y \in C^1$ with bounded derivatives and two random variables Y, X where $Y \in \mathbb{D}^{1,2}$. Suppose $Xv(< DY, v >_R)^{-1} \in Dom\delta$ and $< DY, v >_R \neq 0$ where v an $\mathcal{R}$- Value random variable,then

$$E[y'(Y)X] = E[f(Y)H(Y, X)] \tag{3.3.14}$$

and

$$H(Y, X) = \delta(Xv(< DY, v >_R)^{-1})$$

Remark: In application to finance,
1 If $v = DY$ then

$$E[y'(Y)X] = E[y(Y)\delta(\frac{XDY}{||DY||_R^2})]$$

2 Suppose $X(< DY, v >_R)^{-1} \in \mathbb{D}^{1,2}$ such that

$$Xv(< DY, v >_R)^{-1} \in \mathbb{D}^{1,2}(R) \subset Dom\delta$$

then v is a deterministic process
3 This result form an integral part of the tool used in establishing the results obtained in this work,

3.7 The Divergence Operator

In this section, we introduce the divergence operator which is the adjoint of the derivative operator. The divergence operator in the white noise case is known as the Skorohod integral. The element of $Dom\sigma$ are square integrable stochastic process and the ddivergence $\sigma(u)$ is called the Skorohod integral of the process u.

Let κ be a σ-finite measure so that the underlying Hilbert space $\mathcal{R}$ of the adjoint of the derivative operator also known as the divergence operator is an L^2-space of the form $L^2(B, \mathcal{B}, \mu)$. The adjoint of the derivative operator is both Skorohod integral and stochastic integral in the Brownian motion sense.[Nualart. D (2006)]
Let $(\Omega, \mathcal{A}, P)$ represent a complete probability space on which $Z = Z(r)$ a Gaussian isonormal process is defined where $h \in \mathcal{H}$ the associated Hilbert space. So in the framework of $Z = Z(r), r \in \mathcal{R}$, the divergence operator D is unbounded and closed in $L^2(\Omega; \mathcal{R})$

Definition 3.10:

Let D be the derivative operator and let δ represent it adjoint also known as divergence operator then δ is an unbounded operator on $L^2(\Omega, \mathcal{R})$.This operator satisfies the following assumptions
(i) The domain of δ is represented as Dom δ and its the set of $\mathcal{R}$-valued square integrable random variable $v \in L^2(\Omega; \mathcal{R})$ where

$$|\mathbb{E}(\langle DY, v\rangle_{\mathcal{R}})| \leq c||Y||_{L^2}(\Omega)\forall Y \in \mathbb{D}^{1,2}$$

where c = constant
(ii) Let $v \in Dom\delta$, then $\delta(v) \in L^2(\Omega)$ so that for $Y \in \mathbb{D}^{1,2}$

$$\mathbb{E}(Y(\delta(v)) = \mathbb{E}(\langle DY, v\rangle_{\mathcal{R}}) \qquad (3.3.15)$$

(3.3.15) is called the Duality Relation
From (3.45), if $F = 1$ and $v \in Dom\delta$ then $\mathbb{E}(\delta(v)) = 0$. Suppose $r_j \in \mathcal{R}$ and Y_j are smooth random variables so that

$$v = \sum_{j=1}^{n} Y_j v_j$$

where $v \in S_{\mathcal{R}}$ By the formula of integration by part, we have that for $v \in Dom\delta$,

$$\delta(v) = \sum_{j=1}^{n} Y_j Z(r_j) - \sum_{j=1}^{n} \langle DY_j, r_j \rangle_{\mathcal{R}}$$

Properties of The Divergence Operator: The proof of these properties are shown in [Nualart. D (2009)]

(i) $\mathbb{E}((\delta(v)) = 0$ provided $v \in Dom\delta$

(ii) The operator δ is closed and linear in $Dom\delta$

(iii) if $v \in S_{\mathcal{R}}$, then $v \in Dom\delta$ and

$$\delta(v) = \sum_{j=1}^{n} Y_j Z(r_j) - \sum_{j=1}^{n} \langle DY_j, r_j \rangle_{\mathcal{R}}$$

(iv) Let $v \in S_{\mathcal{R}}$,$Y \in S$ and $r \in \mathcal{R}$,then

$$\langle D(\delta(v)), r \rangle_{\mathcal{R}} = \langle v, r \rangle_{\mathcal{R}} + \delta(\sum_{j=1}^{n} \langle DY_j, r \rangle_{\mathcal{R}} r_j)$$

Lemma 3.6:[Nualart. D (2009)]
Suppose $r \in \mathcal{R}$ and $Y, X \in S$, then

$$E[X\langle DY, r \rangle_{\mathcal{R}}] = E[YXZ(r)] - E[Y\langle DX, r \rangle \mathcal{R}]$$

The implication of this lemma is that, it establish the closability of the operator D

Lemma 2.7: [Malliavin. P and Thalmaier. A (2005)]
Suppose $v \in S_{\mathcal{R}}$ such that

$$v = \sum_{j=1}^{n} Y_j r_j \quad Y \in S \quad r \in \mathcal{R}$$

and

$$D^r(v) = \sum_{j=1}^{n} D^r(Y_j) r_j$$

then the commutativity relationship

$$D^r(\delta(v)) = \langle v, r \rangle_{\mathcal{A}} + \delta(D^r v)$$

holds, but

$$\delta(v) = \sum_{j=1}^{n} Y_j Z(r_j) - \sum_{j=1}^{n} \langle DY_j, r_j \rangle_{\mathcal{R}}$$

so

$$D^r(\delta(v)) = \sum_{j=1}^{n} \langle D(Y_j Z(r_j)) - D\langle DY_j, r_j \rangle_{\mathcal{R}}, r \rangle_{\mathcal{R}}$$

$$= \sum_{j=1}^{n} Y_j \langle r, r_j \rangle_{\mathcal{R}} + \sum_{j=1}^{n} (D^r Y_j Z(r_j) - \langle D(D^r Y_j), r_j \rangle_{\mathcal{R}})$$

$$= \langle v, r \rangle_{\mathcal{R}} + \delta(D^r v)$$

Proposition 3.13: [Pascucci A, (2010)] The equality

$$\delta(Yr) = YZ(r) - D^r Y$$

holds provided Yr is in the domain of δ, $Y \in \mathbb{D}^{1,2}$ and $r \in \mathcal{R}$

3.8 Clark-Ocone Formula

The Clark-Ocone formula is a representation theorem for square integrable random Variables in terms of Ito stochastic integrals in which the integrand is explicitly characterized in terms of the Malliavin derivative. Clark Ocone formula can be applied to find explict formula for hedging portfolio that can be replicated.

Theorem 3.6: (Clark-Ocone formula,(Di-Nunno 2002,2007))
Let $Y \in \mathbb{D}^{1,2}$ be $\mathcal{A}_T$-measurable, then

$$Y = E[Y] + \int_0^T E[D_\tau Y | \mathcal{A}_\tau] dB(\tau)$$

The formula can only be applied to random variables in $\mathbb{D}^{1,2}$ but extension beyond the domain $\mathbb{D}^{1,2}$ to $L^2(P)$ is possible in the white noise framework.

Theorem 3.7: Clark-Ocone formula under change of Measure:(Di-Nunno 2007)
Suppose $X \in \mathbb{D}^{1,2}$ is $\mathcal{A}_T$-measurable and $E_Q[|X|] < \infty$

$$E_Q\left[\int_0^T |D_\tau X|^2 d\tau\right] < \infty,$$

$$E_Q\left[|X| \int_0^T \left(\int_0^T D_\tau v(s) db(s) + \int_0^T v(s) D_\tau U(s) ds\right)^2 d\tau\right] < \infty$$

where the measure $dQ = Z(T)dP$ is the one given by the Girsanov theorem with

$$U(\tau) := \frac{v(\tau) - \rho(\tau)}{\sigma(\tau)}, \ \tau \in [0, T]$$

then

$$X = E_Q[X] + \int_0^T E_Q\left[\left(D_\tau X - X \int_0^T D_\tau U(s) d\tilde{b}(s)\right) | \mathcal{A}_\tau\right] d\tilde{B}(\tau)$$

Under the change of measure framework, the Clark-Ocone formula is applicable to random variables X that are measurable with respect to the filtration generated by the noise. If $\tilde{\mathbb{F}}$ is the $P(\sim)$ Q-augmented filtration generated by $\tilde{B}$, then we have that in general $\tilde{\mathcal{A}}_\tau \subset \mathcal{A}_\tau$ *and* $\tilde{\mathcal{A}}_\tau \neq \mathcal{A}$.

Chapter 4
RESULTS AND DISCUSSION

4.1 Introduction

Rainbow Options are options or derivatives exposed to two or more sources of uncertainty.

Apart from it been a path dependent option, that is, options whose value depend both on the price of the underlying assets, and the path that the asset took during some part or all the life of the option, it is also an option contract linked to the performance of two or more underlying assets. They can speculate on the best performer in the group or minimum performance of all the underlying assets at any time. Each underlying may be called a color so the sum of all these factors makes up a rainbow.

Rainbow options sometimes has many moving paths and all the underlying assets in a rainbow option have to move in the right direction so that the investment will pay off eventually.

The measure of the sensitivity analysis refers to the greeks, and the greeks are quantities that describe the sensitivities of financial derivative with respect to the different parameters of the model. They are vital tools in risk management and hedging.

Definition (Sensitivities):

Suppose $V(t)$ represent the pay off process of some derivatives where $t \in [0, T]$, then

$$\Delta = Delta = \frac{\partial V}{\partial s}$$

This measures the changes in V with respect to the underlying asset initial price.

$$\Gamma = Gamma = \frac{\partial^2 V}{\partial s^2}$$

This quantity estimate the change in terms of delta

$$\rho = rho = \frac{\partial V}{\partial r}$$

This measures the changes in V in terms of the prevailing rate of interest r

$$\theta = theta = -\frac{\partial V}{\partial T}$$

This measures the changes in V with respect to the expiration time

$$\nu = Vega = \frac{\partial V}{\partial \sigma}$$

This measures the changes in V in terms of volatility.

The computation of the greeks are sometime difficult to express in closed form depending on the pay off function, and so, they require numerical methods for their computation.

Malliavin calculus is suitable in calculating greeks especially when the pay off function is strongly discontinuous. Greeks are the measure of changes in the derivative security with respect to the parameters of financial derivative. They are important when considering how stable is the quantity under variation, that is the chosen parameter. If the price of an option is calculated using the measure Q as

$$V = \mathbb{E}[e^{-r(T-\tau)}\varphi(s(\tau))],$$

where the pay off function is represented as $\varphi(x)$, then under the same measure as the price,the greek will be calculated , so that the

$$Greek = \mathbb{E}[e^{-r\tau}\varphi((s(t))) * \psi(x)]$$

where $\psi(x)$ represent the weight function called Malliavin weight.

We consider the stochastic process $\mathtt{S}(t)$ defined on $(\Omega, \mathcal{A}, P, \mathcal{A}_\tau)$, the filtered probability space where $\tau \in [0, T]$

So, if $\mathtt{S}(\tau)$ satisfies the equation

$$\mathtt{S}(\tau) = \mathtt{S}_0 \exp((\kappa - \frac{\sigma^2}{2})\tau + \sigma B(\tau)),$$

then

$$\frac{\partial \mathtt{S}_T}{\partial S_0} = exp((\kappa - \frac{\sigma^2}{2})T + \sigma B(T)) = \frac{\mathtt{S}_T}{S_0}$$

$$\frac{\partial^2 \mathtt{S}_T}{\partial S_0^2} = \frac{-S_0 exp((\kappa - \frac{\sigma^2}{2})T + \sigma B(T))}{s_0^2} = \frac{-\mathtt{S}_T}{S_0^2}$$

$$\frac{\partial \mathtt{S}_T}{\partial \kappa} = S_0 T exp((\kappa - \frac{\sigma^2}{2})T + \sigma B(T)) = T\mathtt{S}_T$$

$$\begin{aligned} \frac{\partial \mathtt{S}_T}{\partial T} &= (\kappa - \frac{\sigma^2}{2}) S_0 exp((\kappa - \frac{\sigma^2}{2})T + \sigma B(T)) \\ &= (\kappa - \frac{\sigma^2}{2})\mathtt{S}_T \end{aligned}$$

$$\frac{\partial \mathtt{S}_T}{\partial \sigma} = (B_T - \sigma T) s_0 exp((\kappa - \frac{\sigma^2}{2})T + \sigma B(T)) = (B_T - \sigma T)\mathtt{S}_T$$

Greeks generally measure the sensitivity of the financial quantity in terms of the changes in the parameter, and these can be calculated using Malliavin calculus integration by part technique defined in equation (3.3.14).

$$\mathbb{E}[y'(Y)X] = \mathbb{E}[y(Y)\delta(Xv(D^v Y)^{-1})]$$

4.2 Greek Delta

Theorem 4.1 (Greek Delta):
Suppose the value of the Rainbow option is represented by $V : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$, where the dynamics of the option underlying asset $\mathtt{S}(\tau)$ is given by

$$d\mathtt{S}(\tau) = \kappa s(\tau)d\tau + \sigma s(\tau)dB(\tau) \qquad \tau \in [0,T]$$

where κ and σ are constant, $B(\tau)$ is defined on the filtered probability space $(\Omega, \mathcal{A}, P, \mathcal{A}_\tau)$, with filtration $\mathcal{A}_\tau$, then greek delta is given by

$$\Delta = e^{-rT}\mathbb{E}(\varphi(\mathtt{S}_T)\psi(x))$$

Proof

$$\Delta = \frac{\partial V_0}{\partial \mathtt{S}_0} \ , \qquad V_0 = \mathbb{E}(e^{-rT}\varphi(\mathtt{S}_T))$$

$$\Delta = \frac{\partial \mathbb{E}(e^{-rT}\varphi(\mathtt{S}_T))}{\partial \mathtt{S}_0},$$

where $\varphi(\mathtt{S}_T)$ represent the payoff function.

$$\begin{aligned}\Delta &= e^{-rT}\frac{\partial}{\partial \mathtt{S}_0}\mathbb{E}(\varphi(\mathtt{S}_T)) \\ &= e^{-rT}\mathbb{E}\left(\varphi'(\mathtt{S}_T)\frac{\partial \mathtt{S}_T}{\partial \mathtt{S}_0}\right) \\ &= e^{-rT}\mathbb{E}\left(\varphi'\mathtt{S}_T)\frac{\mathtt{S}_T}{\mathtt{S}_0}\right).\end{aligned}$$

Here, we apply the Malliavin calculus integration by part technique on the derivative φ' using the relation defined in equation (3.3.15)

$$\mathbb{E}(y'(Y)X) = \mathbb{E}(y(Y)\delta(Xv(D^vY)^{-1}))$$

if we take

$$Y = \mathtt{S}_T \qquad X = \mathtt{S}_T \quad , v = 1$$

then

$$\mathbb{E}(y'(Y)X) = \mathbb{E}(y(\mathtt{S}_T)\delta([S]_T(DY)^{-1}))$$

but

$$D^v\mathtt{S}_T = \int_0^T D_T\mathtt{S}_T d\tau = \int_0^T \sigma\mathtt{S}_T d\tau = \sigma T\mathtt{S}_T$$

Because

$$D_T\mathtt{S}]_{\mathtt{T}} = \sigma[\mathtt{S}_T$$

so

$$\delta\left(\mathtt{S}_T\left(\int_0^T D_T\mathtt{S}_T d\tau\right)^{-1}\right) = \delta\left(\frac{\mathtt{S}_T}{\sigma T\mathtt{S}_T}\right)$$

$$= \delta\left(\frac{1}{\sigma T}\right) = \int_0^T \frac{dB}{\sigma T} = \frac{B_T}{\sigma T}$$

Therefore

$$\begin{aligned}\Delta &= e^{-rT}\mathbb{E}\left(\varphi'(\mathtt{S}_T)\frac{\mathtt{S}_T}{\mathtt{S}_0}\right)\\ &= \frac{e^{-rT}}{\mathtt{S}_0}\mathbb{E}(\varphi'(\mathtt{S}_T)\mathtt{S}_T)\\ &= \frac{e^{-rT}}{\mathtt{S}_0}\mathbb{E}(\varphi(\mathtt{S}_T)\frac{B_T}{\sigma T})\\ &= \frac{e^{-rT}}{\mathtt{S}_0\sigma T}\mathbb{E}(\varphi(\mathtt{S}_T)B_T)\end{aligned}$$

Where

$$\frac{B_T}{\mathtt{S}_0\sigma T} = Weight\ function.$$

So for European call option with payoff described as

$$\varphi(\mathtt{S}_T) = (\mathtt{S}_T - \mathbf{K})^+$$

we have

$$\Delta = \frac{e^{-rT}}{\mathtt{S}_0\sigma T}\mathbb{E}(\mathtt{S}_T - \mathbf{K})^+ B_T)$$

For an Asian options whose payoff is described as

$$\varphi(\mathtt{S}_T) = \frac{1}{T}\int_0^T \mathtt{S}_T d\tau$$

We have

$$\Delta = \frac{e^{-rT}}{\mathtt{S}_0\sigma T}\mathbb{E}\left(\frac{1}{T}\int_0^T \mathtt{S}_T d\tau \cdot B_T\right)$$

Here $v(s) = \mathtt{S}_s$, $\quad Y = \tilde{\mathtt{S}}_T$ (average of $\mathtt{S}_T$), $\quad X = \frac{\partial \tilde{\mathtt{S}_T}}{\partial \mathtt{S}_0} = \frac{\tilde{\mathtt{S}}_T}{\mathtt{S}_0}$
This means that

$$\mathbb{E}(y'(Y)X) = \mathbb{E}(y(Y)\delta(Xv(D^vY)^{-1}))$$

can be expressed as

$$\mathbb{E}(y'(\tilde{\mathtt{S}}_T)\frac{\tilde{\mathtt{S}}_T}{\mathtt{S}_0}) = \mathbb{E}\left(y(\tilde{\mathtt{S}}_T)\delta\left(\frac{\frac{\tilde{\mathtt{S}}_T\cdot\mathtt{S}_\tau}{\mathtt{S}_0}}{D^v\tilde{\mathtt{S}}_T}\right)\right)$$

$$\mathbb{E}(y'(\tilde{\mathtt{S}}_T)\frac{\tilde{\mathtt{S}}_T}{\mathtt{S}_0}) = \mathbb{E}\left(y(\tilde{\mathtt{S}}_T)\delta\left(\frac{\tilde{\mathtt{S}}_T \cdot \mathtt{S}_\tau}{\mathtt{S}_0} * \frac{1}{\sigma T\tilde{\mathtt{S}}_T}\right)\right)$$

But $\quad D^u\tilde{\mathtt{S}}_T = \sigma T\mathtt{S}_T$

so

$$\delta(Xv(D^vY)^{-1}) = \delta\left(\frac{Xv}{D^vY}\right)$$

$$= \delta\left(\frac{\frac{\tilde{\mathtt{S}}_T\mathtt{S}_\tau}{\mathtt{S}_0}}{\int_0^T \mathtt{S}_s D_s\tilde{\mathtt{S}}_T d\tau}\right)$$

$$= \delta\left(\frac{\tilde{\mathtt{S}}_T\mathtt{S}_\tau}{\mathtt{S}_0\int_0^T \mathtt{S}_s D_s\tilde{\mathtt{S}}_T d\tau}\right)$$

$$\delta(\frac{\mathtt{S}_\tau}{\mathtt{S}_0\sigma T}) = \int_0^T \frac{\mathtt{S}_\tau}{\mathtt{S}_0\sigma T}dB$$

$$= \frac{1}{\mathtt{S}_0\sigma T}\int_0^T \mathtt{S}_\tau dB = \frac{1}{\mathtt{S}_0\sigma T}\left[\frac{1}{2}(\mathtt{S}_T^2 - T)\right]$$

so

$$\Delta = \frac{e^{-rT}}{\mathtt{S}_0\sigma T}\mathbb{E}(\varphi(\mathtt{S}_T)\frac{1}{2}(\mathtt{S}_T^2 - T))$$

$$= \frac{e^{-rT}}{2\mathtt{S}_0\sigma T}\mathbb{E}\left(\varphi(\mathtt{S}_T)(\mathtt{S}_T^2 - T)\right),$$

where $\frac{\mathtt{S}_T^2 - T}{2\mathtt{S}_0\sigma T}$ represent the Weight function ψ

For a best of asset call whose payoff is described as

$$\varphi(\mathtt{S}_T) = max(\mathtt{S}_i - \mathbf{K}), \mathbf{1}_{\mathtt{S}_i > \mathtt{S}_j} \quad {\scriptstyle i\neq j,} \quad {\scriptstyle i,j=1,2,\ldots n} \quad i = 1\ldots n.,$$

we have

$$\Delta = \frac{e^{-rT}}{\mathtt{S}_0\sigma T}\mathbb{E}(max(\mathtt{S}_i - \mathbf{K})B_T)$$

4.3 Greek Gamma

Theorem 4.2 (Greek Gamma):
Suppose the value of the Rainbow option is represented by $V : [0, T] \times \mathbb{R} \longrightarrow \mathbb{R}$, where the dynamics of the option underlying asset $\mathtt{S}(\tau)$ is given by

$$d\mathtt{S}(\tau) = \kappa s(\tau)d\tau + \sigma s(\tau)dB(\tau) \qquad \tau \in [0, T]$$

where κ and σ are constant, $B(\tau)$ is defined on the filtered probability space $(\Omega, \mathcal{A}, P, \mathcal{A}_\tau)$, with filtration $\mathcal{A}_\tau$, then Greek gamma is given by

$$\Gamma = e^{-rT}\mathbb{E}(\varphi(\mathtt{S}_T)\psi(x))$$

Proof

$$\Gamma = \frac{\partial^2 V}{\partial \mathtt{S}^2}, \quad V_0 = \mathbb{E}(e^{-rT}\varphi(\mathtt{S}_T))$$

$$\Gamma = \frac{\partial^2}{\partial \mathtt{S}_0^2}\mathbb{E}(e^{-rT}\varphi(\mathtt{S}_T))$$

$$= e^{-rT}\frac{\partial^2}{\partial \mathtt{S}_0^2}\mathbb{E}(\varphi(\mathtt{S}_T)) = e^{-rT}\mathbb{E}\left(\varphi'(\mathtt{S}_T)\frac{\partial^2 \mathtt{S}_T}{\partial \mathtt{S}_0}\right)$$

$$= e^{-rT}\mathbb{E}(\varphi'(\mathtt{S}_T)\frac{\mathtt{S}_T^2}{\mathtt{S}_0^2})$$

we have

$$\frac{e^{-rT}}{\mathtt{S}_0^2}\mathbb{E}(\varphi'(\mathtt{S}_T)\mathtt{S}_T^2)$$

$$\Gamma = \frac{-e^{-rT}}{\mathtt{S}_0^2}\mathbb{E}\left(\varphi'(\mathtt{S}_T)\mathtt{S}_T(\frac{B_T}{\sigma T} - 1)\right)$$

$$y' = \varphi', \quad Y = \mathtt{S}_T, \quad X = \mathtt{S}_T(\frac{B_T}{\sigma T} - 1), \quad v = 1$$

$$\Gamma = \frac{-e^{-rT}}{\mathtt{S}_0^2}\mathbb{E}\left(\varphi'(\mathtt{S}_T)\mathtt{S}_T(\frac{B_T}{\sigma T} - 1)\right)$$

$$y' = \varphi', \quad Y = \mathtt{S}_T, \quad X = \mathtt{S}_T(\frac{B_T}{\sigma T} - 1), \quad v = 1$$

$$\Gamma = \frac{-e^{-rT}}{\mathtt{S}_0^2}\mathbb{E}\left(\varphi(\mathtt{S}_T)\delta(\mathtt{S}_T(\frac{B_T}{\sigma T}-1)(\sigma T\mathtt{S}_T)^{-1})\right)$$
$$= \frac{-e^{-rT}}{\mathtt{S}_0^2}\mathbb{E}\left(\varphi(\mathtt{S}_T)\delta\left(\frac{\mathtt{S}_T(\frac{B_T}{\sigma T}-1)}{\sigma T\mathtt{S}_T}\right)\right)$$
$$\frac{-e^{-rT}}{\mathtt{S}_0^2}\mathbb{E}\left(\varphi(\mathtt{S}_T)\delta\left((\frac{B_T}{\sigma T}-1)\times\frac{1}{\sigma T}\right)\right)$$
$$= \frac{-e^{-rT}}{\mathtt{S}_0^2}\mathbb{E}\left(\varphi(\mathtt{S}_T)\delta(\frac{B_T}{(\sigma T)^2}-\frac{1}{\sigma T})\right)$$
$$\frac{-e^{-rT}}{\mathtt{S}_0^2}\mathbb{E}_Q\left(\varphi(\mathtt{S}_T)\frac{1}{(\sigma T)^2}(B_T^2-T\right)\frac{1}{2}-\frac{B_T}{\sigma T}$$

The weight function is

$$\frac{B_T^2-T}{2\mathtt{S}_0^2(\sigma T)^2}-\frac{B_T}{\sigma T}$$

So for European call option whose payoff is described as

$$\varphi(\mathtt{S}_T) = (\mathtt{S}_T-\mathbf{K})^+$$

We have

$$\Gamma = \frac{-e^{-rT}}{\mathtt{S}_0^2}\mathbb{E}\left[(\mathtt{S}_T-\mathbf{K})^+\frac{1}{(\sigma T)^2}\frac{1}{2}(B_T^2-T)-\frac{B_T}{\sigma T}\right]$$

For an Asian option whose payoff is described as

$$\varphi(\mathtt{S}_T) = \frac{1}{T}\int_0^T \mathtt{S}_T d\tau$$

we have

$$\Gamma = \frac{-e^{-rT}}{\mathtt{S}_0^2}\mathbb{E}\left[\frac{1}{T}\int_0^T \mathtt{S}_T d\tau\frac{1}{(\sigma T)^2}\frac{1}{2}(B_T^2-T)-\frac{B_T}{\sigma T}\right]$$

For Best of asset option with payoff

$$\varphi(\mathtt{S}_T) = max(\mathtt{S}_i-\mathbf{K}), \mathbf{1}_{\mathtt{S}_i>\mathtt{S}_j} \quad {\scriptstyle i\neq j,} \quad {\scriptstyle i,j=1,2,\ldots n} \quad i=1\ldots n$$

we have

$$\Gamma = \frac{-e^{-rT}}{\mathtt{S}_0^2}\mathbb{E}\left[max(\mathtt{S}_i-\mathbf{K})\frac{1}{(\sigma T)^2}\frac{1}{2}(B_T^2-T)-\frac{B_T}{\sigma T}\right]$$

4.4 Greek Rho

Theorem 4.3 (Greek Rho):
Suppose the value of the Rainbow option is represented by $V : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$, where the dynamics of the option underlying asset $\mathtt{S}(\tau)$ is given by

$$d\mathtt{S}(\tau) = \kappa s(\tau)d\tau + \sigma s(\tau)dB(\tau) \qquad \tau \in [0,T]$$

where κ and σ are constant, $B(\tau)$ is defined on the filtered probability space $(\Omega, \mathcal{A}, P, \mathcal{A}_\tau)$, with filtration $\mathcal{A}_\tau$, then greek rho is given by

$$\rho = e^{-rT}\mathbb{E}(\varphi(\mathtt{S}_T)\psi(x))$$

Proof

$$\rho = \frac{\partial V}{\partial \kappa}, \qquad V_0 = \mathbb{E}(e^{-rT}\varphi(\mathtt{S}_T))$$

$$\begin{aligned}\rho &= \frac{\partial \mathbb{E}(e^{-rT}\varphi(\mathtt{S}_T))}{\partial \kappa} \\ &= e^{-rT}\frac{\partial \mathbb{E}(\varphi(\mathtt{S}_T))}{\partial \kappa} \\ &= e^{-rT}\mathbb{E}(\varphi'(\mathtt{S}_T)\frac{\partial \mathtt{S}_T}{\partial \kappa}) \\ &= e^{-rT}\mathbb{E}(\varphi'(\mathtt{S}_T)T\mathtt{S}_T)\end{aligned}$$

Here, using

$$\varphi = y, \quad \mathtt{S}_T = Y, \quad X = T\mathtt{S}_T \qquad v = 1$$

in equation (3.45)

$$\mathbb{E}(y'(Y)X = \mathbb{E}(y(Y)\delta(Xv(D^vY)^{-1}))$$

we have

$$\begin{aligned}\mathbb{E}(y'(Y)X) &= \mathbb{E}(y(\mathtt{S}_T)\delta(\frac{T\mathtt{S}_T}{\sigma T\mathtt{S}_T})) \\ &= \mathbb{E}(\varphi(\mathtt{S}_T)\delta(\frac{1}{\sigma})) \\ &= \mathbb{E}(\varphi(\mathtt{S}_T)\frac{B_T}{\sigma})\end{aligned}$$

So

$$\rho = \frac{e^{-rT}}{\sigma}\mathbb{E}(\varphi(\mathtt{S}_T), B_T)$$

The weight function is

$$\psi = \frac{B_T}{\sigma}$$

So for European call options whose payoff is described as

$$\varphi(\mathtt{S}_T) = (\mathtt{S}_T - \mathbf{K})^+$$

, we have

$$\rho = \frac{-e^{-rT}}{\sigma}\mathbb{E}((\mathtt{S}_T - \mathbf{K})^+ B_T)$$

For Asian options whose payoff is described as

$$\varphi(\mathtt{S}_T) = \frac{1}{T}\int_0^T \mathtt{S}_T d\tau,$$

then

$$\rho = \frac{e^{-rT}}{\sigma}\mathbb{E}\left(\frac{1}{T}\int_0^T \mathtt{S}_T d\tau B_T\right)$$

For a best of asset call whose payoff is described as

$$\varphi(\mathtt{S}_T) = max(\mathtt{S}_i - [K]), i = 1, 2...$$

So

$$\rho = \frac{e^{-rT}}{\sigma}\mathbb{E}\left(max(\mathtt{S}_i - \mathbf{K})\mathbf{1}_{\mathbf{S}_i > \mathbf{S}_j \ i \neq j, \ i,j=1,2,...n} B_T\right)$$

4.5 Greek Theta

Theorem 4.4 (Greek Theta):
Suppose the value of the Rainbow option is represented by $V : [0, T] \times \mathbb{R} \longrightarrow \mathbb{R}$, where the dynamics of the option underlying asset $\mathtt{S}(\tau)$ is given by

$$d\mathtt{S}(\tau) = \kappa s(\tau)d\tau + \sigma s(\tau)dB(\tau) \qquad \tau \in [0, T]$$

where κ and σ are constant, $B(\tau)$ is defined on the filtered probability space

$(\Omega, \mathcal{A}, P, \mathcal{A}_\tau)$, with filtration $\mathcal{A}_\tau$, then greek theta is given by

$$\theta = e^{-rT}\mathbb{E}(\varphi(\mathtt{S}_T)\psi(x))$$

Proof

$$\theta = \frac{\partial V}{\partial T}, \qquad V_0 = \mathbb{E}(e^{-rT}\ \varphi(\mathtt{S}_T))$$

$$\begin{aligned}\theta &= \frac{\partial \mathbb{E}(e^{-rT}\varphi(\mathtt{S}_T))}{\partial T} \\ &= e^{-rT}\frac{\partial \mathbb{E}(\varphi(\mathtt{S}_T))}{\partial T} \\ &= e^{-rT}\mathbb{E}(\varphi'(\mathtt{S}_T)\frac{\partial \mathtt{S}_T}{\partial T}) \\ &= e^{-rT}\mathbb{E}(\varphi'(\mathtt{S}_T)(\kappa - \frac{\sigma^2}{2})\mathtt{S}_T)\end{aligned}$$

Here, using

$$y = \varphi, \quad Y = \mathtt{S}_T, \quad v = 1, \quad X = (\kappa - \sigma^2/2)\mathtt{S}_T$$

in equation (3.45), we have

$$\begin{aligned}\mathbb{E}(y'(Y)X) &= \mathbb{E}(\varphi(\mathtt{S}_T)\delta(Xv(D^vY)^{-1})) \\ &= \mathbb{E}\left(\varphi(\mathtt{S}_T)\delta\left((\kappa - \frac{\sigma^2}{2})\frac{[S]_T}{\sigma T \mathtt{S}_T}\right)\right) \\ &= \mathbb{E}\left(\varphi(\mathtt{S}_T)\delta\left(\frac{\kappa - \frac{\sigma^2}{2}}{\sigma T}\right)\right) \\ &= \mathbb{E}\left(\varphi(\mathtt{S}_T)\left(\frac{\kappa - \frac{\sigma^2}{2}}{\sigma T}\right)\int_0^T dB\right) \\ &= \mathbb{E}\left(\varphi(\mathtt{S}_T)\left(\frac{\kappa - \frac{\sigma^2}{2}}{\sigma T}\right)B_T\right)\end{aligned}$$

so

$$\theta = e^{-rT}\mathbb{E}\left(\varphi(\mathtt{S}_T)\left(\frac{\kappa - \frac{\sigma^2}{2}}{\sigma T}\right)B_T\right)$$

The weight function is

$$\psi = \left(\frac{\kappa - \frac{\sigma^2}{2}}{\sigma T}\right)B_T$$

For an European case,

$$\theta = e^{-rT}\mathbb{E}\left((\mathtt{S}_T - \mathbf{K})^+ \left(\frac{\kappa - \frac{\sigma^2}{2}}{\sigma T}\right) B_T\right)$$

For an Asian option

$$\theta = e^{-rT}\mathbb{E}\left(\frac{1}{T}\int_0^T \mathtt{S}_T d\tau \left(\frac{\kappa - \frac{\sigma^2}{2}}{\sigma T}\right) B_T\right)$$

For best of asset call option

$$\theta = e^{-rT}\mathbb{E}\left(max(\mathtt{S}_i - \mathbf{K})^+)\mathbf{1}_{\mathtt{S}_i > \mathtt{S}_j} \quad {}_{i,j=1,\dots n} \left(\frac{\kappa - \frac{\sigma^2}{2}}{\sigma T}\right) B_T\right)$$

4.6 Greek Vega

Theorem 4.5 (Greek Vega):
Suppose the value of the Rainbow option is represented by $V : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$, where the dynamics of the option underlying asset $\mathtt{S}(\tau)$ is given by

$$d\mathtt{S}(\tau) = \kappa s(\tau)d\tau + \sigma s(\tau)dB(\tau) \qquad \tau \in [0,T]$$

where κ and σ are constant, $B(\tau)$ is defined on the filtered probability space $(\Omega, \mathcal{A}, P, \mathcal{A}_\tau)$, with filtration $\mathcal{A}_\tau$, then greek delta is given by

$$\vartheta = e^{-rT}\mathbb{E}(\varphi(\mathtt{S}_T)\psi(x))$$

Proof

$$\vartheta = \frac{\partial V}{\partial \sigma}, \qquad V_0 = \mathbb{E}(e^{-rT}\varphi(\mathtt{S}_T))$$

$$
\begin{aligned}
\vartheta &= \frac{\partial \mathbb{E}(e^{-rT}\varphi(\mathtt{S}_T))}{\partial \sigma} \\
&= e^{-rT}\frac{\partial \mathbb{E}_Q(\varphi(\mathtt{S}_T))}{\partial \sigma} \\
&= e^{-rT}\mathbb{E}\left(\varphi'(\mathtt{S}_T\frac{\partial \mathtt{S}_T}{\partial \sigma}\right) \\
&= e^{-rT}\mathbb{E}\left(\varphi'(\mathtt{S}_T)\mathtt{S}_T(B_T-\sigma T)\right)
\end{aligned}
$$

Here using

$$
Y=\mathtt{S}_T,\ v=1,\ Y=\mathtt{S}_T(B_T-\sigma T)
$$

in equatiom (3.45), we get

$$
\begin{aligned}
\mathbb{E}(y'(Y)X) &= \mathbb{E}(\varphi(\mathtt{S}_T)\delta(\mathtt{S}_T\frac{(B_T-\sigma T)}{\sigma T\mathtt{S}_T}) \\
&= \mathbb{E}\left(\varphi(\mathtt{S}_T)\delta\left(\frac{B_T-\sigma T}{\sigma T}\right)\right) \\
&= \mathbb{E}\left(\varphi(\mathtt{S}_T)\delta\left(\frac{B_T}{\sigma T}-1\right)\right) \\
&= \mathbb{E}\left(\varphi(\mathtt{S}_T)\frac{1}{\sigma T}\left(\frac{1}{2}\left(B_T^2-T\right)\right)\right)
\end{aligned}
$$

So

$$
\begin{aligned}
\vartheta &= e^{-rT}\mathbb{E}\left[\varphi(\mathtt{S}_T)\frac{1}{\sigma T}\left(\frac{1}{2}\left(B_T^2-T\right)B_T\right)\right] \\
&= e^{-rT}\mathbb{E}\left[\varphi(\mathtt{S}_T)\frac{1}{2\sigma T}\left(B_T^2-T-2B_T\right)\right] \\
&= \frac{e^{-rT}}{2\sigma T}\mathbb{E}\left[\varphi(\mathtt{S}_T)\left(B_T^2-T-2B_T\right)\right]
\end{aligned}
$$

For European case,

$$
\vartheta = \frac{e^{-rT}}{2\sigma T}\mathbb{E}\left[(\mathtt{S}_T-\mathbf{K})^+\left(B_T^2-T-2B_T\right)\right]
$$

For Asian call option

$$
\vartheta = \frac{e^{-rT}}{2\sigma T}\mathbb{E}\left[\frac{1}{T}\int_0^T \mathtt{S}_T dt\left(B_T^2-T-2B_T\right)\right]
$$

For a best of asset call option

$$\vartheta = \frac{e^{-rT}}{2\sigma T}\mathbb{E}\left[max(\mathtt{S}_i - \mathbf{K})\mathbf{1}_{\mathtt{S}_i > \mathtt{S}_j \;\; i \neq j \;\; i,j=1,\ldots n}\left(B_T^2 - T - 2B_T\right)\right]$$

4.7 Chain Rule

Theorem(Closability): Assume $Y_k \in \mathbb{D}^{1,2}$ where $Y \in L^2(P)$ and $k = 1, 2, ...$
(1.) $Y_k \to Y$ in $L^2(P)$ as $k \to \infty$
(2.) If $Y \in \mathbb{D}^{1,2}$ and $D_\tau Y_k \to D_\tau Y$ in $L^2(P \times \lambda)$ then $\{D_\tau Y_k\}_{k=1}^{\infty} \to L^2(P \times \lambda)$, as $k \to \infty$

Theorem (Chain rule) Let $P \geq 1$ and $Y^i \in \mathbb{D}^{1,p}$ such that $Y = (Y^1, ...Y^d)$ is a random vector, then $g(Y) \in \mathbb{D}^{1,p}$ where $g : \mathbb{R}^d \to \mathbb{R}$ is a function in C^1 with bounded partial derivatives and

$$D(g(Y)) = \sum_{i=1}^{d} \partial_i g(Y) DY^i$$

Proof: Let $Y^j \in (\mathbb{D}^{1,p})^d$, and given a sequence $\{Y_k^j\}_{k \geq 1}$ with $Y_k \in S$, S is a set of smooth random variables where $[Y_k = y_k(Z(r_1)\ldots Z(r_{n_k}))]$ and $Y = Y(Z(r_1)\ldots Z(r_n))$ where $Y_k \in C_p^\infty(\mathbb{R})^n$ and it converges to Y in $L^p(\Omega)$

$$g(Y_k^j) = g(Y_k^1, Y_k^2, \ldots Y_k^n)$$

$$Y_k^j = y_k^1(Z(r_1)\ldots Z(r_{n_k})), y_k^2(Z(r_1)\ldots Z(r_{n_k})), \ldots,$$
$$y_k^n(Z(r_1)\ldots Z(r_{n_k}))$$
$$Y_k^j = Y_k^1,\; Y_k^2, \ldots Y_k^n = Y$$
$$Y_k^1 \to Y^1,\; Y_k^2 \to Y^2, \ldots Y_k^n \to Y^n \; as \; k \to \infty$$

.

$$Y^1, Y^2, \ldots Y^n \in L^p(\Omega)$$

and the sequence $DY_k^j \to Y^j \epsilon L^p(\Omega, R) \; as \; k \to \infty$.

$$g(Y_k^j) = g(Y_k^1, Y_k^2, \ldots Y_k^n)$$

$$g'(Y_k^1) = \frac{\partial g}{\partial Y_k^1} + \frac{\partial g}{\partial Y_k^2} + \dots \frac{\partial g}{\partial Y_k^n}$$

$$D(g(Y_k^j)) = \frac{\partial g}{\partial Y_k^1} \cdot DY_k^1 + \frac{\partial g}{\partial Y_k^2} \cdot DY_k^2 + \dots + \frac{\partial g}{\partial Y_k^n} \cdot DY_k^n$$

$$= g'(Y_k^1)DY_k^1 + g'(Y_k^2)DY_k^2 + \dots + g'(F_k^n)DF_k^n$$

$$= \sum_{j=1}^{n} g'(Y_k^j)DY_k^j \quad k \geq 1 \quad = \sum_{j=1}^{n} \partial_j g(Y)DY$$

4.8 Computation and Analysis

The greeks play a major role when hedging a financial derivatives.It provides the tool for risk management which help investor in taking right and appropriate decisions concerning their investment. We discretize the investment period from 0 to 5 into 50 discretes (i.e 0, 0.1, 0.2, 0.3...5.0) and then express the underlying asset price in discret form by the Euler-Maruyana method then, we simulate with MatLab and Excel computational softwares to generate our values.

Definition [Call Option]
If the holder of a certain option is given a right in the option contract to buy the option at a specified time τ at a fixed strike price $\mathbf{K}$, such an option is known as a call option. The call option has a payoff described by

$$Payoff = max[(\mathtt{S}_T - \mathbf{K}), 0]$$

$\mathtt{S}_T$ is the price of the underlying asset at the expiration date or time

Definition [Put Option] An option is called put if the option at a particular time τ gives the holder the right to sell at specified strike price $\mathbf{K}$ but not the obligation . The put option has a payoff described by

$$Payoff = max[(\mathbf{K} - \mathtt{S}_T), 0]$$

$\mathtt{S}_T$ is the price of the underlying asset at the expiration date or time

- If $\mathbf{K} < \mathtt{S}_\tau$ (call option) or $\mathbf{K} > \mathtt{S}_\tau$ (put option), then the option is said to be IN-THE-MONEY
- If $\mathbf{K} > \mathtt{S}_\tau$ (call option) and $\mathbf{K} < \mathtt{S}_\tau$ (put option), then the option is said to be 0UT-OF -THE-MONEY

- If $\mathbf{K} = \mathtt{S}_\tau$ (call option) and (put optoon), then the option is said to be AT-THE-MONEY
- An investor can take either a long or a short position on an option of any kind

4.9 Greeks

4.9.1 Delta

Let $C_E = max[(\mathtt{S}_T - \mathbf{K}), 0]$ be the pay off process of an European call and suppose $V(\tau)$ represent the option value, where $\tau \in [0, T]$, then the measures of changes in V in terms of initial price of the asset is given as

$$\Delta = \frac{\partial V}{\partial S}$$

$$\Delta_1 = \frac{e^{-rT}}{\mathtt{S}_0 \sigma T} \mathbb{E}(\mathtt{S}_T - \mathbf{K})^+ B_T)$$

$\mathtt{S}_\tau$ satisfies the SDE described as

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t) \qquad S(0) = S_0$$

with Brownian motion $B(\tau)$ defined on $(\Omega, \mathcal{A}, P, \mathcal{A}_\tau)$, with filtration $\mathcal{A}_\tau$.
So we can descritize the solution of the SDE as

$$\mathtt{S}_{j+1} = \mathtt{S}_j + a\mathtt{S}_j h + b\mathtt{S}_j \sqrt{h}\mathcal{Z}_j, \qquad j = 0, 1, 2, \cdots n$$

where $\mathcal{Z}_j \sim N(0, t)$
Also, we have

$$B_T = B_0 + \sum_{j=1}^{T-1} \sqrt{h}\mathcal{Z}_j$$

When we put these together we have

$$\Delta_1 = \frac{e^{-rT}}{\mathtt{S}_0 \sigma T} \mathbb{E}[(\mathtt{S}_j + a\mathtt{S}_j h + b\mathtt{S}_j \sqrt{h}\mathcal{Z}_j - \mathbf{K})^+)(B_0 + \sum_{j=1}^{T-1} \sqrt{h}\mathcal{Z}_j)]$$

.

$$= \frac{e^{-rT}}{\mathtt{S}_0 \sigma T}[(\mathtt{S}_j + a\mathtt{S}_j h - \mathbf{K})B_0]$$

AO Graph.pdf

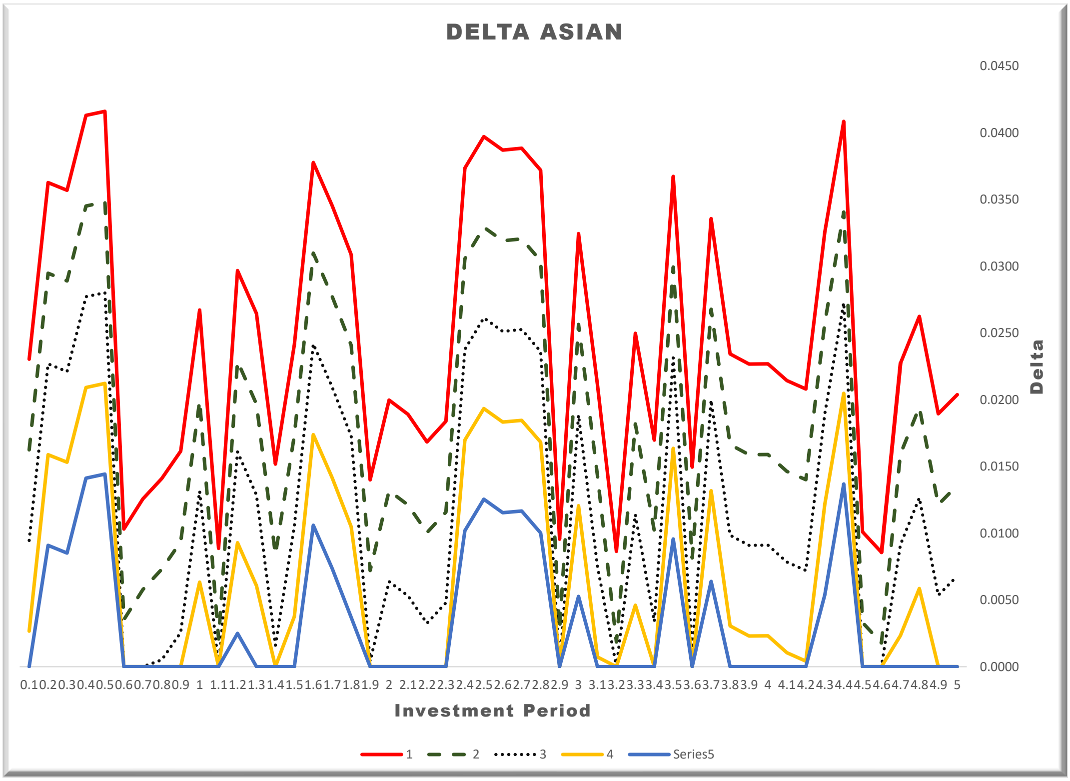

Figure 4.1: Delta AO Graph

Let $C_A = [Max(\frac{1}{T}\int_0^T \mathtt{S}_T d\tau - \mathbf{K}), 0]$ be the pay off process of an Asian call and suppose $V(\tau)$ represent the option value where $\tau \in [0, T]$, then the measures of changes in V with respect to the asset price is given as

$$\Delta_2 = \frac{\partial V}{\partial S}$$

$$\Delta_2 = \frac{e^{-rT}}{2\mathtt{S}_0 \sigma T}\mathbb{E}[(\frac{1}{T}\int_0^T \mathtt{S}_T d\tau - \mathbf{K})(\mathtt{S}_T^2 - T)]$$

.

AO.pdf

	Deltas for Asian Option						
	Strike Prices						Initial Spot price
Investment Period	71	72	73	74	75	Sj	70
0.1	0.0146	0.0078	0.0010	0.0000	0.0000	73.1527	
0.2	0.0145	0.0077	0.0009	0.0000	0.0000	73.1296	
0.3	0.0277	0.0209	0.0141	0.0073	0.0005	75.0792	
0.4	0.0118	0.0050	0.0000	0.0000	0.0000	72.7327	
0.5	0.0300	0.0232	0.0164	0.0096	0.0028	75.4096	
0.6	0.0177	0.0109	0.0042	0.0000	0.0000	73.6115	
0.7	0.0093	0.0025	0.0000	0.0000	0.0000	72.3625	
0.8	0.0190	0.0122	0.0054	0.0000	0.0000	73.7928	
0.9	0.0232	0.0164	0.0096	0.0028	0.0000	74.4084	
1	0.0374	0.0306	0.0238	0.0170	0.0102	76.5055	
1.1	0.0238	0.0170	0.0102	0.0034	0.0000	74.5076	
1.2	0.0224	0.0156	0.0088	0.0020	0.0000	74.2988	
1.3	0.0285	0.0217	0.0149	0.0081	0.0013	75.1924	
1.4	0.0395	0.0327	0.0259	0.0191	0.0123	76.8120	
1.5	0.0135	0.0067	0.0000	0.0000	0.0000	72.9880	
1.6	0.0243	0.0175	0.0107	0.0039	0.0000	74.5702	
1.7	0.0404	0.0336	0.0269	0.0201	0.0133	76.9521	
1.8	0.0373	0.0305	0.0238	0.0170	0.0102	76.4958	
1.9	0.0357	0.0289	0.0221	0.0153	0.0085	76.2487	
2	0.0124	0.0056	0.0000	0.0000	0.0000	72.8202	
2.1	0.0173	0.0105	0.0037	0.0000	0.0000	73.5428	
2.2	0.0195	0.0127	0.0059	0.0000	0.0000	73.8741	
2.3	0.0374	0.0306	0.0238	0.0171	0.0103	76.5097	
2.4	0.0230	0.0162	0.0094	0.0026	0.0000	74.3887	
2.5	0.0382	0.0314	0.0246	0.0178	0.0110	76.6153	
2.6	0.0131	0.0063	0.0000	0.0000	0.0000	72.9329	
2.7	0.0120	0.0052	0.0000	0.0000	0.0000	72.7701	
2.8	0.0234	0.0166	0.0098	0.0030	0.0000	74.4445	
2.9	0.0202	0.0134	0.0066	0.0000	0.0000	73.9664	
3	0.0280	0.0212	0.0144	0.0076	0.0008	75.1146	
3.1	0.0128	0.0060	0.0000	0.0000	0.0000	72.8851	
3.2	0.0227	0.0159	0.0091	0.0023	0.0000	74.3347	
3.3	0.0127	0.0059	0.0000	0.0000	0.0000	72.8628	
3.4	0.0211	0.0143	0.0075	0.0007	0.0000	74.1003	
3.5	0.0326	0.0258	0.0190	0.0122	0.0054	75.7936	
3.6	0.0202	0.0134	0.0066	0.0000	0.0000	73.9723	
3.7	0.0264	0.0196	0.0128	0.0060	0.0000	74.8784	
3.8	0.0355	0.0287	0.0219	0.0151	0.0084	76.2294	
3.9	0.0395	0.0327	0.0259	0.0191	0.0123	76.8079	
4	0.0190	0.0122	0.0054	0.0000	0.0000	73.7893	
4.1	0.0351	0.0283	0.0215	0.0147	0.0079	76.1608	
4.2	0.0239	0.0171	0.0103	0.0035	0.0000	74.5155	
4.3	0.0240	0.0172	0.0104	0.0036	0.0000	74.5336	
4.4	0.0125	0.0057	0.0000	0.0000	0.0000	72.8352	
4.5	0.0312	0.0244	0.0177	0.0109	0.0041	75.5982	
4.6	0.0251	0.0183	0.0115	0.0048	0.0000	74.6996	
4.7	0.0299	0.0231	0.0163	0.0095	0.0028	75.4051	
4.8	0.0357	0.0289	0.0221	0.0153	0.0085	76.2581	
4.9	0.0377	0.0309	0.0241	0.0173	0.0105	76.5468	
5	0.0257	0.0189	0.0121	0.0053	0.0000	74.7801	

Table 4.1: Delta AO Data

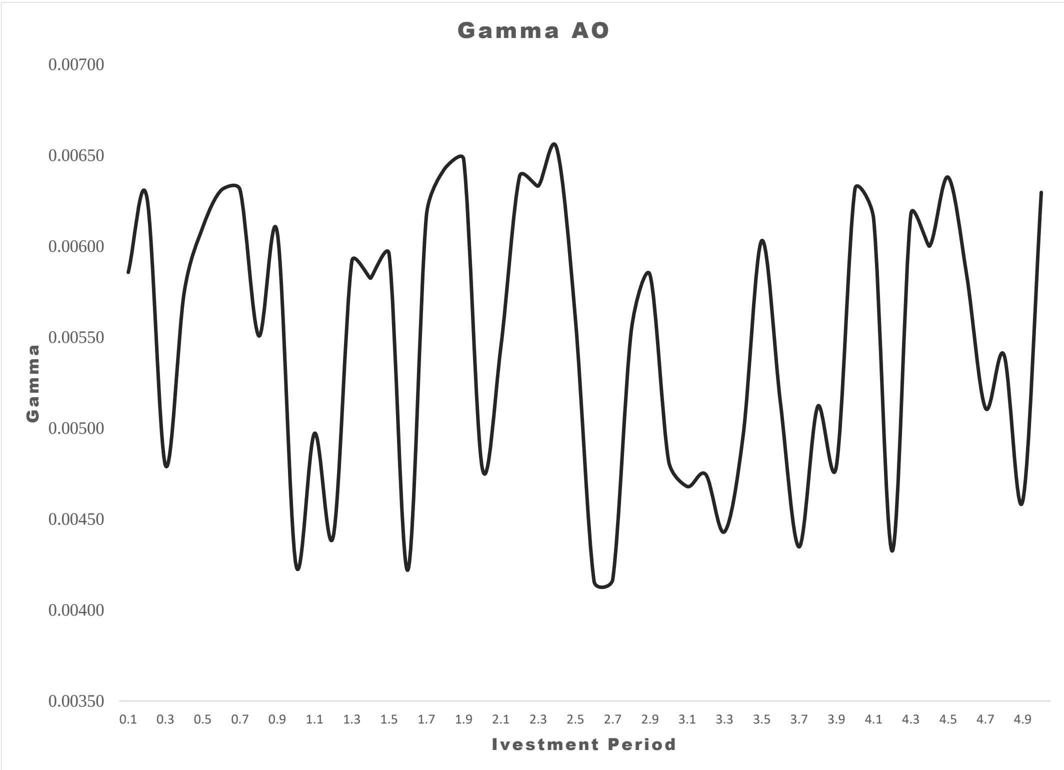

Table 4.2: Delta BOA Data

BOA Graph.pdf

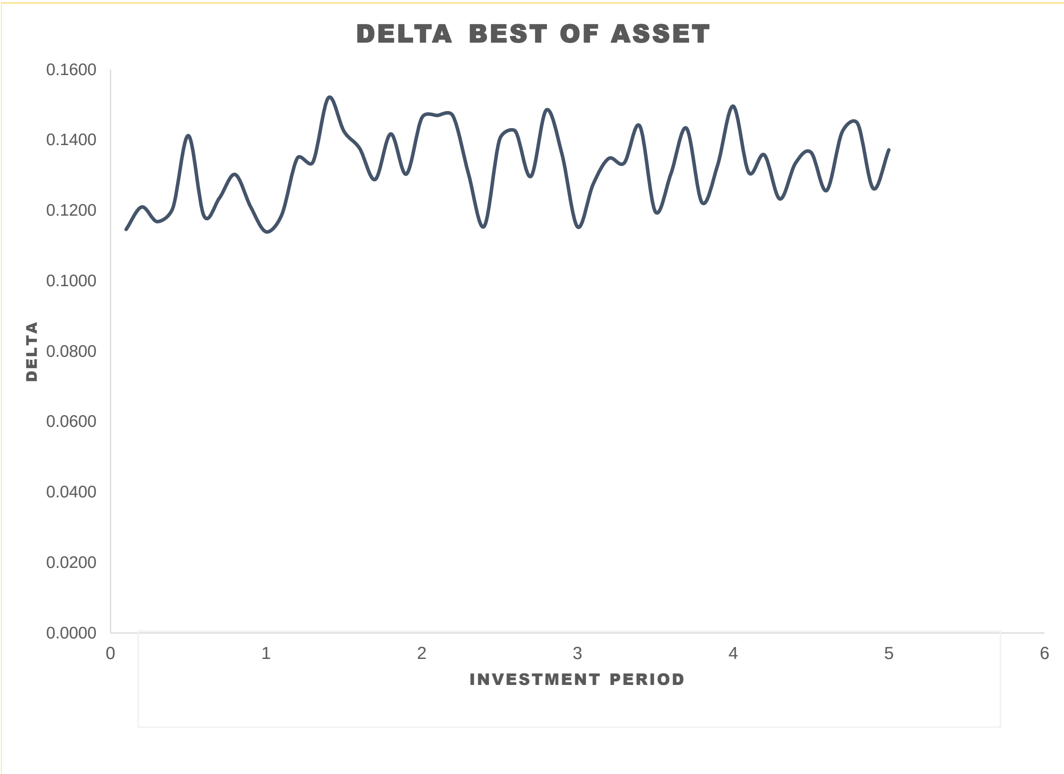

Figure 4.2: Delta BOA Graph

We discretize

$$(\frac{1}{T}\int_0^T \mathtt{S}_T d\tau - \mathbf{K}) = (\frac{1}{m}\sum_{j=1}^{m} \mathtt{S}_{\tau_j} - \mathbf{K}), \qquad 0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_n = T$$

So we have

$$\begin{aligned}\Delta_2 &= \frac{e^{-rT}}{2\mathtt{S}_0\sigma T}\mathbb{E}_Q[(\frac{1}{m}\sum_{j=1}^{m}\mathtt{S}_{\tau_j} - \mathbf{K})(\mathtt{S}_T^2 - T)] \\ &= \frac{e^{-rT}}{2\mathtt{S}_0\sigma T}[(\frac{1}{m}\sum_{j=1}^{m}(\mathtt{S}_j + a\mathtt{S}_j h - \mathbf{K})(\mathtt{S}_j + a\mathtt{S}_j h)^2 - T)]\end{aligned}$$

Let $C_B = [Max(\mathtt{S}_i - \mathbf{K}), 0]\mathbf{1}_{\mathtt{S}_i > \mathtt{S}_j} \quad {}_{i\neq j,} \quad {}_{i,j=1,2,\ldots n}$ be the pay off process of a Best of Asset call option and suppose $V(\tau)$ represent the option value at time τ, $\tau \in [0,T]$, then the measures of changes in V in terms of the initial price of the asset is given as

$$\begin{aligned}\Delta_3 &= \frac{\partial V}{\partial s} \\ &= \frac{e^{-rT}}{\mathtt{S}_0\sigma T}\mathbb{E}[(Max(\mathtt{S}_i - \mathbf{K}))B_T] \\ &= \frac{e^{-rT}}{\mathtt{S}_0\sigma T}[(Max(\mathtt{S}_j + a\mathtt{S}_j h - \mathbf{K}))B_0]\end{aligned}$$

4.9.2 Gamma

Let $C_E = max[(\mathtt{S}_T - \mathbf{K}), 0]$ be the pay off process of an European call and suppose $V(\tau)$ represent the option value where $\tau \in [0,T]$, then the measures of changes in V with respect to the chnges in delta is given as

$$\Gamma = \frac{\partial^2 V}{\partial S^2}$$

$$\Gamma_1 = \frac{e^{-rT}}{\mathtt{S}_0^2}\mathbb{E}[(\mathtt{S}_T - \mathbf{K})^+)\frac{1}{(\sigma T)^2}(B_T^2 - T)\frac{1}{2} - \frac{B_T}{\sigma T}]$$

so,

$$\Gamma_1 = \frac{e^{-rT}}{\mathtt{S}_0^2}[(\mathtt{S}_j + a\mathtt{S}_j h - \mathbf{K})\frac{1}{(\sigma T)^2}(B_0^2 - T)\frac{1}{2} - \frac{B_0}{\sigma T}]$$

		Asset Prices				
Investment Period	GAMMA AO	75	80	85	65	70
0.1	0.00586	81.05977	86.46376	91.86774	70.25180	75.65579
0.2	0.00627	81.92658	87.38835	92.85012	71.00303	76.46480
0.3	0.00479	78.82418	84.07913	89.33408	68.31429	73.56924
0.4	0.00575	80.82543	86.21380	91.60216	70.04871	75.43707
0.5	0.00610	81.58271	87.02156	92.46040	70.70501	76.14386
0.6	0.00631	82.01356	87.48113	92.94870	71.07842	76.54599
0.7	0.00631	82.01384	87.48143	92.94902	71.07866	76.54625
0.8	0.00551	80.32591	85.68097	91.03603	69.61579	74.97085
0.9	0.00607	81.51709	86.95157	92.38604	70.64815	76.08262
1	0.00425	77.67325	82.85147	88.02969	67.31682	72.49504
1.1	0.00497	79.19676	84.47654	89.75632	68.63719	73.91697
1.2	0.00440	78.00062	83.20066	88.40070	67.60054	72.80058
1.3	0.00592	81.18862	86.60119	92.01377	70.36347	75.77604
1.4	0.00582	80.99184	86.39130	91.79075	70.19293	75.59238
1.5	0.00595	81.26720	86.68502	92.10283	70.43158	75.84939
1.6	0.00422	77.61263	82.78680	87.96098	67.26428	72.43845
1.7	0.00617	81.71433	87.16195	92.60957	70.81908	76.26671
1.8	0.00643	82.26512	87.74946	93.23380	71.29644	76.78078
1.9	0.00648	82.36746	87.85863	93.34979	71.38514	76.87630
2	0.00478	78.79094	84.04367	89.29640	68.28548	73.53821
2.1	0.00545	80.20118	85.54792	90.89467	69.50769	74.85443
2.2	0.00638	82.17125	87.64934	93.12742	71.21509	76.69317
2.3	0.00633	82.05893	87.52953	93.00012	71.11774	76.58834
2.4	0.00654	82.49329	87.99284	93.49239	71.49418	76.99374
2.5	0.00560	80.51346	85.88103	91.24859	69.77833	75.14590
2.6	0.00416	77.48301	82.64855	87.81408	67.15194	72.31748
2.7	0.00417	77.50571	82.67276	87.83981	67.17162	72.33867
2.8	0.00554	80.39073	85.75011	91.10949	69.67196	75.03134
2.9	0.00584	81.03049	86.43253	91.83456	70.22643	75.62846
3	0.00482	78.87610	84.13451	89.39292	68.35929	73.61770
3.1	0.00468	78.58049	83.81919	89.05789	68.10309	73.34179
3.2	0.00474	78.71794	83.96580	89.21366	68.22221	73.47008
3.3	0.00443	78.05708	83.26089	88.46469	67.64947	72.85328
3.4	0.00496	79.17241	84.45057	89.72873	68.61609	73.89425
3.5	0.00603	81.42658	86.85502	92.28346	70.56970	75.99814
3.6	0.00514	79.55143	84.85486	90.15829	68.94457	74.24800
3.7	0.00435	77.88240	83.07456	88.26672	67.49808	72.69024
3.8	0.00512	79.50684	84.80729	90.10775	68.90593	74.20638
3.9	0.00479	78.81229	84.06644	89.32060	68.30399	73.55814
4	0.00631	82.02360	87.49184	92.96008	71.08712	76.55536
4.1	0.00615	81.68273	87.12824	92.57376	70.79170	76.23721
4.2	0.00432	77.83437	83.02333	88.21228	67.45645	72.64541
4.3	0.00617	81.72938	87.17801	92.62663	70.83213	76.28076
4.4	0.00600	81.36322	86.78743	92.21165	70.51479	75.93900
4.5	0.00638	82.16102	87.63843	93.11583	71.20622	76.68362
4.6	0.00584	81.02305	86.42459	91.82612	70.21998	75.62151
4.7	0.00511	79.48972	84.78903	90.08835	68.89109	74.19040
4.8	0.00540	80.10461	85.44491	90.78522	69.42399	74.76430
4.9	0.00460	78.40960	83.63690	88.86421	67.95498	73.18229
5	0.00630	81.98528	87.45096	92.91665	71.05391	76.51959

Table 4.3: Gamma Data AO

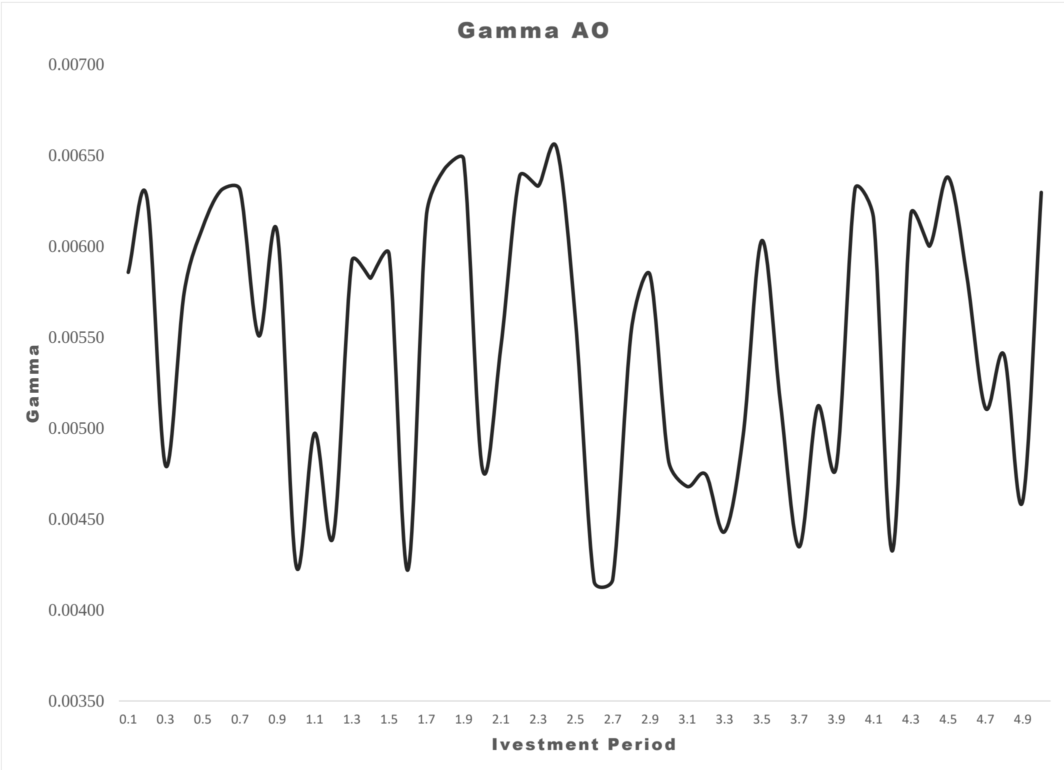

Figure 4.3: Gamma Graph AO

Data.pdf

		Asset Prices				
Investment Period	**GAMMA**	**75**	**80**	**85**	**65**	**70**
0.1	0.0096	80.98696	86.3861	91.78523	70.1887	75.58783
0.2	0.0101	81.88494	87.34394	92.80293	70.96695	76.42594
0.3	0.0088	79.48733	84.78649	90.08564	68.88902	74.18818
0.4	0.0099	81.44507	86.87475	92.30442	70.58573	76.0154
0.5	0.0098	81.36035	86.78438	92.2084	70.5123	75.93633
0.6	0.0099	81.4794	86.91136	92.34332	70.61548	76.04744
0.7	0.0084	78.69437	83.94066	89.18695	68.20179	73.44808
0.8	0.0081	78.11943	83.32739	88.53535	67.7035	72.91147
0.9	0.0087	79.2378	84.52032	89.80284	68.67276	73.95528
1	0.0093	80.2893	85.64192	90.99453	69.58406	74.93668
1.1	0.0094	80.65512	86.03212	91.40913	69.9011	75.27811
1.2	0.0079	77.74239	82.92522	88.10805	67.37674	72.55957
1.3	0.0100	81.72414	87.17242	92.62069	70.82759	76.27587
1.4	0.0103	82.23878	87.72137	93.20396	71.27361	76.7562
1.5	0.0093	80.35018	85.70686	91.06354	69.63682	74.9935
1.6	0.0094	80.57521	85.94689	91.31857	69.83185	75.20353
1.7	0.0086	79.04386	84.31346	89.58305	68.50468	73.77427
1.8	0.0094	80.50705	85.87418	91.24132	69.77277	75.13991
1.9	0.0104	82.49459	87.99422	93.49386	71.49531	76.99495
2	0.0091	79.99371	85.32663	90.65954	69.32788	74.6608
2.1	0.0085	78.80156	84.05499	89.30843	68.29468	73.54812
2.2	0.0101	81.90093	87.36099	92.82105	70.9808	76.44086
2.3	0.0091	80.04279	85.37898	90.71517	69.37042	74.70661
2.4	0.0098	81.26768	86.68553	92.10337	70.43199	75.84984
2.5	0.0079	77.72188	82.90334	88.0848	67.35897	72.54042
2.6	0.0087	79.14127	84.41736	89.69344	68.5891	73.86519
2.7	0.0082	78.30243	83.52259	88.74275	67.86211	73.08227
2.8	0.0098	81.30424	86.72452	92.1448	70.46367	75.88395
2.9	0.0092	80.09849	85.43839	90.77829	69.41869	74.75859
3	0.0087	79.31725	84.60506	89.89288	68.74161	74.02943
3.1	0.0088	79.34755	84.63739	89.92723	68.76788	74.05772
3.2	0.0090	79.71457	85.02888	90.34318	69.08596	74.40027
3.3	0.0090	79.8896	85.21558	90.54155	69.23766	74.56363
3.4	0.0091	80.03076	85.36614	90.70152	69.35999	74.69537
3.5	0.0101	81.94876	87.41201	92.87526	71.02226	76.48551
3.6	0.0092	80.16069	85.50474	90.84878	69.4726	74.81665
3.7	0.0102	82.12967	87.60498	93.08029	71.17904	76.65435
3.8	0.0086	79.05725	84.32773	89.59821	68.51628	73.78676
3.9	0.0101	81.99592	87.46231	92.92871	71.06313	76.52952
4	0.0085	78.87228	84.13044	89.38859	68.35598	73.61413
4.1	0.0086	78.93677	84.19922	89.46167	68.41187	73.67432
4.2	0.0098	81.254	86.67094	92.08787	70.42014	75.83707
4.3	0.0101	81.97298	87.43785	92.90271	71.04325	76.50812
4.4	0.0101	81.99033	87.45635	92.92237	71.05828	76.5243
4.5	0.0090	79.87201	85.19681	90.52161	69.2224	74.5472
4.6	0.0090	79.86076	85.18482	90.50887	69.21266	74.53671
4.7	0.0080	77.85162	83.04172	88.23183	67.4714	72.66151
4.8	0.0092	80.24856	85.59847	90.94837	69.54876	74.89866
4.9	0.0093	80.46057	85.82461	91.18864	69.73249	75.09653
5	0.0092	80.24274	85.59225	90.94177	69.5437	74.89322

Table 4.4: Gamma Data BOA

BOA Graph.pdf

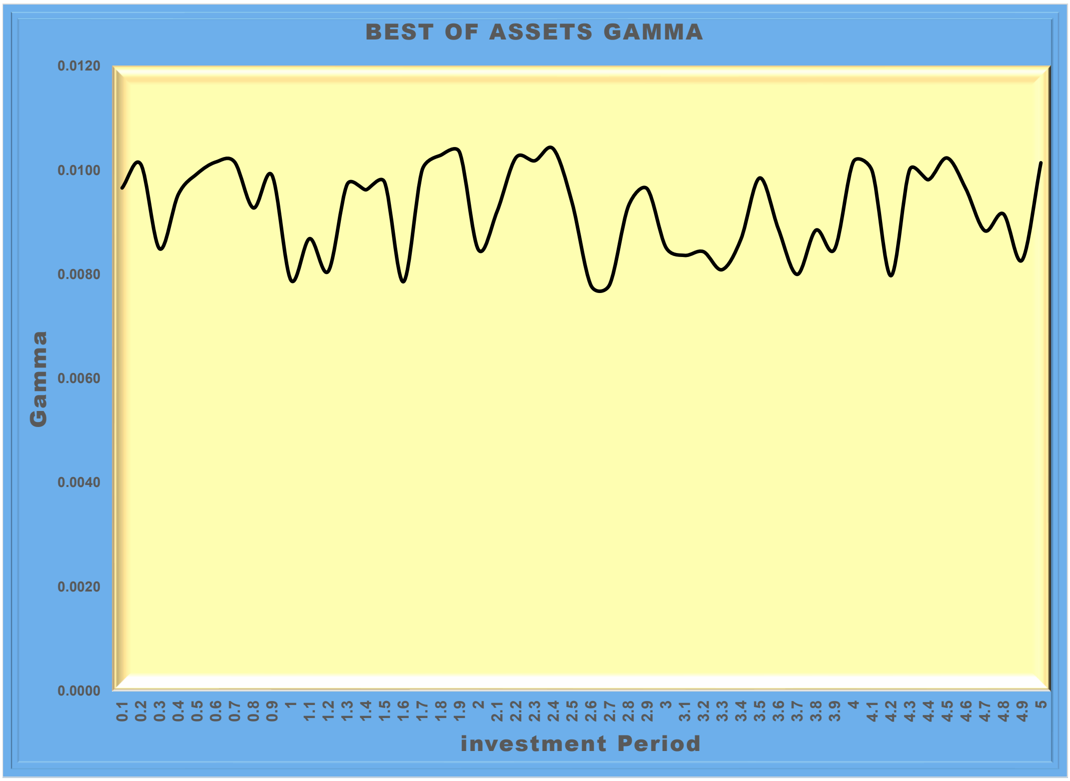

Figure 4.4: Gamma BOA Graph

Let $C_A = [Max(\frac{1}{T}\int_0^T \mathtt{S}_T d\tau - \mathbf{K}), 0]$ be the pay off process of an Asian call option and suppose $V(\tau)$ represent the option value where $\tau \in [0,T]$, then the measures of changes in V with respect to the changes in delta is given as

$$\Gamma = \frac{\partial^2 V}{\partial s^2}$$

$$\Gamma_2 = \frac{e^{-rT}}{\mathtt{S}_0^2}\mathbb{E}_Q[(\frac{1}{T}\int_0^T \mathtt{S}_\tau d\tau - \mathbf{K})\frac{1}{(\sigma T)^2}(B_T^2 - T)\frac{1}{2} - \frac{B_T}{\sigma T}]$$

so,

$$\Gamma_2 = \frac{e^{-rT}}{\mathtt{S}_0^2}[(\frac{1}{m}\sum_{j=1}^{m}(\mathtt{S}_j + a\mathtt{S}_j h - \mathbf{K})\frac{1}{(\sigma T)^2}(B_0^2 - T)\frac{1}{2} - \frac{B_0}{\sigma T}]$$

Let $C_B = [Max(\mathtt{S}_i - \mathbf{K}), 0]\mathbf{1}_{\mathtt{S}_i > \mathtt{S}_j \quad i \neq j, \quad i,j=1,2,\ldots n}$ be the payoff process of Best of Assets call option and let $V(\tau), \tau \in [0,T]$ be the value of the option at time τ, then the measures the sensitivity of the option with respect to changes in delta is given as

$$\Gamma = \frac{\partial^2 V}{\partial s^2}$$

$$\Gamma_3 = \frac{e^{-rT}}{\mathtt{S}_0^2}\mathbb{E}_Q[Max(\mathtt{S}_i - \mathbf{K}), 0]\mathbf{1}_{\mathtt{S}_i > \mathtt{S}_j \quad i \neq j}\frac{1}{(\sigma T)^2}(B_T^2 - T)\frac{1}{2} - \frac{B_T}{\sigma T}]$$

so,

$$\Gamma_3 = \frac{e^{-rT}}{\mathtt{S}_0^2}[(Max(\mathtt{S}_i + a\mathtt{S}_i h - \mathbf{K})\frac{1}{(\sigma T)^2}(B_0^2 - T)\frac{1}{2} - \frac{B_0}{\sigma T}]$$

4.9.3 Rho

Let $C_E = max[(\mathtt{S}_T - \mathbf{K}), 0]$ be the pay off process of an European call and suppose $V(\tau)$ represent the option value where $\tau \in [0,T]$, then the measures of changes in V in terms of rate of interest is given as

$$\rho = \frac{\partial V}{\partial r}$$

$$\rho_1 = \frac{e^{-rT}}{\sigma}\mathbb{E}_Q[(\mathtt{S}_T - \mathbf{K})^+)B_T]$$

so,

$$\rho_1 = \frac{e^{-rT}}{\sigma}[(\mathtt{S}_j + a\mathtt{S}_j h - \mathbf{K})B_0]$$

Let $C_A = [Max(\frac{1}{T}\int_0^T \mathtt{S}_T d\tau - \mathbf{K}), 0]$ be the pay off process of an Asian call

		Asset Prices			
Investment Period	Rho AO	75	80	85	65
0.1	23.50324	80.66530	86.04298	91.42067	69.90992
0.2	25.60503	81.73443	87.18340	92.63236	70.83651
0.3	24.77652	81.31299	86.73385	92.15472	70.47125
0.4	24.30634	81.07382	86.47874	91.88366	70.26398
0.5	23.84605	80.83968	86.22899	91.61830	70.06105
0.6	26.87360	82.30057	87.78728	93.27398	71.32716
0.7	19.83352	78.79858	84.05182	89.30506	68.29210
0.8	21.10465	79.44518	84.74153	90.03787	68.85249
0.9	19.82114	78.79229	84.04511	89.29792	68.28665
1	25.16699	81.51161	86.94572	92.37982	70.64339
1.1	20.30949	79.04070	84.31008	89.57946	68.50194
1.2	18.55713	78.14931	83.35926	88.56922	67.72940
1.3	17.90155	77.81583	83.00355	88.19128	67.44039
1.4	21.71518	79.75574	85.07279	90.38984	69.12164
1.5	25.12462	81.49006	86.92273	92.35540	70.62472
1.6	25.54195	81.70235	87.14917	92.59599	70.80870
1.7	19.02609	78.38786	83.61372	88.83958	67.93615
1.8	21.99635	79.89877	85.22536	90.55194	69.24560
1.9	25.78861	81.82782	87.28301	92.73819	70.91744
2	17.78361	77.75584	82.93956	88.12328	67.38839
2.1	18.21744	77.97652	83.17495	88.37338	67.57965
2.2	22.65270	80.23264	85.58149	90.93033	69.53496
2.3	23.56811	80.69829	86.07818	91.45806	69.93852
2.4	26.78722	82.26425	87.74853	93.23282	71.29568
2.5	20.23451	79.00256	84.26940	89.53623	68.46888
2.6	24.27238	81.05654	86.46031	91.86408	70.24900
2.7	20.87497	79.32835	84.61690	89.90546	68.75123
2.8	24.84238	81.34649	86.76959	92.19269	70.50029
2.9	18.25818	77.99724	83.19706	88.39687	67.59761
3	17.22561	77.47199	82.63679	87.80159	67.14240
3.1	17.70554	77.71612	82.89720	88.07827	67.35397
3.2	19.32828	78.54158	83.77768	89.01379	68.06937
3.3	18.10042	77.91699	83.11146	88.30593	67.52806
3.4	22.87646	80.34646	85.70289	91.05932	69.63360
3.5	19.15119	78.45150	83.68160	88.91170	67.99130
3.6	22.91781	80.36750	85.72533	91.08317	69.65183
3.7	20.81256	79.29660	84.58304	89.86948	68.72372
3.8	17.68191	77.70411	82.88438	88.06465	67.34356
3.9	18.11692	77.92538	83.12041	88.31544	67.53533
4	21.69441	79.74518	85.06153	90.37787	69.11249
4.1	23.77200	80.80201	86.18881	91.57561	70.02841
4.2	23.89300	80.86356	86.25446	91.64537	70.08175
4.3	25.39652	81.62837	87.07026	92.51215	70.74458
4.4	22.01752	79.90954	85.23684	90.56414	69.25493
4.5	19.11216	78.43164	83.66042	88.88919	67.97409
4.6	25.63496	81.74966	87.19964	92.64961	70.84970
4.7	25.31413	81.58646	87.02555	92.46465	70.70826
4.8	23.06762	80.44370	85.80662	91.16953	69.71788
4.9	22.43941	80.12415	85.46576	90.80737	69.44093
5	27.23274	82.45160	87.94837	93.44514	71.45805

Table 4.5: Rho AO Data

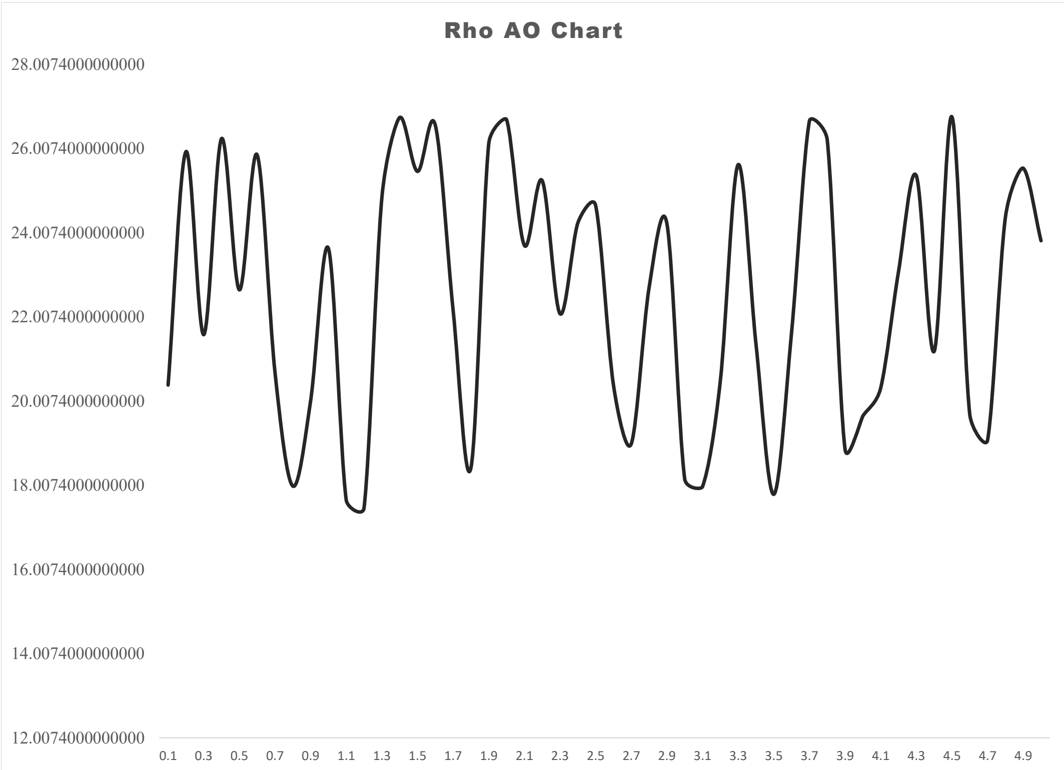

Figure 4.5: Rho AO Graph

			Asset Prices				
Investment Period	Rho		75.0000	80.0000	85.0000	65.0000	70.0000
0.1000	46.7697		80.0003	85.3337	90.6671	69.3336	74.6670
0.2000	51.0455		81.5868	87.0260	92.4651	70.7086	76.1477
0.3000	49.7324		81.0996	86.5062	91.9129	70.2863	75.6930
0.4000	50.1767		81.2645	86.6821	92.0997	70.4292	75.8468
0.5000	48.0856		80.4886	85.8545	91.2204	69.7568	75.1227
0.6000	51.6405		81.8076	87.2614	92.7152	70.8999	76.3537
0.7000	47.1276		80.1331	85.4753	90.8175	69.4487	74.7909
0.8000	45.1046		79.3825	84.6747	89.9669	68.7982	74.0904
0.9000	42.7861		78.5223	83.7571	88.9919	68.0527	73.2875
1.0000	46.2135		79.7940	85.1136	90.4332	69.1548	74.4744
1.1000	43.4694		78.7758	84.0275	89.2792	68.2724	73.5241
1.2000	41.6191		78.0893	83.2952	88.5012	67.6774	72.8833
1.3000	40.7190		77.7553	82.9390	88.1227	67.3879	72.5716
1.4000	48.1689		80.5195	85.8875	91.2554	69.7836	75.1515
1.5000	43.6918		78.8583	84.1155	89.3728	68.3439	73.6011
1.6000	48.2264		80.5408	85.9102	91.2796	69.8021	75.1714
1.7000	52.8617		82.2607	87.7448	93.2288	71.2926	76.7767
1.8000	53.6837		82.5657	88.0701	93.5744	71.5569	77.0613
1.9000	44.6830		79.2261	84.5079	89.7896	68.6626	73.9444
2.0000	42.8151		78.5330	83.7686	89.0041	68.0620	73.2975
2.1000	49.9272		81.1719	86.5833	91.9948	70.3490	75.7604
2.2000	52.7556		82.2213	87.7028	93.1842	71.2585	76.7399
2.3000	50.3947		81.3454	86.7684	92.1914	70.4993	75.9223
2.4000	42.0627		78.2539	83.4708	88.6877	67.8200	73.0370
2.5000	40.7869		77.7805	82.9659	88.1512	67.4098	72.5951
2.6000	44.9500		79.3252	84.6135	89.9018	68.7485	74.0368
2.7000	39.8561		77.4351	82.5975	87.7598	67.1105	72.2728
2.8000	48.2570		80.5522	85.9223	91.2925	69.8119	75.1820
2.9000	46.6462		79.9545	85.2848	90.6151	69.2939	74.6242
3.0000	49.7869		81.1198	86.5278	91.9358	70.3039	75.7119
3.1000	47.7984		80.3820	85.7408	91.0996	69.6644	75.0232
3.2000	46.0944		79.7498	85.0664	90.3831	69.1165	74.4331
3.3000	40.9523		77.8419	83.0313	88.2208	67.4630	72.6524
3.4000	41.6225		78.0906	83.2966	88.5026	67.6785	72.8845
3.5000	43.1386		78.6531	83.8966	89.1402	68.1660	73.4095
3.6000	46.4774		79.8919	85.2180	90.5441	69.2396	74.5657
3.7000	47.4271		80.2442	85.5939	90.9435	69.5450	74.8946
3.8000	45.3666		79.4797	84.7784	90.0770	68.8824	74.1811
3.9000	51.4185		81.7252	87.1736	92.6219	70.8285	76.2769
4.0000	45.7540		79.6235	84.9317	90.2400	69.0070	74.3153
4.1000	51.8787		81.8960	87.3557	92.8154	70.9765	76.4363
4.2000	41.8341		78.1690	83.3803	88.5916	67.7465	72.9578
4.3000	52.2042		82.0167	87.4845	92.9523	71.0812	76.5489
4.4000	42.8204		78.5350	83.7707	89.0063	68.0637	73.2993
4.5000	40.7257		77.7578	82.9416	88.1255	67.3901	72.5739
4.6000	49.7816		81.1179	86.5257	91.9336	70.3022	75.7100
4.7000	52.2116		82.0195	87.4875	92.9554	71.0836	76.5515
4.8000	53.1822		82.3796	87.8716	93.3636	71.3957	76.8876
4.9000	49.2101		80.9058	86.2995	91.6932	70.1184	75.5121
5.0000	41.5088		78.0483	83.2516	88.4548	67.6419	72.8451

Table 4.6: Rho BOA Data

BOA Graph.pdf

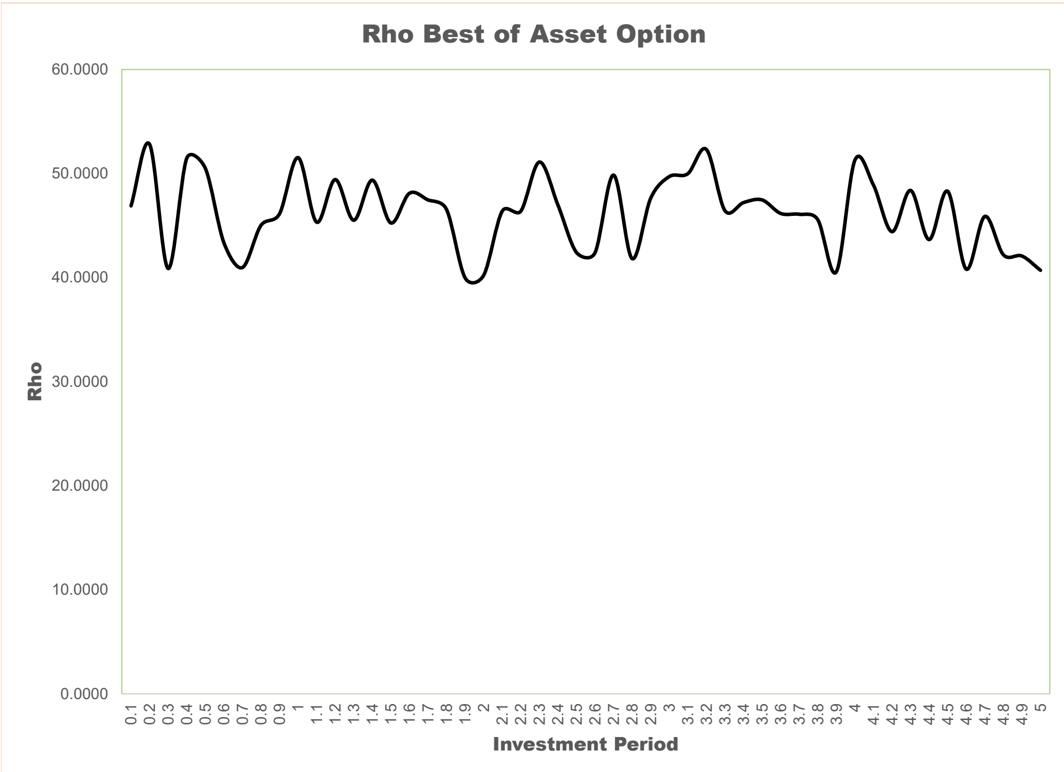

Figure 4.6: Rho BOA Graph

and suppose $V(\tau)$ represents the option value where $\tau \in [0, T]$, then the measures of changes in V in terms of rate of interest is given as

$$\rho = \frac{\partial V}{\partial r}$$

$$\rho_2 = \frac{e^{-rT}}{\sigma} \mathbb{E}_Q[(\frac{1}{T}\int_0^T \mathtt{S}_\tau d\tau - \mathbf{K})B_T]$$

so,

$$\rho_2 = \frac{e^{-rT}}{\sigma}[(\frac{1}{m}\sum_{j=1}^{m}(\mathtt{S}_j + a\mathtt{S}_j h - \mathbf{K})B_0]$$

Let $C_B = [Max(\mathtt{S}_i - \mathbf{K}), 0]\mathbf{1}_{\mathtt{S}_i > \mathtt{S}_j \quad i \neq j, \quad i,j=1,2,\ldots n}$ be the payoff process of Best of Assets call option and let $V(\tau), \tau \in [0, T]$ be the value of the option at time τ, then the measures the sensitivity of the option with respect to changes in the rate of interest is given as

$$\rho = \frac{\partial V}{\partial r}$$

$$\rho_3 = \frac{e^{-rT}}{\sigma}\mathbb{E}[Max(\mathtt{S}_i - \mathbf{K}), 0]\mathbf{1}_{\mathtt{S}_i > \mathtt{S}_j \quad i \neq j} B_T]$$

so,

$$\rho_3 = \frac{e^{-rT}}{\sigma}[(Max(\mathtt{S}_i + a\mathtt{S}_i h - \mathbf{K})B_0]$$

4.9.4 Theta

Let $C_E = max[(\mathtt{S}_T - \mathbf{K}), 0]$ be the pay off process of an European call and suppose $V(\tau)$ represents the option value, at time τ, $\tau \in [0, T]$, then the measures of changes in V in terms of expiration time is given as

$$\Theta = \frac{\partial V}{\partial T}$$

$$\Theta_1 = \frac{e^{-rT}}{\sigma T}\mathbb{E}[(\mathtt{S}_T - \mathbf{K})^+)(\kappa - \frac{\sigma^2}{2})B_T]$$

so,

$$\Theta_1 = \frac{e^{-rT}}{\sigma T}[(\mathtt{S}_j + a\mathtt{S}_j h - \mathbf{K})(\kappa - \frac{\sigma^2}{2})B_0]$$

Let $C_A = [Max(\frac{1}{T}\int_0^T \mathtt{S}_T d\tau - \mathbf{K}), 0]$ be the pay off process of an Asian call and suppose $V(\tau)$ represent the option value where $\tau \in [0, T]$, then the measures

			Asset Prices				
Investment Period	Theta AO		75	80	85	65	70
0.1	0.33914		79.49167	84.79112	90.09056	68.89278	74.19223
0.2	0.39814		81.36756	86.79206	92.21657	70.51855	75.94305
0.3	0.32319		78.98470	84.25034	89.51599	68.45341	73.71905
0.4	0.37287		80.56415	85.93510	91.30604	69.82227	75.19321
0.5	0.42087		82.06118	87.53192	93.00267	71.11969	76.59043
0.6	0.31794		78.81779	84.07231	89.32683	68.30875	73.56327
0.7	0.33745		79.43803	84.73389	90.02976	68.84629	74.14216
0.8	0.35962		80.14290	85.48576	90.82862	69.45718	74.80004
0.9	0.35287		79.92842	85.25698	90.58554	69.27130	74.59986
1.0	0.29472		78.07945	83.28474	88.49004	67.66885	72.87415
1.1	0.34505		79.67954	84.99151	90.30348	69.05560	74.36757
1.2	0.35412		79.96797	85.29917	90.63037	69.30558	74.63677
1.3	0.39594		81.29774	86.71759	92.13744	70.45804	75.87789
1.4	0.34451		79.66241	84.97324	90.28407	69.04076	74.35158
1.5	0.31819		78.82578	84.08083	89.33588	68.31567	73.57072
1.6	0.27302		77.38979	82.54911	87.70843	67.07115	72.23047
1.7	0.29876		78.20784	83.42170	88.63556	67.78013	72.99399
1.8	0.27210		77.36025	82.51760	87.67495	67.04555	72.20290
1.9	0.32660		79.09297	84.36583	89.63870	68.54724	73.82010
2.0	0.40874		81.70466	87.15164	92.59862	70.81071	76.25768
2.1	0.41761		81.97559	87.44063	92.90567	71.04551	76.51055
2.2	0.42398		82.14293	87.61912	93.09532	71.19054	76.66673
2.3	0.28618		77.80801	82.99521	88.18241	67.43361	72.62081
2.4	0.30905		78.53526	83.77094	89.00662	68.06389	73.29957
2.5	0.39363		81.22407	86.63901	92.05394	70.39419	75.80913
2.6	0.35736		80.07114	85.40921	90.74729	69.39498	74.73306
2.7	0.41909		82.01450	87.48213	92.94976	71.07923	76.54686
2.8	0.32362		78.99841	84.26497	89.53153	68.46529	73.73185
2.9	0.41419		81.87795	87.33648	92.79501	70.96089	76.41942
3.0	0.36438		80.29423	85.64718	91.00013	69.58833	74.94128
3.1	0.38876		81.06941	86.47404	91.87866	70.26015	75.66478
3.2	0.33500		79.36027	84.65096	89.94164	68.77890	74.06959
3.3	0.31264		78.64937	83.89266	89.13595	68.16279	73.40608
3.4	0.30036		78.25871	83.47596	88.69321	67.82422	73.04146
3.5	0.31566		78.74541	83.99510	89.24480	68.24602	73.49571
3.6	0.28604		77.80375	82.99066	88.17758	67.42991	72.61683
3.7	0.38969		81.09874	86.50533	91.91191	70.28558	75.69216
3.8	0.42213		82.09438	87.56734	93.04030	71.14847	76.62142
3.9	0.36638		80.35788	85.71507	91.07227	69.64350	75.00069
4.0	0.40892		81.71013	87.15747	92.60481	70.81544	76.26278
4.1	0.30498		78.40560	83.63264	88.85968	67.95152	73.17856
4.2	0.41305		81.84140	87.29750	92.75359	70.92921	76.38531
4.3	0.31106		78.59913	83.83907	89.07901	68.11925	73.35919
4.4	0.42085		82.06061	87.53132	93.00202	71.11919	76.58990
4.5	0.39189		81.16887	86.58012	91.99138	70.34635	75.75761
4.6	0.38412		80.92177	86.31655	91.71134	70.13220	75.52698
4.7	0.42029		82.04608	87.51582	92.98556	71.10660	76.57634
4.8	0.36275		80.24225	85.59173	90.94121	69.54328	74.89276
4.9	0.35154		79.88607	85.21181	90.53755	69.23460	74.56034
5.0	0.29492		78.08583	83.29155	88.49727	67.67438	72.88011

Table 4.7: Theta AO Data

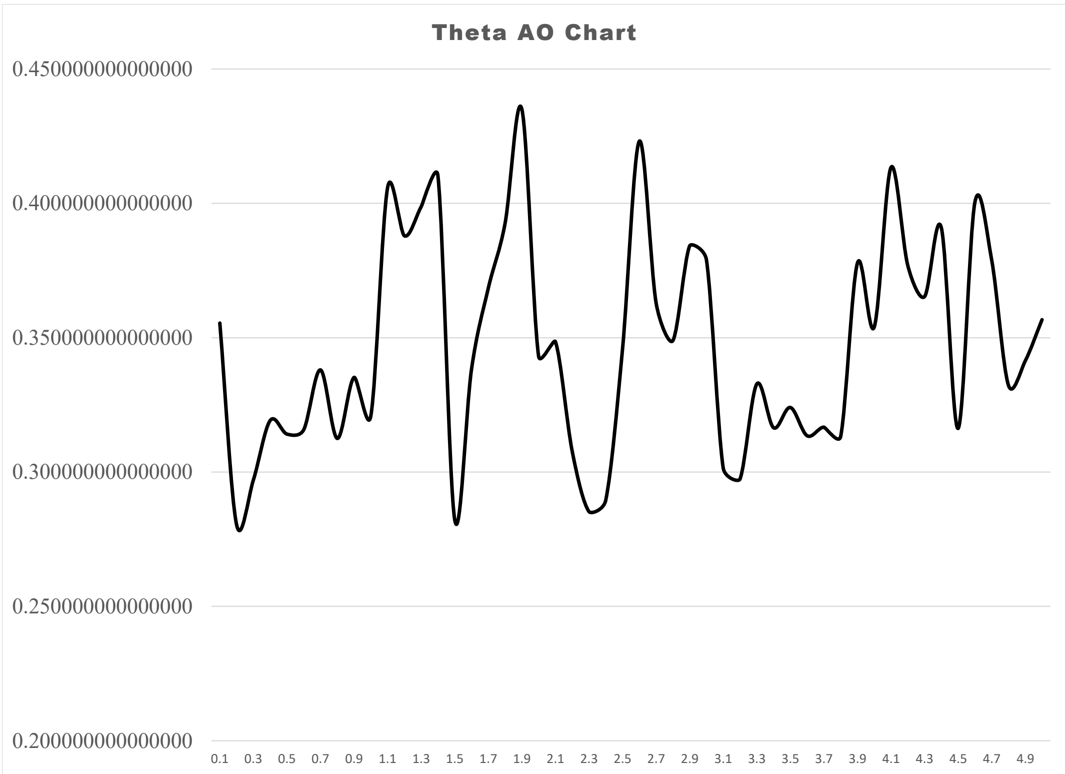

Figure 4.7: Theta AO Graph

			Asset Prices				
Investment Period	**Theta**		**75.0000**	**80.0000**	**85.0000**	**65.0000**	**70.0000**
0.1000	0.8111		81.4562	86.8867	92.3171	70.5954	76.0258
0.2000	0.7546		80.1470	85.4902	90.8333	69.4608	74.8039
0.3000	0.7552		80.1607	85.5047	90.8488	69.4726	74.8166
0.4000	0.8317		81.9344	87.3967	92.8590	71.0098	76.4721
0.5000	0.6512		77.7477	82.9309	88.1140	67.3813	72.5645
0.6000	0.7311		79.6006	84.9073	90.2140	68.9872	74.2939
0.7000	0.6911		78.6733	83.9182	89.1631	68.1835	73.4284
0.8000	0.7630		80.3403	85.6963	91.0523	69.6282	74.9842
0.9000	0.7029		78.9462	84.2093	89.4724	68.4201	73.6832
1.0000	0.7521		80.0892	85.4285	90.7678	69.4107	74.7500
1.1000	0.7611		80.2966	85.6498	91.0029	69.5904	74.9435
1.2000	0.8164		81.5781	87.0167	92.4552	70.7011	76.1396
1.3000	0.6881		78.6028	83.8430	89.0832	68.1225	73.3627
1.4000	0.7550		80.1564	85.5001	90.8439	69.4689	74.8126
1.5000	0.7685		80.4678	85.8324	91.1969	69.7388	75.1033
1.6000	0.7147		79.2197	84.5011	89.7824	68.6571	73.9384
1.7000	0.7282		79.5335	84.8357	90.1379	68.9290	74.2312
1.8000	0.7848		80.8475	86.2373	91.6271	70.0678	75.4576
1.9000	0.8469		82.2868	87.7726	93.2583	71.3152	76.8010
2.0000	0.6521		77.7686	82.9532	88.1378	67.3995	72.5840
2.1000	0.6566		77.8745	83.0661	88.2578	67.4912	72.6829
2.2000	0.8352		82.0161	87.4839	92.9516	71.0806	76.5484
2.3000	0.7773		80.6716	86.0497	91.4278	69.9154	75.2935
2.4000	0.7975		81.1403	86.5497	91.9590	70.3216	75.7309
2.5000	0.7478		79.9875	85.3200	90.6525	69.3225	74.6550
2.6000	0.6908		78.6659	83.9103	89.1547	68.1771	73.4215
2.7000	0.7650		80.3883	85.7475	91.1067	69.6698	75.0291
2.8000	0.8453		82.2485	87.7317	93.2149	71.2820	76.7652
2.9000	0.8339		81.9856	87.4513	92.9170	71.0542	76.5199
3.0000	0.8501		82.3606	87.8513	93.3420	71.3792	76.8699
3.1000	0.8469		82.2876	87.7735	93.2593	71.3159	76.8018
3.2000	0.8397		82.1202	87.5948	93.0695	71.1708	76.6455
3.3000	0.8077		81.3774	86.8026	92.2278	70.5271	75.9523
3.4000	0.6645		78.0561	83.2599	88.4636	67.6487	72.8524
3.5000	0.8315		81.9285	87.3904	92.8522	71.0047	76.4666
3.6000	0.6436		77.5731	82.7446	87.9161	67.2300	72.4015
3.7000	0.7216		79.3807	84.6728	89.9648	68.7966	74.0887
3.8000	0.7067		79.0351	84.3041	89.5731	68.4971	73.7661
3.9000	0.7807		80.7524	86.1359	91.5194	69.9854	75.3689
4.0000	0.8502		82.3623	87.8531	93.3439	71.3807	76.8715
4.1000	0.7963		81.1127	86.5202	91.9277	70.2977	75.7052
4.2000	0.7605		80.2829	85.6351	90.9872	69.5785	74.9307
4.3000	0.6590		77.9302	83.1256	88.3209	67.5395	72.7349
4.4000	0.6686		78.1515	83.3616	88.5717	67.7313	72.9414
4.5000	0.8008		81.2173	86.6318	92.0463	70.3884	75.8028
4.6000	0.8532		82.4318	87.9272	93.4227	71.4409	76.9363
4.7000	0.8498		82.3526	87.8428	93.3330	71.3723	76.8625
4.8000	0.8329		81.9625	87.4267	92.8909	71.0342	76.4984
4.9000	0.7639		80.3617	85.7192	91.0766	69.6468	75.0043
5.0000	0.6513		77.7508	82.9342	88.1176	67.3840	72.5674

Table 4.8: Theta BOA Data

BOA Graph.pdf

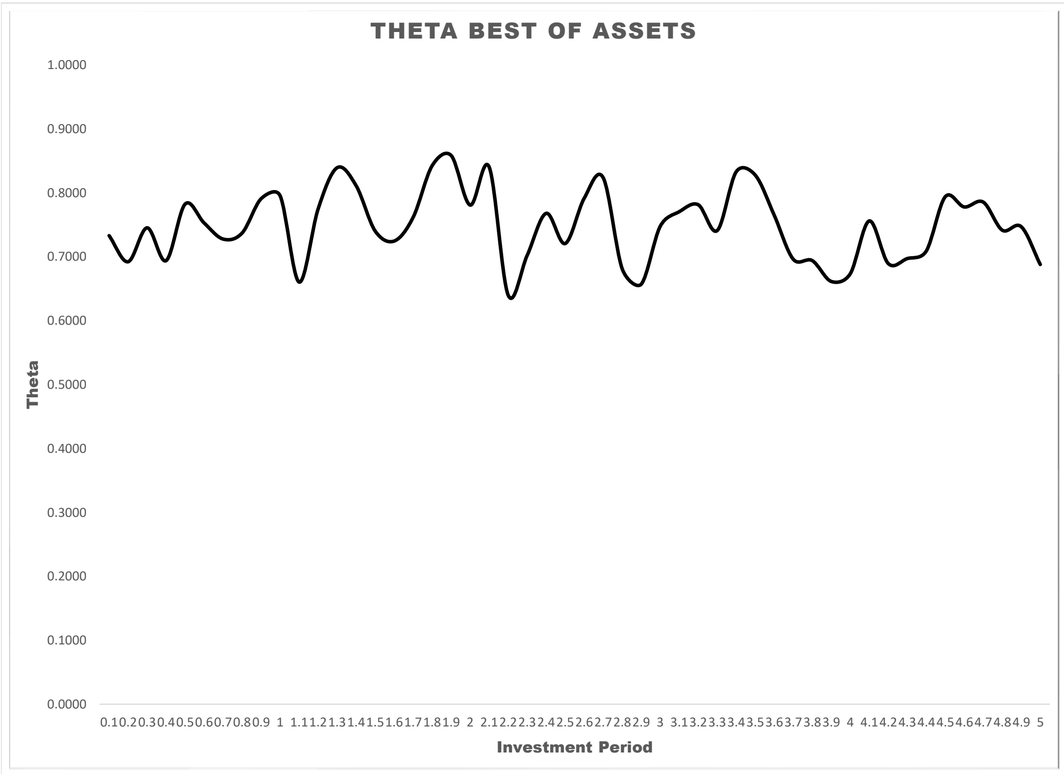

Figure 4.8: Theta BOA Graph

of changes in V in terms of expiration time is given as

$$\Theta = \frac{\partial V}{\partial T}$$

$$\Theta_2 = \frac{e^{-rT}}{\sigma T}\mathbb{E}[(\frac{1}{T}\int_0^T \mathtt{S}_\tau d\tau - \mathbf{K})))(\kappa - \frac{\sigma^2}{2})B_T]$$

so,

$$\Theta_2 = \frac{e^{-rT}}{\sigma}[(\frac{1}{m}\sum_{j=1}^{m}(\mathtt{S}_j + a\mathtt{S}_j h - \mathbf{K})))(\kappa - \frac{\sigma^2}{2})B_0]$$

Let $C_B = [Max(\mathtt{S}_i - \mathbf{K}), 0]\mathbf{1}_{\mathtt{S}_i > \mathtt{S}_j \quad i \neq j, \quad i,j=1,2,\ldots n}$ be the payoff process of Best of Assets call option and

Let $V(\tau), \tau \in [0, T]$ be the value of the option at time τ, then the measures the sensitivity of the option with respect to changes in the time to expiration is given as

$$\Theta = \frac{\partial V}{\partial T}$$

$$\Theta_3 = \frac{e^{-rT}}{\sigma T}\mathbb{E}[Max(\mathtt{S}_i - \mathbf{K}), 0]\mathbf{1}_{\mathtt{S}_i > \mathtt{S}_j \quad i \neq j}(\kappa - \frac{\sigma^2}{2})B_T]$$

so,

$$\Theta_3 = \frac{e^{-rT}}{\sigma T}[(Max(\mathtt{S}_i + a\mathtt{S}_i h - \mathbf{K})(\kappa - \frac{\sigma^2}{2})B_0]$$

4.9.5 Vega

Let $C_E = max[(\mathtt{S}_T - \mathbf{K}), 0]$ be the pay off process of an European call and suppose $V(\tau)$ represent the option value where $\tau \in [0, T]$, then the measures of changes in V with respect to changes in the volatility is given as

$$\Theta = \frac{\partial V}{\partial \sigma}$$

$$\vartheta_1 = \frac{e^{-rT}}{2\sigma T}\mathbb{E}[(\mathtt{S}_T - \mathbf{K})^+)(B_T^2 - T - 2B_T)]$$

so,

$$\vartheta_1 = \frac{e^{-rT}}{2\sigma T}[(\mathtt{S}_j + a\mathtt{S}_j h - \mathbf{K})(B_0^2 - T - 2B_0)]$$

Let $C_A = [Max(\frac{1}{T}\int_0^T \mathtt{S}_T d\tau - \mathbf{K}), 0]$ be the pay off process of an Asian call and suppose $V(\tau)$ represent the option value where $\tau \in [0, T]$, then the measures

			Asset Prices				
Investment Period	Vega AO		75	80	85	65	70
0.1	1.22398		81.16197	86.57277	91.98356	70.34037	75.75117
0.2	1.09328		79.83230	85.15445	90.47661	69.18799	74.51015
0.3	0.99922		78.87532	84.13368	89.39203	68.35861	73.61697
0.4	1.09872		79.88756	85.21339	90.53923	69.23588	74.56172
0.5	1.15503		80.46044	85.82447	91.18850	69.73238	75.09641
0.6	1.20554		80.97435	86.37264	91.77093	70.17777	75.57606
0.7	1.26719		81.60154	87.04165	92.48175	70.72134	76.16144
0.8	1.29809		81.91589	87.37695	92.83801	70.99377	76.45483
0.9	1.09309		79.83028	85.15230	90.47432	69.18625	74.50827
1	1.29698		81.90462	87.36493	92.82524	70.98400	76.44431
1.1	1.10498		79.95126	85.28134	90.61143	69.29109	74.62117
1.2	1.33447		82.22313	87.70467	93.18621	71.26004	76.74159
1.3	1.36330		82.46561	87.96332	93.46103	71.47020	76.96791
1.4	0.86539		77.51385	82.68144	87.84903	67.17867	72.34626
1.5	0.98696		78.75058	84.00062	89.25066	68.25050	73.50054
1.6	0.97599		78.63903	83.88163	89.12423	68.15382	73.39643
1.7	1.01752		79.06155	84.33232	89.60309	68.52001	73.79078
1.8	0.93016		78.17277	83.38429	88.59581	67.74974	72.96126
1.9	1.24203		81.34557	86.76861	92.19164	70.49949	75.92253
2	1.36779		82.50333	88.00355	93.50378	71.50289	77.00311
2.1	0.93455		78.21743	83.43193	88.64642	67.78844	73.00294
2.2	1.32159		82.11477	87.58909	93.06341	71.16614	76.64046
2.3	1.07218		79.61763	84.92547	90.23331	69.00195	74.30979
2.4	1.21216		81.04174	86.44452	91.84731	70.23618	75.63896
2.5	1.13096		80.21563	85.56334	90.91105	69.52021	74.86792
2.6	1.15893		80.50014	85.86681	91.23349	69.76678	75.13346
2.7	0.86469		77.50663	82.67374	87.84085	67.17241	72.33952
2.8	1.03266		79.21553	84.49656	89.77760	68.65346	73.93449
2.9	1.30246		81.95394	87.41754	92.88113	71.02675	76.49034
3	1.10428		79.94414	85.27375	90.60336	69.28492	74.61453
3.1	1.12243		80.12880	85.47072	90.81264	69.44496	74.78688
3.2	1.20351		80.95365	86.35056	91.74747	70.15983	75.55674
3.3	1.22602		81.18266	86.59484	92.00701	70.35830	75.77048
3.4	1.15956		80.50657	85.87367	91.24078	69.77236	75.13946
3.5	1.27325		81.66321	87.10743	92.55164	70.77478	76.21900
3.6	1.04248		79.31549	84.60319	89.89089	68.74009	74.02779
3.7	1.20476		80.96646	86.36422	91.76199	70.17093	75.56869
3.8	0.91256		77.99368	83.19326	88.39284	67.59453	72.79410
3.9	0.96309		78.50779	83.74164	88.97550	68.04009	73.27394
4	1.09914		79.89185	85.21798	90.54410	69.23961	74.56573
4.1	1.10691		79.97096	85.30236	90.63376	69.30817	74.63957
4.2	1.24088		81.33389	86.75615	92.17841	70.48937	75.91163
4.3	1.22708		81.19348	86.60638	92.01928	70.36768	75.78058
4.4	0.97494		78.62829	83.87018	89.11207	68.14452	73.38641
4.5	0.85163		77.37381	82.53206	87.69031	67.05730	72.21555
4.6	1.15265		80.43625	85.79866	91.16108	69.71141	75.07383
4.7	1.28044		81.73632	87.18541	92.63450	70.83814	76.28723
4.8	1.37552		82.56837	88.07293	93.57749	71.55925	77.06381
4.9	1.15776		80.48826	85.85414	91.22002	69.75649	75.12237
5	1.02210		79.10807	84.38194	89.65581	68.56033	73.83420

Table 4.9: Vega AO Data

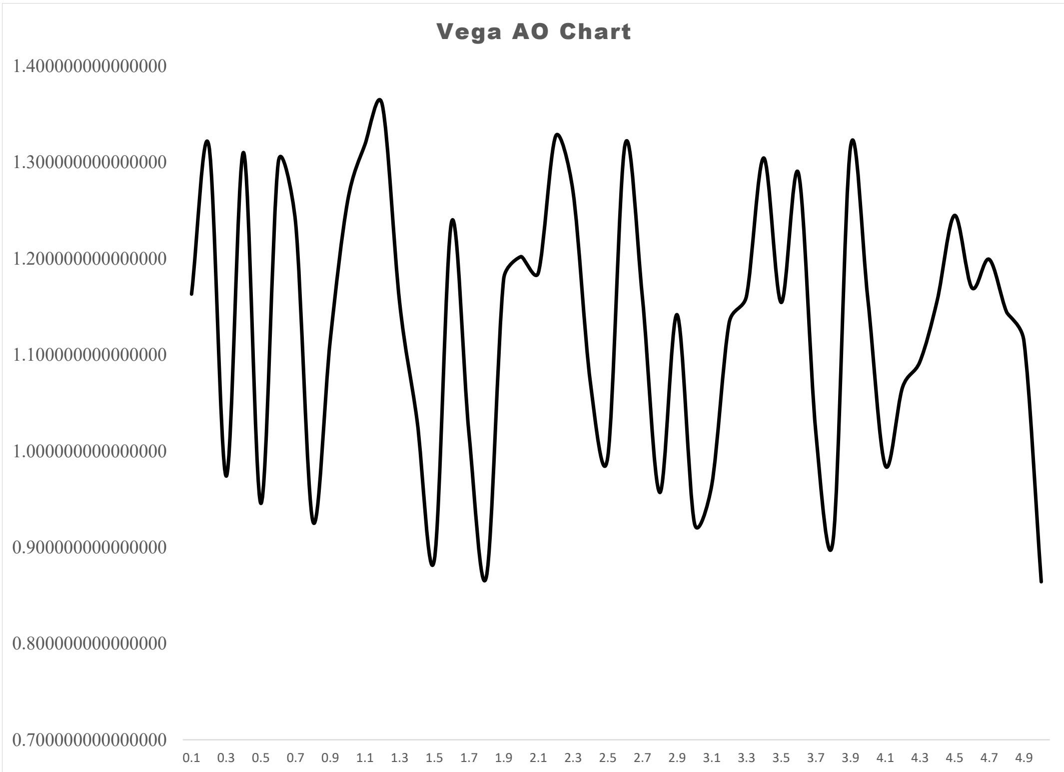

Figure 4.9: Vega AO Graph

Data.pdf

Investment Period	Vega	Asset Prices				
		75.0000	**80.0000**	**85.0000**	**65.0000**	**70.0000**
0.1	2.099709763	78.2285	83.4437	88.6589	67.7980	73.0132
0.2	2.5510197	81.5775	87.0160	92.4545	70.7005	76.1390
0.3	2.436824233	80.7301	86.1121	91.4941	69.9661	75.3481
0.4	2.075732231	78.0505	83.2539	88.4573	67.6438	72.8472
0.5	1.998531473	77.4776	82.6428	87.8080	67.1473	72.3125
0.6	2.623128541	82.1126	87.5868	93.0610	71.1643	76.6384
0.7	2.239088962	79.2627	84.5469	89.8311	68.6944	73.9786
0.8	2.430047463	80.6798	86.0585	91.4371	69.9225	75.3011
0.9	2.479653032	81.0479	86.4511	91.8543	70.2415	75.6447
1	2.648351329	82.2998	87.7864	93.2731	71.3265	76.8131
1.1	2.497353011	81.1793	86.5912	92.0032	70.3554	75.7673
1.2	2.174365323	78.7825	84.0346	89.2868	68.2781	73.5303
1.3	2.217171568	79.1001	84.3734	89.6468	68.5534	73.8268
1.4	2.081942999	78.0966	83.3031	88.5095	67.6837	72.8902
1.5	2.339690595	80.0093	85.3432	90.6772	69.3414	74.6753
1.6	2.609807248	82.0138	87.4813	92.9489	71.0786	76.5462
1.7	2.237198	79.2487	84.5320	89.8152	68.6822	73.9655
1.8	2.451604985	80.8398	86.2291	91.6184	70.0611	75.4505
1.9	2.216879727	79.0979	84.3711	89.6443	68.5515	73.8247
2	2.404187768	80.4879	85.8538	91.2196	69.7562	75.1220
2.1	2.172586255	78.7693	84.0205	89.2718	68.2667	73.5180
2.2	2.026163458	77.6827	82.8615	88.0404	67.3250	72.5038
2.3	2.476152941	81.0219	86.4234	91.8249	70.2190	75.6205
2.4	2.395564431	80.4239	85.7855	91.1471	69.7007	75.0623
2.5	2.581779047	81.8058	87.2595	92.7132	70.8983	76.3520
2.6	2.260029037	79.4181	84.7127	90.0072	68.8291	74.1236
2.7	2.07580555	78.0511	83.2545	88.4579	67.6443	72.8477
2.8	2.493575911	81.1512	86.5613	91.9714	70.3311	75.7411
2.9	2.272249398	79.5088	84.8094	90.1100	68.9076	74.2082
3	2.441451258	80.7644	86.1487	91.5330	69.9958	75.3801
3.1	2.347382396	80.0664	85.4041	90.7419	69.3909	74.7286
3.2	2.399284358	80.4515	85.8150	91.1784	69.7246	75.0881
3.3	2.028640691	77.7011	82.8811	88.0612	67.3409	72.5210
3.4	2.35668011	80.1354	85.4777	90.8201	69.4506	74.7930
3.5	2.580551524	81.7967	87.2498	92.7029	70.8904	76.3435
3.6	2.426546133	80.6538	86.0307	91.4077	69.9000	75.2769
3.7	2.168352437	78.7378	83.9870	89.2362	68.2395	73.4886
3.8	2.615006251	82.0523	87.5225	92.9926	71.1120	76.5822
3.9	2.332329475	79.9547	85.2850	90.6153	69.2940	74.6244
4	2.501337065	81.2088	86.6227	92.0367	70.3810	75.7949
4.1	2.265082961	79.4556	84.7527	90.0497	68.8616	74.1586
4.2	2.384240601	80.3399	85.6959	91.0519	69.6279	74.9839
4.3	2.208406309	79.0351	84.3041	89.5731	68.4971	73.7661
4.4	2.085795065	78.1252	83.3335	88.5419	67.7085	72.9168
4.5	2.628250995	82.1506	87.6273	93.1040	71.1972	76.6739
4.6	2.666021264	82.4309	87.9263	93.4217	71.4401	76.9355
4.7	2.034993471	77.7482	82.9314	88.1146	67.3818	72.5650
4.8	2.609196365	82.0092	87.4765	92.9438	71.0747	76.5419
4.9	2.062019252	77.9488	83.1453	88.3419	67.5556	72.7522
5	2.042181283	77.8015	82.9883	88.1751	67.4280	72.6148

Table 4.10: Vega BOA Data

BOA Graph.pdf

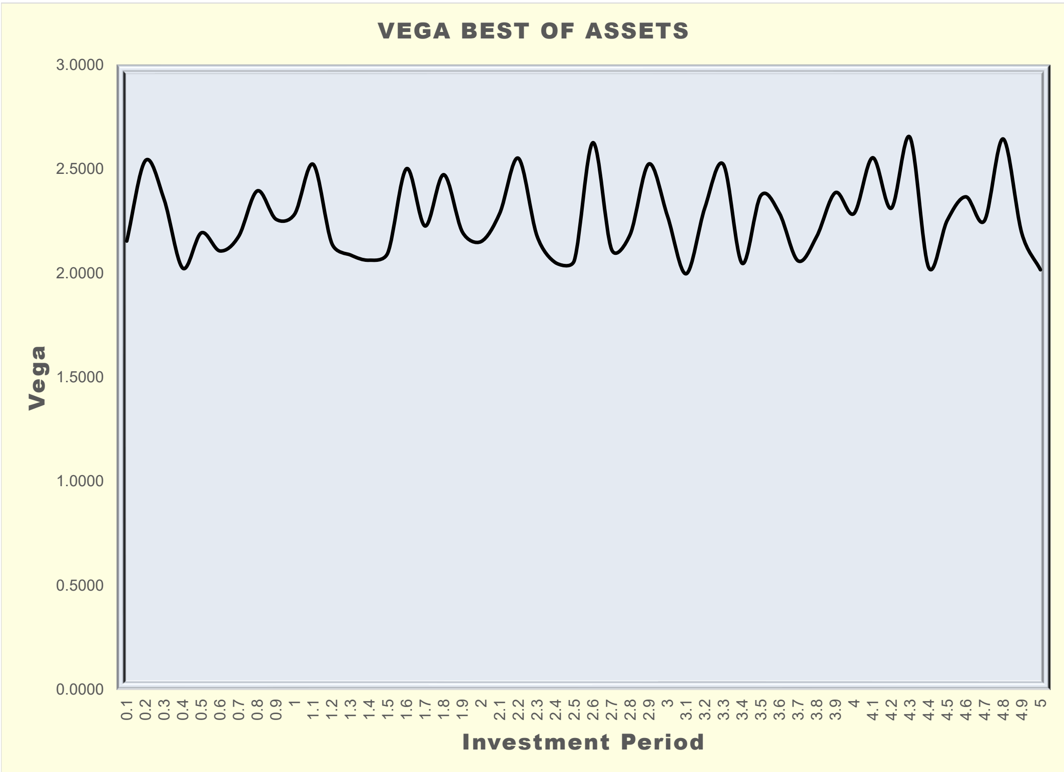

Figure 4.10: Vega BOA Graph

of changes in V with respect to changes in the volatility is given as

$$\vartheta = \frac{\partial V}{\partial \sigma}$$

$$\vartheta_2 = \frac{e^{-rT}}{2\sigma T}\mathbb{E}[(\frac{1}{T}\int_0^T \mathtt{S}_\tau d\tau - \mathbf{K})(B_T^2 - T - 2B_T)]$$

so,

$$\vartheta_2 = \frac{e^{-rT}}{2\sigma T}[(\frac{1}{m}\sum_{j=1}^{m}(\mathtt{S}_j + a\mathtt{S}_j h - \mathbf{K})((B_0^2 - T - 2B_0)]$$

Let $C_B = [Max(\mathtt{S}_i - \mathbf{K}), 0]\mathbf{1}_{\mathtt{S}_i > \mathtt{S}_j \quad i \neq j, \quad i,j=1,2,\ldots n}$ be the payoff process of Best of Assets call and suppose $V(\tau)$ represent the option value where $\tau \in [0, T]$, then the measures the sensitivity of the option with respect to changes in the volatility is given as

$$\vartheta = \frac{\partial V}{\partial \sigma}$$

$$\vartheta_3 = \frac{e^{-rT}}{2\sigma T}\mathbb{E}[Max(\mathtt{S}_i - \mathbf{K}), 0]\mathbf{1}_{\mathtt{S}_i > \mathtt{S}_j \quad i \neq j}(B_T^2 - T - 2B_T)]$$

so,

$$\vartheta_3 = \frac{e^{-rT}}{2\sigma T}[(Max(\mathtt{S}_i + a\mathtt{S}_i h - \mathbf{K})(B_0^2 - T - 2B_0)]$$

4.10 Discussion

In this section, we analyse and discuss the results obtained for the various Greeks and their implications to an investors

4.10.1 Delta

Delta values are always between -1 and 1.The delta value of a Call option stands between 0 and 1, while the delta values of a Put option always stands between 0 and -1. When delta value of a Call option is between 0 and 0.5, delta is said to be strong and consequently, risk is minimized. But when delta value of a Call option is between 0.5 and 1, delta is said to be weak and consequently, risk is high.

In table 4.1, we used the following values for the computation, $\sigma = 0.2$, $r = 0.01$, $S_0 = 70$, $\kappa = 0.3$, $h = 0.1$, $B_0 = 0.5$, $T = 5$, but we used different values for K, that is $K = 71$, $K = 72$, $K = 73$, $K = 74$, and $K = 75$. When K is allowed to take different values, and the the value of S_j is taken randomly, it was observed that delta is higher when K is the smallest. Delta is better when it value increases from zero towards 0.2.

In table 4.2, we used the following values for the computation, $\sigma = 0.2$, $r = 0.01$, $S_0 = 70$, $\kappa = 0.3$, $h = 0.1$, $B_0 = 0.5$, $T = 5$, and $K = 71$. The result here indicate that, if K is fixed and we allowed S_j to take different values, it was observed that delta is higher at 0.1502 when the asset values are high at 82.1524, 87.6293, 93.1061, 71.1988 and 76.6756. This is expected for a Call option because, as the underlying asset value increases, the difference between the underlying asset value and the strike price increases also. This is what an investor wants since this increment is likely to be positive.This positive difference is like making profit on the investment.

4.10.2 Gamma

In table 4.3, we used the following values for the computation, $\sigma = 0.2$, $r = 0.01$, $S_0 = 70$, $\kappa = 0.3$, $h = 0.1$, $B_0 = 0.5$, $T = 5$, and $K = 71$.
Gamma is the derivative of delta with respect to the underlying asset. This means that, the value of gamma is expected to be less when compare with corresponding values of delta. It can be observe that, 0.00654 is the highest value of gamma,and this value is obtained when the underlying asset value is highest at 82.4939, 87.99284, 93.49239, 71.49418 and 76.99374. When the value of Gamma reduces over the investment period compare to the value of delta, and the gamma value is between 0 and 0.1, then gamma is strong. If gamma is strong, then risk is minimal.

In table 4.4, we used the following values for the computation, $\sigma = 0.2$, $r = 0.01$, $S_0 = 70$, $\kappa = 0.3$, $h = 0.1$, $B_0 = 0.5$, $T = 5$, and $K = 71$.
Gamma is the derivative of delta with respect to the underlying asset. This means that, the value of gamma is expected to be less when compare with corresponding values of delta. It can be observe that, 0.0104 is the highest value of gamma,and this value is obtained when the underlying asset value is highest at 82.49459, 87.99422, 93.49386, 71.49531 and 76.99495. When the value of Gamma reduces over the investment period compare to the value of delta, and the gamma value is between 0 and 0.1, then gamma is strong. If gamma is strong, then risk is minimal.

This is expected for a Call option because, as the underlying asset value increases, the difference between the underlying asset value and the strike price increases also. This is what an investor wants since this increment is likely to be positive.This positive difference is like making profit on the investment.

4.10.3 Rho

Rho measured the effect of changes in the interest rate on the value of the option. When the interest rate is high, the holder of a Call is happy because the condition is favourable to him or her. This is because, the value of Call will increase, but this position is not favourable to the holder of a Put option. This high interest rate will lead to high value of rho, and this is only attainable when undelying asset value is high compare to the strike price.

In table 4.5, we used the following values for the computation, $\sigma = 0.2$, $r = 0.01$, $S_0 = 70$, $\kappa = 0.3$, $h = 0.1$, $B_0 = 0.5$, $T = 5$, and $K = 71$. Rho is highest with value 27.23274. This value is obtained when the underlying asset values are respectively 82.45160, 87.94837, 93.44514, and 71.45805. The difference between these values and the strike price is the highest, and when this happened, the holder of a Call option is at advantage because the condition is favourable.

In table 4.6, we used the following values for the computation, $\sigma = 0.2$, $r = 0.01$, $S_0 = 70$, $\kappa = 0.3$, $h = 0.1$, $B_0 = 0.5$, $T = 5$, and $K = 71$. Rho is highest with value 53.6837. This value is obtained when the underlying asset values are respectively 82.5657, 88.0701, 93.5733, 71.5569, and 77.0613. The difference between these values and the strike price is the highest, and when this happened, the holder of a Call option is at advantage because the condition is favourable.

4.10.4 Theta

Theta measures the effect of changes on the option with respect to the time to expiration. The value of theta is expected to lies between 0 and 1 for a Call option and between -1 and 0 for a Put option. Theta is expected to increase for option that is in the money, that is when the underlying asset value is greater than the strike price. As the difference between the underlying asset value and the strike price increases, the value of theta is also expected to increase.

In table 4.7, we used the following values for the computation, $\sigma = 0.2$, $r = 0.01$, $S_0 = 70$, $\kappa = 0.3$, $h = 0.1$, $B_0 = 0.5$, $T = 5$, and $K = 71$. Theta is highest with value 0.42398. This value is obtained when the underlying asset values are respectively 82.14293, 87.61912, 93.09532, 71.19054 and 76.66673 . The difference between these values and the strike price is the highest, and when this happened, the holder of a Call option is at advantage because the condition is favourable.

In table 4.8, we used the following values for the computation, $\sigma = 0.2$, $r = 0.01$, $S_0 = 70$, $\kappa = 0.3$, $h = 0.1$, $B_0 = 0.5$, $T = 5$, and $K = 71$. Theta is highest with value 0.8501. This value is obtained when the underlying asset values are respectively 82.3606, 87.8513, 93.3420, 71.3792 and 76.8699 . The difference between these values and the strike price is the highest, and when this happened, the holder of a Call option is at advantage because the condition is favourable.

4.10.5 Vega

Vega measures the effect of changes in the option with respect to the volatility. Vega takes positive values when volatility is high. When this happened, the financial market is said to be highly volatile. This condition is favourable to a holder of a Call option. This is because, increase in volatility leads to increase in the option value, and the increase in the option value is due to increase in the value of the underlying asset compare to the strike price.

In table 4.9, we used the following values for the computation, $\sigma = 0.2$, $r = 0.01$, $S_0 = 70$, $\kappa = 0.3$, $h = 0.1$, $B_0 = 0.5$, $T = 5$, and $K = 71$. Vega value is highest at 1.37552 when the underlying asset values becomes 82.56837, 88.07293, 93.57749, 71.55925 and 77.06381.

In table 4.10, we used the following values for the computation, $\sigma = 0.2$, $r = 0.01$, $S_0 = 70$, $\kappa = 0.3$, $h = 0.1$, $B_0 = 0.5$, $T = 5$, and $K = 71$. Vega value is highest at 2.66602 when the underlying asset values becomes 82.4309, 87.9263, 93.4217, 71.4401 and 76.9355.

Chapter 5

SUMMARY AND CONCLUSIONS

5.1 Introduction

In this section, we summarise our results and conclude as folows;

5.2 Summary

Delta:

- Changes in the option value with respect to changes in the value of the underlying asset.
- The value of delta is such that $-1 \leq \Delta \leq 1$.
- Δ of a call stand between 0 and 1 while for put, it stands between 0 and -1.
- As the underlying prices increases, Δ also increase towards 1.
- if the value of the underlying asset Increase, call is positive and put is negative.

Gamma

- This measure the changes in delta.
- Gamma is positive for long position and negative for short position.
- Gamma is smallest for deep out of money option and deep in the money option.
- As the market move higher,delta becomes more negative.

Rho

- This measure the changes in option value with respect to changes is interest rate.
- Increase in interest rate make call expensive and put less expensive.

Theta

- This measure the changes in option value with respect to the expiration time T.
- Theta decreases for out of the money option.
- Theta is least at the money.
- As theta decreases, it has negative effect on a holder with a long position.
- If T increases, call is positive and put is negative.

Vega

- This measure the changes in option value with respect to the volatility.
- Increase in the volatility increase the option value and it end up in the money.
- The writer is favoured when volatility falls and Vega becomes negative. This is because a writer want price to decline.
- Long call is favourable when the volatility rise.

5.3 Recommendation

In this work, the theory of Malliavin calculus was used to obtained the sensitivities of options with multiple underlying assets with non-smooth payoff. We assume that the underlying asset used in this work has a dynamics with constant drift which represent the interest rate and constant volatility. For future research, random drift and random volatility may be consider. Also, we could consider the possibility of having correlation among the underlying assets vis a vis the possibility of using it to analyse risk.

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