GRONWALL-BELLMAN-BIHARI TYPE INEQUALITY AND HYERS-ULAM AND HYERS-ULAM-RASSIAS STABILITIES OF CERTAIN CLASSES OF NONLINEAR SECOND AND THIRD ORDER DIFFERENTIAL EQUATIONS

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Certification

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Dedication

This thesis is dedicated to the glory of Almighty God who has been the Alpha and Omega of my life all through the period of this research.

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Abstract

The concept of stability in differential equations is of immense importance particularly for determination of properties of solutions of nonlinear equations which cannot be readily solved to obtain closed analytic solutions. Stability in the sense of Hyers-Ulam(H-U) and Hyers-Ulam-Rassias(H-U-R) has been considered for linear Ordinary Differential Equations(ODE) due to their solutions which are easily determined. However, cases of nonlinear ODE of second and third orders have received little attention. This research was therefore designed to establish the stability of nonlinear second and third order in the sense of H-U and H-U-R.

Variants of the perturbed second order ODE of the form
$u''(t) + f(t, u(t), u'(t)) = P(t, u(t), u'(t))$ and third order ODE of the form
$u'''(t) + f(t, u(t), u'(t), u''(t)) = P(t, u(t), u'(t))$ were reduced to their equivalent integral equations, where t is an independent real variable; f, P, and u are continuous functions of their argument. Extension of Gronwall-Bellman-Bihari(G-B-B) type inequalities having the same number of integrals as the equivalent integral equations were developed. These integral inequalities were used to prove the existence of H-U and H-U-R stability. They were also used to estimate the H-U and H-U-R constants for each of the variants of the equations.

The newly developed G-B-B type inequalities of nonlinear integrals obtained were:
$u(t) \leq u_0 + L\int_{t_0}^{t} f(s)u(s)ds + \int_{t_0}^{t} f(s)r(s)(\int_{t_0}^{s} \rho(\tau)\varpi(u(\tau))d\tau)ds$
and $u(t) \leq u_0 + T\int_{t_0}^{t} r(s)\beta(s)ds + L\int_{t_0}^{t} h(s)\varpi u(s))ds$
where $f,\ r,\ \rho,\ \varpi,\ \beta$ and h are continuous functions and $T,\ L$ and u_0 are positive constants. The nonlinear second order ODE were found to possess H-U stability and H-U constants
$K_{21} = L(1 + \frac{1}{2}\lambda^2 + |q(\xi, u(\xi), u'(\xi), u''(\xi))| + |p(u(\rho), u'(\rho))|\Omega^{-1}(\Omega(1) + ne^{M})$

and $K_{22} = (\frac{L}{\alpha(\xi)} + \frac{\lambda^2}{2\alpha(\xi)})\Omega^{-1}(\Omega(1) + \frac{\lambda^2}{\alpha(\xi)} n\varpi(F^{-1}(F(1) + \frac{\lambda^2}{\alpha(\xi)} m))F^{-1}(F(1) + m\frac{\lambda^2}{\alpha(\xi)})$, respectively, where $q, p, \Omega, \Omega^{-1}, F, F^{-1}, \varpi, h$ are functions of their argument and λ, ξ, ρ, M, n and m are constants. The newly developed G-B-B type inequality for two nonlinear integrals $u(t) \leq \rho(t) + T\int_{t_0}^{t} r(s)\beta(s)ds + L\int_{t_0}^{t} h(s)\varpi(u(s))ds$. where $\rho(t)$ a monotonic, nonnegative, continuous function, with this nonlinear second order ODE were found to possess H-U-R stability and the H-U-R constants
$C_{\varphi_{21}} = \Omega(\Omega(1) + m(\eta + \eta^2)\varpi(F^{-1}(F(1) + l))F^{-1}(F(1) + l)$ and
$C_{\varphi_{22}} = \Gamma^{-1}(\Gamma(1) + m\eta^n\gamma(F^{-1}(F(1) + l)))F^{-1}(F(1) + l)$ where
Γ, Γ^{-1} and γ are functions of their argument and η, l are constants. The new G-B-B type inequality for three nonlinear integrals was:
$u(t) \leq D + T\int_{t_0}^{t} r(s)\beta(u(s))ds + B\int_{t_0}^{t} h(s)\varpi(u(s))ds + L\int_{t_0}^{t} g(s)\gamma(u(s))ds$, where B and D are positive constants. The nonlinear third order ODE were found to possess H-U stability with H-U constants given by $K_{31} = \frac{L+L\psi}{\delta}F^{-1}(F(1) + d(\lambda)\frac{\lambda}{\delta}\phi(SX))SX$, where $S = \Omega^{-1}(\Omega(1) + n\frac{\lambda}{\delta}\gamma(X))X$ and $X = F^{-1}(F(1) + m\frac{\lambda^2}{\delta})$ and $K_{32} = \frac{L+|r(u(\kappa))|L}{2\phi|u'(\xi)|}\Gamma^{-1}(\Gamma(1) + C_4 qr(\Omega^{-1}(\Omega(1) + C_3 mg(H))\Omega^{-1}(\Omega(1) + C_2 n)))$
$\Omega^{-1}(\Omega(1) + C_3 mg)H$, where $H = F^{-1}(F(1) + C_2 n)$, $C_2 = \frac{2|u'(\eta)|\lambda}{2\phi|u'(\xi)|}$, $C_3 = \frac{\lambda^2}{2\phi|u'(\xi)|}$ and $C_4 = \frac{\lambda^{n+1}}{2\phi|u'(\xi)|}, \delta, \psi$ constants and g, r, ϕ are functions of their argument. The newly developed G-B-B type inequality with the three nonlinear integrals :
$u(t) \leq \rho(t) + T\int_{t_0}^{t} r(s)\beta(u(s))ds + B\int_{t_0}^{t} h(s)\varpi(u(s))ds + L\int_{t_0}^{t} g(s)\gamma(u(s))ds$, where $\rho(t)$ a monotonic, nonnegative, nondecreasing and continuous function. The nonlinear third order ODE were also found to possess H-U-R stability with H-U-R constants
$C_{\varphi_{31}} = \frac{1}{\delta}\Upsilon^{-1}(\Upsilon(1) + \frac{\eta^n}{\delta}\rho_1\gamma(\Omega^{-1}(\Omega(1) + \frac{1}{\delta}\rho_3\alpha(Y))y)\Omega^{-1}(\Omega(1) + \frac{1}{\delta}\rho_3\alpha(Y))Y$, where $Y = F^{-1}(F(1) + \frac{\eta}{\delta}\rho_1)$ and $C_{\varphi_{32}} = \frac{1}{\delta}\Omega^{-1}(\Omega(1) + \frac{d_1\lambda^n}{\eta}\omega(F^{-1}(F(1) + \frac{d_2 h(\lambda)}{\eta})))F^{-1}(F(1) + \frac{d_2 h(\lambda)}{\eta})$ with ρ_1, ρ_2 and ρ_3 are constants.

A generalisation of the existing results on Hyers-Ulam and Hyers-Ulam-Rassias stability to nonlinear ordinary differential equations was achieved. This can also be used to achieve the stability of the other differential equations.
Key Words: Hyers-Ulam-Rassias stability, Integral equations, Perturbed second order nonlinear differential equation, Perturbed third order nonlinear differential equation.

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Notations

Unless or otherwise stated first, second and third order derivatives of $u(t)$ with regard to the variable t is represented by $u'(t), u''(t)$ and $u'''(t)$ respectively. Let $\mathbf{R}$, $\mathbf{R}_+$ and $\mathbf{I}$ denote the intervals $(-\infty, \infty)$, $0 \le t < \infty$ and $(0 < t \le b)$. $\mathbf{R}$ denotes set of real numbers. For $u(t) \in \mathbf{R}$, $|u(t)|$ is an absolute value $u(t)$.

List of Special Symbols

The symbols list below are followed by a brief statement of their meaning;

$C(\mathbf{R}_+)$	space of continuously function,
$C^1(\mathbf{I}, \mathbf{R}_+)$	space of continuously one df,
$C^2(\mathbf{I}, \mathbf{R}_+)$	space of continuously twice df,
$C^3(\mathbf{I}, \mathbf{R}_+)$	space of continuously thrice df.

and

$\in$	belong to,	$<, \le, >, \ge$	inequality signs,
$\|u(t)\|$	absolute value of $u(t)$,	$\sum$	summation sign,
lim	limit,	$\forall$	for all,
$\to$	converges to,	$\ni$	such that,
$\int$	integral sign,	exp or e	exponential function,
$\equiv$	equivalent to,	$\exists$	there exists
$\mathbf{R}^n$	n-Euclidean space,	$\|X\|$	is an Euclidean norm of, X,
∞	Infinity,	$C(\mathbf{I}, \mathbf{R}_+)$	space of continuous function
G-B-B	Gronwall-Bellman-Bihari,	H-U	Hyers-Ulam
H-U-R	Hyers-Ulam-Rassias,	DE	differential equation
ODE	Ordinary Differential Equation	df	differential function
m.c	nondecreasing and continuous	m.n	monotonic nondecreasing
R.H.S	Right Hand Side	L.H.S	Left Hand Side

CHAPTER ONE

INTRODUCTION

1.0 Introduction

The introduction is discussed under the following sections, namely, background of the study, preliminaries on basic concepts, statement of the problem, aim and specific objectives of the study, justification for the study and the outline of the whole work.

1.1 Background of the Study

The importance of inequalities has long been recognized in the field of mathematics. The mathematical foundations of the theory of inequalities were established during the 18th and 19th century. This had played vital role in mathematical models of most dynamic processes in engineering, physical and biological sciences which often conveniently expressed in the form of linear or nonlinear ordinary Differential equations (DEs). Behaviour of such solutions of the systems are easily considered by the use of Gronwall-Bellman inequality or its extensions. Integral inequalities have application to questions of stability, uniqueness of solutions, dichotomy, asymptotic behaviour of solutions of differential equations.

Researchers that worked on integral inequalities and their applications include; Gronwall (1919), Bellman (1943), Bihari (1956), Azebelev and Tsalyuk (1962), Willet (1965), Willet and Wong (1967), Lakshimikantham and Leela (1969), Deo and Murdeshewar (1972), Dhongade and Deo (1973), Pachpatte (1973), Pachpatte (1974),Deo and Murdeshewar (1972), Beesack (1976), Agarwal and Thandapani (1981), Young (1982), Dannan (1985), Dannan (1986), Young (1985), Popenda (1986), Tsalyuk (1988), Pinto(1990), Oguntuase(2000), Abdelain(2011) and a host of others.

This thesis is an investigation of stability that involves second and third order nonlinear ordinary differential equations through Hyers-Ulam (H-U) and Hyers-

Ulam-Rassias (H-U-R) stabilities which are not common in the literature. This study rests on the extensions of Grownwall-Bellman-Bihari(G-B-B) type inequalities.

Hyers-Ulam stability started with the problem of functional equation which began with the question concerning stability of group homomorphism proposed by Ulam (1940) in Ulam(1960) during a talk before a Mathematics Club of the University of Wincosin. In 1941, Hyers (1941) gave a partial solution of Ulam's problem for the case of approximate additive mappings in the context of Banach spaces and called the result H-U stability. Some years later precisely in 1978, Rassias (1978) generalised the result of Hyers. The phenomenon of stability that was introduced by Rassias is called the Hyers-Ulam-Rassias stability (or Generalised H-U Stability). Thereafter, many authors have studied the stability problems of functional equations which include: Aoki(1950), Bourgin (1951), Hyers et al.(1998), Jun and Lee(1999), Park,(2002), Forti(2007), Jung(2011) and a host of others.

Researchers such as: Obloza(1993), Obloza(1997), Alsina and Ger(1998) were first set of group who considered H-U stability of linear ordinary differential equations. Besides, several researchers have looked into the H-U stability of the various linear differential equation of first order Takahasi et al.(2002), Miura et al.(2007), Jung(2004), Jung(2005), Jung(2006), Wang, Zhou and Sun(2008), Popa and Rasa (2011), Onitsuka and Shoji(2017) and a host of others. While the following researchers Li and Shen (2009), Jung(2010), Li(2010), Li and Shen(2010), Javadian et al.(2011), Gavruta, Jung and Li(2011), Xue (2014) , Ghaemi et al.(2012) and host of others, investigated Hyers-Ulam and Hyers-Ulam-Rassias stability of differential equations which are linear.

Recently, H-U and H-U-R of third and fourth order linear differential equations were investigated by the following researchers Abdollahpour and Najati (2012), Abdollahpour et al.(2012), Tuns and Bicer(2013), Tripathy and Satapathy (2014), Murali and Ponmanaselvan (2018a), Murali and Ponmanaselvan (2018). A few of the researchers such as Rus(2009), Rus(2010), Gachpazan and Baghani(2010), Qarawani(2012a), Huang et al.(2015), Ravi et al.(2016), Bicer and Tunc(2018) and a host of others, have been able to investigate stability of second order nonlinear ordinary differential equations in the sense of H-U stability using different approaches. Only Algifiary and Jung(2014) discussed Hyers-Ulam of nonlinear differential equations using Gronwall lemma. The aforementioned researchers were unable to discuss the Hyers-Ulam and Hyers-Ulam-Rassias stability of perturbed nonlinear second and third order ordinary differential equations by the use of G-B-B type inequalities which is the main problem tackled in this thesis.

1.2 Definition of Terms

This section is divided into two, the first part deals with definitions of some basic concepts. Second part deals with some basic theorems and lemmas.

1.2.1 Definition of Some Basic Concepts

This subsection contains definition of some basic concepts that are required for developing the extensions of G-B-B type inequalities, H-U and H-U-R stability of second and third order nonlinear differential equation.

Definition 1.1:

A function $\varpi : \mathbf{R}_+ \to \mathbf{R}_+$ is said to belong to class Ψ if

(i) $\varpi(u) > 0$ is nondecreasing and continuous for $u \geq 0$

(ii) $\frac{1}{n}\varpi(u) \leq \varpi\left(\frac{u}{n}\right)$, $n > 0$ is a monotonic, nondecreasing and continuous function on $\mathbf{R}_+$.

(iii)there exist a function ϕ, continuous on $[0, \infty)$ with $\varpi(\alpha u) \leq \phi(\alpha)\varpi(u)$ for $\alpha \geq 0$

Definition 1.2:

A function $f(t)$ defined on a domain $\mathbf{I} = (0 < t \leq b)$.

(i) Is said to be submultiplicative if

$$f(t_1 \times t_2) \leq f(t_1) \times f(t_2) \; \forall t_1, \; t_2 \in \mathbf{I}. \tag{1.1}$$

(ii) Is said to be subadditive if

$$f(t_1 + t_2) \leq f(t_1) + f(t_2) \; \forall t_1, \; t_2 \in \mathbf{I}. \tag{1.2}$$

1.2.2 Useful Basic Theorem and Lemma

Basic theorem and lemma which are needed in this work are stated as follows;

Theorem 1.1 (Generalised First Mean Value Theorem)

Murray(1974), Stephenson(1973):

If $f(t)$ and $g(t)$ are continuous in $[t_0, t] \subseteq \mathbf{I}$ and $f(t)$ does not change sign in the interval, then there is a point $\xi \in [t_0, t]$ such that

$$\int_{t_0}^{t} g(s)f(s)ds = g(\xi)\int_{t_0}^{t} f(s)ds$$

Lemma 1.1 Ince(1926):

Let $f(t)$ be an integrable function then the n-successive integration of f over the interval $[t_0, t]$ is given by

$$\int_{t_0}^{t}\cdots\int_{t_0}^{t} f(s)ds^n = \frac{1}{(n-1)!}\int_{t_0}^{t}(t-s)^{n-1}f(s)ds. \tag{1.3}$$

1.3 Statement of the Problem

It is amazing that many of the researchers who investigated H-U and H-U-R stability gave maximum attention to first, second, third, fourth and nth order of linear ordinary differential equations while few of them gave little attention to nonlinear differential equation using the methods that are only limited to ordinary differential equation such as

$$u''(t) = f(t, u(t)).$$

This means that H-U and H-U-R stability of perturbed nonlinear ordinary differential equations of the form

$$u''(t) + f(t, u(t), u'(t)) = P(t, u(t), u'(t))$$

and

$$u'''(t) + f(t, u(t), u'(t), u''(t)) = P(t, u(t), u'(t))$$

, and their variants through G-B-B are not paid attention to, which are the focused of this thesis.

1.4 Aims and Objectives of the Study

The aim of this research work is to examine the Hyers-Ulam and Hyers-Ulam-Rassias stability of perturbed nonlinear second and third order ordinary differential equations. The specific objectives of this research work include the following :

(a) to develop the different extensions of Gronwall-Bellman-Bihari type inequality consisting of one, two and three nonlinear integral terms.

(b) to illustrate the application of integral inequalities for investigating Hyers-Ulam and Hyers-Ulam-Rassias stability.

(c) to convert all the nonlinear second and third order differential equations consider to integral inequalities

(d) to investigate the stability in term of Hyers-Ulam and Hyers-Ulam-Rassias of second and third order nonlinear differential equations with forcing terms.

(e) to obtain the positive Hyers-Ulam and Hyers-Ulam-Rassias constants of every nonlinear second and third order differential equations considered.

1.5 Justification of the Study

Stability played crucial role in the field of mathematics, especially in nonlinear differential equations. Hence, different methods have been be employed in physical and biological sciences, engineering and any other relevant areas to consider stability of the behaviour of solutions of linear and nonlinear ordinary differential equations. However, the study of stability of certain classes of nonlinear second and third order ordinary differential equations in the sense of Hyers-Ulam and Hyers-Ulam-Rassias stabilities through Gronwall-Bellman-Bihari type inequality has not been extensively studied.

For this cause, this research work is developed to study the stability of certain classes of nonlinear second and third order ordinary differential equations in the sense of Hyers-Ulam and Hyers-Ulam-Rassias stabilities by converting them to integral inequalities and later use the Gronwall-Bellman-Bihari type inequality, to the best of our knowledge this has not been reported in the literature.

1.6 Motivation for the study

This study is motivated by the work of Algfiary and Jung (2014), who considered Hyers-Ulam stability of linear and nonlinear second order differential equations through Gronwall's lemma which is not sufficient to determine the Hyers-Ulam and Hyers-Ulam-Rassias stabilities of the extension of nonlinear second order differential equations considered by Algfiary and Jung (2014) and Qarawani (2012a). Also, through the work of Algfiary and jung(2014), this research work has extended the scope of the study of Hyers-Ulam and Hyers-Ulam-Rassias stabilities to damped, lienard, forcing terms and Euler type of third order nonlinear differential equations. Therefore, the major tool employs in this study to establish Hyers-Ulam and Hyers-Ulam-Rassias stabilities of certain classes of nonlinear second and third order ordinary differential equations is Gronwall-Bellman-Bihari type inequality.

1.7 Outline

Here, we give a brief outline of how this research work is arranged. *Chapter one*, contains background of the study, preliminaries and basic concepts, statement of the problem, aims and specific objectives of the study, justification for the study and outline. *Chapter two*, contains a review of literature consulted in this research work. *Chapter three*, contains the methodology adopted in obtaining the results of this work. That is the development of the integral inequalities with one, two and three integral terms. *Chapter four*, contains Hyers-Ulam and Hyers-Ulam-Rassias stability of perturbed nonlinear second and third order differential equations. *Chapter five*, contains Hyers-Ulam and Hyers-Ulam-Rassias stabilities of perturbed nonlinear third order differential equations. *Chapter six*, contains summary, conclusion and recommendation, and further research.

CHAPTER TWO

LITERATURE REVIEW

2.0 Introduction

The mathematical foundation of the theory of inequalities were established by the following Mathematicians who saw the need. Gronwall (1919) established a result termed as Gronwall inequality or Gronwall Lemma. Reid (1930) developed an integral inequality similar to Gronwall inequality. Quade (1942) gave an integral inequality which was an extension of Gronwall inequality. Bellman (1943) improved on the result of Gronwall while Bihari (1956)extended the Bellman's inequality to nonlinear form. Thus, several researchers extended the result of Bihari to G-B-B inequalities.

Ulam (1940) in Ulam(1960) started his study on Stability at the University of Wincosin, which was extended by Hyers(1941) and the result was called Hyers-Ulam stability. Further extension of their result was achieved by Rassias (1978) and the result was called H-U-R stability. Thereafter, various extensions of their results were discovered by many researchers.

2.1 Gronwall-Bellman-Bihari Type Inequalities

In this section, the work of Gronwall, Bellman, Bihari and other researchers on integral inequalities are reviewed.

Lemma 2.1 Gronwall (1919)

Let $f(t)$ and $u(t)$ be nonnegative continuous functions on $\mathbf{I} = [t_0, \infty)$ for which the inequality

$$u(t) \leq c + \int_{t_0}^{t} f(s)u(s)ds, \quad t_0 \leq t \leq \infty \tag{2.1}$$

holds, where c is a nonnegative constant. Then

$$u(t) \leq c \exp\left(\int_{t_0}^{t} f(s)ds\right), \; a \leq t \leq \infty. \tag{2.2}$$

The following integral inequalities were developed from Gronwall's lemma:

Theorem 2.1 Reid (1930)

Let $u(t)$ and $f(t)$ be nonnegative continuous functions on $(\mathbf{I}, \mathbf{R}_+)$, and suppose

$$u(t) \leq c + \int_{t_0}^{t} f(s)u(s)|ds| \quad t \in \mathbf{I} \tag{2.3}$$

where $t \in \mathbf{I}$ and c is a constant. Then

$$u(t) \leq c\left(\int_{t_0}^{t} f(s)|ds|\right) \quad t \in \mathbf{I}. \tag{2.4}$$

Corollary 2.1 Quade (1942)

Let $f(t)$ be a continuous function for $t \geq \alpha$, and suppose

$$u(t) \leq ae^{-\gamma(t-s)}[bu(s) + c]ds \quad t \geq t_0 \tag{2.5}$$

where $t \geq t_0, a, c$ and $\gamma \neq t$ are constants. Then

$$u(t) \leq ae^{(b-\gamma)(t-t_0)} + \frac{c}{\gamma - b}[1 - e^{(b-\gamma)(t-t_0)}] \; t \geq t_0. \tag{2.6}$$

Lemma 2.2 Bellman (1943)

Let $u(t)$ and $f(t)$ be positive continuous functions on $(\mathbf{I}, \mathbf{R}_+)$. Let $N > 0$ and $M \geq 0$ be constants then the inequality

$$u(t) \leq N + M\int_{\alpha}^{t} f(s)u(s)ds, \quad t \in \mathbf{I} \tag{2.7}$$

implies that

$$u(t) \leq N\exp\left(M\int_{\alpha}^{t} f(s)ds\right), \quad t \in \mathbf{I}. \tag{2.8}$$

Lemma 2.3 Bellman(1953)

Let $u(t)$ and $f(t)$ be positive continuous functions on $(\mathbf{I}, \mathbf{R}_+)$, suppose

$$u(t) \leq u(t_0) + \int_{t_0}^{t} f(s)u(s)ds, \quad t, t_0 \in \mathbf{I}. \tag{2.9}$$

Then for $t_0 \leq t \leq b$

$$u(t_0)\exp\left(-\int_{a}^{t} f(s)ds\right) \leq u(t) \leq u(t_0)\exp\left(\int_{a}^{t} f(s)ds\right). \tag{2.10}$$

Lemma 2.4 Bihari(1956)

Suppose $u(t)$, $f(t)$ be positive continuous functions defined on $t_0 \leq t \leq b(\leq \infty)$ and $N > 0$, $M \geq 0$, further let $\varpi(u)$ be a nonnegative, nondecreasing continuous function for $u \geq 0$, if the inequality

$$u(t) \leq N + M\int_{t_0}^{t} f(s)\varpi(u(s))ds, \quad t_0 \leq t < b \tag{2.11}$$

holds, then

$$u(t) \leq \Omega^{-1}\left(\Omega(N) + M\int_{t_0}^{t} f(s)ds\right), \quad t_0 \leq t \leq b' \leq b, \tag{2.12}$$

where

$$\Omega(u) = \int_{u_0}^{u} \frac{dt}{\varpi(t)}, \quad 0 < u_0 < u \tag{2.13}$$

In the case $\varpi(0) > 0$ or $\Omega(0+)$ is finite, one may take $u_0 = 0$, and Ω^{-1} is the inverse function of Ω and t must be in the subinterval $[t_0, b']$ of $[t_0, b]$ such that

$$\Omega(N) + M\int_{t_0}^{t} f(s)ds \in Dom(\Omega^{-1})$$

Theorem 2.2 Bellman and Cook (1963)

Let $n(t)$ be positive, monotonic, nondecreasing function and $u(t) > 0, \quad f(t) > 0$. If all these functions are continuous and if

$$u(t) \leq n(t) + \int_{t_0}^{t} f(s)u(s)ds, \quad t_0 \leq t \leq b, \tag{2.14}$$

then

$$u(t) \leq n(t) \exp\left(\int_{t_0}^{t} f(s)ds\right). \quad t_0 \leq t \leq .b \tag{2.15}$$

Theorem 2.3 Willett (1965)

Suppose

$$u(t) \leq \omega_*(t) + \omega(t) \int_{t_0}^{t} v(s)u(s)ds \quad t \in \mathbf{I}, \tag{2.16}$$

where $v\omega, v\omega_*,$ and vu are locally integrable on $\mathbf{I}$. Then

$$u(t) \leq \omega_*(t) + \omega(t) \left(\exp \int_{t_0}^{t} v\omega\right) \left(\int_{t_0}^{t} v\omega_*\right) \quad t \in \mathbf{I} \tag{2.17}$$

Theorem 2.4 Willett (1965)

Suppose that

$$u(t) \leq \omega_0(t) + \sum_{i=1}^{n} \omega_i(t) \int_{t_0}^{t} v_i(s)u(s)ds \quad t \in \mathbf{I}, \tag{2.18}$$

where $v_i\omega_j(i = 1, 2, ..., n, \ j = 0, 1......, n)$ and $v_iu(i = 1, 2, 3, ..., n)$ are locally integrable on $\mathbf{I}$. Then

$$u(t) \leq E_n\omega_0 \tag{2.19}$$

where $E_i(i = 0, 1, 2......n)$ is defined inductively as the composition of $i+1$ functional operators,that is $E_i = D_iD_{i-1}...D_n$

$$D_0\omega = \omega$$

$$D_i\omega = \omega + (E_{j-1}\omega_i) \left(\exp \int_{t_0}^{t} v_jE_{j-1}\omega_i ds\right) \int_{t_0}^{t} v_j\omega ds \ j = 1, 2, ..n). \tag{2.20}$$

Chu and Metcalf presented the following integral inequality which improved on the results of Jones(1964), Willett(1965) and the similar results in Condington and Levinson (1955)

Theorem 2.5 Chu and Metcalf (1967)

Let the functions u and f be continuous on the interval $[0, 1]$: let the function K be continuous and nonnegative on the region $0 \leq s \leq t \leq 1$. If

$$u(t) \leq f(t) + \int_{0}^{t} K(t, s)u(s)ds, \quad t \in [0, 1], \tag{2.21}$$

then

$$u(t) \leq f(t) + \int_{0}^{t} H(t, s)f(s)ds, \quad t \in [0, 1], \tag{2.22}$$

where $H(t, s) = \sum_{i=1}^{\infty} K_i(t, s), \ 0 \leq s \leq t \leq 1$, is the resolvent kernel and the $K_i(i = 1, 2, ...)$ are the iterated kernels of K.

Gollwitzer (1969) considered the following two functional inequalities as improvement on the results of previous researchers.

$$u(t) \leq f(t) + g(t)G^{-1}\left(\int_{t_0}^{t} G(u(s))h(s)ds\right) \quad t_0 \leq x \leq t \leq b, \tag{2.23}$$

$$u(t) \geq f(t) - g(t)G^{-1}\left(\int_{t_0}^{t} G(u(s))h(s)ds\right) \quad t_0 \leq x \leq t \leq b, \tag{2.24}$$

where the u, f, g and h are nonnegative and continuous on the interval $[t_0, b]$. The function $G(u)$ is continuous and strictly increasing for $u \geq 0$, $G(0) = 0$, $\lim_{u\to\infty} G(u) = \infty$ and G^{-1} denotes the inverse function of G. Deo and Murdeshwar (1970) developed the following integral inequalities with their estimate bounds:

$$u_i(t) \leq k_i + M_i \int_0^t F_i(s)\omega(u(s))ds, \quad t \in [0, \alpha] \tag{2.25}$$

$$u_i(t) \geq k_i - M_i \int_0^t F_i(s)\omega(u(s))ds, \quad t \in [0, \alpha^-] \tag{2.26}$$

$$f(u_i(t)) \leq k_i + M_i \int_{t_0}^t F_i(s)\omega(u(s))ds, \quad t \in [0, \alpha] \tag{2.27}$$

where u_i, F_i are nonnegative continuous functions on

$\mathbf{R}_+$ and $f(0) = u_{i0}$, $f(\infty) = \infty$

Deo and Murdeshwar obtained the upper and lower bounds for unknown function $u(t)$ in the following integral inequalities:

Theorem 2.6 Deo and Murdeshwar (1971)

(i) u, η and F are positive continuous functions on $\mathbf{I}$,

(ii) ϖ a positive, continuous, subadditive and nondecreasing function on $\mathbf{I}$

(iii)$\psi : \mathbf{I} \to \mathbf{I}$ a non-decreasing continuous function on and

(iv) $u(t) \leq \eta(t) + \psi\left(\int_0^t F(s)\varpi(u(s))ds\right) \quad t \in (t_0, \infty)$,

then, for $t \in \mathbf{I}$

$$u(t) \leq \eta(t) + \psi\left(G^{-1}\left[G\left(\int_0^t F(s)\varpi(\eta(s))ds\right) + \int_0^t F(s)ds\right]\right) \tag{2.28}$$

where

$$G(u) = \int_{u_0}^{u} \frac{ds}{\varpi(\psi(s))} \quad 0 < u_0 \leq u, \tag{2.29}$$

G^{-1} is the inverse function of G and t is in the subinterval $(0, b]$ of $\mathbf{I}$ so that $G\left(\int_{t_0}^t F(s)\varpi(\eta(s))ds\right) + \int_{t_0}^t F(s)ds \in Dom(G^{-1})$

Theorem 2.7 Deo and Murdeshwar (1971)

If in addition to the assumptions(i),(ii) and (iii) of Theorem 2.6 let ϖ be an even function on (∞, ∞), let

(iv') $u(t) \geq \eta(t) - \psi\left(\int_0^t F(s)\varpi(u(s))ds\right)$, $t \in (0, \infty)$, then, for $0 < t\infty$

$$u(t) \geq \eta(t) - \psi\left(G^{-1}\left[G\left(\int_0^t F(s)\varpi(\eta(s))ds\right) + \int_0^t F(s)ds\right]\right) \tag{2.30}$$

where the function G is defined as equation(2.29)

Theorem 2.8 Dhongade and Deo(1973)

Let (i) $u(t), r(t), g(t) : (0,\infty) \to (0,\infty)$ are continuous on $(0,\infty)$

(ii) $\varpi(u)$ is a nonnegative, monotonic nondecreasing, continuous, submultiplicative function for $u > 0$

if

$$u(t) \le k + \int_0^t r(s)u(s)ds + \int_0^t g(s)\varpi(u(s))ds, \quad 0 < t < \infty \tag{2.31}$$

for $k > 0$, is a constant, then

$$u(t)\exp\left(-\int_0^t r(s)ds\right)$$
$$\le \Omega^{-1}\left(\Omega(k) + \int_0^t g(s)\varpi\left(\exp\int_0^s r(\delta)d\delta\right)ds\right) \quad 0 < t \le b \tag{2.32}$$

where $\Omega(u)$ is defined in(2.13) and Ω^{-1} is the inverse of Ω and t is in the subinterval $(0,b)$ of $(0,\infty)$ so that

$$\Omega(k) + \int_0^t g(s)\varpi\left(\exp\int_0^s r(\delta)d\delta\right)ds \in Dom(\Omega^{-1}) \tag{2.33}$$

Theorem 2.9 Dhongade and Deo (1973) Let

i $u(t), r(t) : (0,\infty) \to (0,\infty)$ and continuous on $(0,\infty)$,

ii $\varpi \in \Psi$

iii $n > 0$ be monotonic, nondecreasing and continuous on $(0,\infty)$

if

$$u(t) \le n(t) + \int_0^t f(s)\varpi(u(s))ds, \quad 0 < t < \infty, \tag{2.34}$$

then

$$u(t) \le n(t)\Omega^{-1}\left(\Omega(1) + \int_0^t f(s)ds\right) \quad 0 < t \le b, \tag{2.35}$$

where $(0,b) \subset (0,\infty)$, where $\Omega(u)$ is defined in (2.13) and Ω^{-1} is the inverse of Ω and t is in the subinterval $(0,b)$ is so chosen that

$$\Omega(1) + \int_0^t f(s)ds \in Dom(\Omega^{-1})$$

If the constant $k > 0$ is replaced by a continuous function $y(t)$ in equation (2.31), the function $y(t)$ in equation (2.31) is continuous and additive.

Theorem 2.10 Dhongade and Deo (1973)

Let in addition to assumptions (i),(ii) of Theorem 2.7, the function ϖ be subadditive, the function $y(t) > 0$, $\psi(t) \ge 0$ be nondecreasing in t and continuous on $\mathbf{R}^+$ for $t > 0$

$$u(t) \le y(t) + \int_0^t f(s)u(s)ds + \psi\left(\int_0^t g(s)\varpi(u(s))ds\right) \quad 0 < t < \infty \tag{2.36}$$

then

$$u(t)\exp\left(-\int_0^t f(s)ds\right)$$

$$\leq y(t)+\psi\left[G^{-1}\left(G\left(\int_0^t g(s)\varpi\left(y(s)\exp\int_0^s f(\delta)d\delta\right)ds\right)\right.\right. \tag{2.37}$$

$$\left.\left.+\int_0^t g(s)\varpi\left(\exp\int_0^s f(\delta)d\delta\right)ds\right)\right],\ 0<t\leq b$$

where $G(u)$ is defined as equation (2.29) and G^{-1} is the inverse of G and t is the subinterval $(0,\infty)$ so that

$$G\left(\int_0^t g(s)\varpi\left(y(s)\exp\int_0^s f(\delta)d\delta\right)ds\right)$$

$$+\int_0^t g(s)\varpi\left(\exp\int_0^s f(\delta)d\delta\right)ds\in Dom(G^{-1})$$

The next theorem provides a nonlinear generalisation of the Bihari lemma.

Theorem 2.11 Dhongade and Deo (1973)

Let condition (i),(ii) of Theorem 2.8 hold, and $\omega\in\Psi$,

if

$$u(t)\leq k+\int_0^t f(s)\beta(u(s))ds+\int_0^t g(s)\varpi(u(s))ds,\ \ 0<t\leq\infty \tag{2.38}$$

for $k>0$, a constant, then

$$u(t)\left(\Omega^{-1}\left(\Omega(1)+\int_0^t f(s)ds\right)^{-1}\right)$$

$$\leq F^{-1}\left[F(k)+\int_0^t g(s)\varpi\left[\Omega^{-1}\left(\Omega(1)+\int_0^s f(\delta)d\delta\right)\right]ds\right]\ t_0<t\leq b \tag{2.39}$$

where $\Omega(u)$ is defined in equation (2.13), F is defined as

$$F(u)=\int_{u_0}^u \frac{ds}{\beta(s)},\ \ 0<u_0\leq u, \tag{2.40}$$

and Ω^{-1}, F^{-1} are inverses of Ω, F respectively and t is the subinterval $(0,b)$ of $(0,\infty)$ such that

$$\Omega(1)+\int_0^t f(s)ds\ Dom(G^{-1}),$$

and

$$F(k)+\int_0^t g(s)\varpi\left[\Omega^{-}\left(\Omega(1)+\int_0^s f(\delta)d\delta\right)\right]ds\in Dom(F^{-1})$$

. If the constant $k>0$ in equation (2.38) is replaced by a monotonically nondecreasing function $y(t)$, we require ϖ to be subadditive.

Theorem 2.12 Dehongade and Deo (1973)

If, in addition to the assumption of Theorem 2.11, ϖ is subadditive, and the function $y(t)>0$, $\psi(t)\geq 0$ be nondecreasing, continuous on $(0,\infty)$, and if

$$u(t)\leq y(t)+\int_0^t f(s)\beta(u(s))ds+\psi\left(\int_0^t g(s)\varpi(u(s))ds\right),\ \ 0<t<\infty \tag{2.41}$$

then

$$u(t)\left(\Omega^{-1}\left(\Omega(1)+\int_0^t f(s)ds\right)^{-1}\right)$$

$$\leq y(t)+\psi\left[G^{-1}\left(G\left(\int_{t_0}^t g(s)\varpi\left(y(s).\Omega^{-1}\left(\Omega(1)+\int_0^s f(\delta)d\delta\right)\right)ds\right)\right.\right. \tag{2.42}$$

$$\left.\left.+\int_0^t g(s)\varpi\left(\Omega^{-1}\left(\Omega(1)+\int_0^s f(\delta)d\delta\right)\right)ds\right)\right] \quad 0<t\leq b$$

where Ω is defined as equation (2.13) and G is defined in equation (2.29) where Ω^{-1}, G^{-1} have the same meaning as in Theorem 2.9 and t is in subinterval $(0, b]$ so that

$\Omega(1)+\int_0^t f(s)ds \in Dom(\Omega^{-1})$

and

$$G\left(\int_0^t g(s)\varpi\left(y(s).\Omega^{-1}\left(\Omega(1)+\int_0^s f(\delta)d\delta\right)\right)ds\right)$$

$$+\int_0^t g(s)\varpi\left(\Omega^{-1}\left(\Omega(1)+\int_0^s f(\delta)d\delta\right)\right)ds \in Dom(G^{-1})$$

In 1973, Pachpatte developed the following integral inequalities as generalisation of Bellman(1953).

Theorem 2.13 Pachpatte (1973)

Let $u(t)$, $f(t)$ and $g(t)$ be real-valued nonnegative continuous functions defined on $\mathbf{I}$, for which the inequality

$$u(t)\leq u_0+\int_0^t f(s)u(s)ds+\int_0^t f(s)\left(\int_0^s g(\tau)u(\tau)d\tau\right)ds, \quad t\in\mathbf{I} \tag{2.43}$$

holds, where u_0 is a nonnegative constant. Then

$$u(t)\leq u_0\left(1+\int_0^t f(s)\exp\left(\int_0^s (f(\tau)+g(\tau))d\tau\right)ds\right) \quad t\in\mathbf{I} \tag{2.44}$$

Theorem 2.14 Pachpatte(1973)

Let $u(t)$, $f(t)$ and $g(t)$ be real-valued nonnegative continuous functions defined on $\mathbf{I}$, for which the inequality

$$u(t)\leq u_0+\int_0^t f(s)u(s)ds+\int_0^t f(s)\left(\int_0^s g(\tau)u^p(\tau)d\tau\right)ds, \quad t\in\mathbf{I} \tag{2.45}$$

holds, where u_0 is a nonnegative constant and $0\leq p<1$. Then

$$u(t)\leq u_0+\int_0^t f(s)\exp\left(\int_0^s f(\tau)d\tau\right)$$

$$.\left[u_0^{1-p}+(1-p)\int_0^s g(\tau)\exp\left(-(1-p)\int_0^\tau f(\eta)d\eta\right)d\tau\right]^{\frac{1}{1-p}} ds \;\forall t\in\mathbf{I}. \tag{2.46}$$

Let u_0 in Theorem 2.13 be replaced with a positive, monotonic, nondecreasing continuous function $n(t)$ defined on $\mathbf{I}$.

Theorem 2.15 Pachpatte (1973)

Let $u(t)$, $f(t)$ and $g(t)$ be real-valued nonnegative continuous functions defined on $\mathbf{I}$, and $n(t)$ be a positive, monotonic, nondecreasing continuous function defined on **I**, for which the inequality

$$u(t) \le n(t) + \int_0^t f(s)u(s)ds + \int_0^t f(s)\left(\int_0^s g(\tau)u(\tau)d\tau\right)ds, \quad t \in \mathbf{I} \tag{2.47}$$

holds. Then

$$u(t) \le n(t)\left(1 + \int_0^t f(s)\exp\left(\int_0^s (f(\tau)+g(\tau))\,d\tau\right)ds\right) \quad t \in \mathbf{I} \tag{2.48}$$

Pachpatte applied Theorem 2.15 to establish the following integral inequalities which consist of nonlinear term(s)

Theorem 2.16 Pachpatte (1975a)

Let $u(t)$, $f(t)$, $h(t)$ and $g(t)$ be real-valued nonnegative continuous functions defined on **I**, $\varpi(u)$ be a positive, continuous, monotonic, nondecreasing and submultiplicative function for $u > 0$ $\varpi(0) = 0$ and suppose further that the inequality

$$\begin{aligned} u(t) &\le u_0 + \int_0^t f(s)u(s)ds \\ &+ \int_0^t f(s)\left(\int_0^s g(\tau)u(\tau)d\tau\right)ds + \int_0^t h(s)\varpi(u(s))ds, \quad t \in \mathbf{I} \end{aligned} \tag{2.49}$$

is satisfied for all $t \in \mathbf{I}$, where u_0 is a positive constant. Then

$$\begin{aligned} u(t) &\le \Omega^{-1}\Big[\Omega(u_0) \\ &+ \int_0^t h(s)\varpi\left(1 + \int_0^s f(\tau)u\exp\left(\int_0^\tau (f(k)+g(k))dk\right)d\tau\right)ds\Big] \\ &\cdot\left[1 + \int_0^t f(s)\exp\left(\int_0^s (f(\tau)+g(\tau))\,d\tau\right)ds\right], \quad 0 \le t \le b \end{aligned} \tag{2.50}$$

where Ω is defined as equation (2.13) and Ω^{-1} is the inverse function of Ω, and t is in the subinterval [0, b] of **I** so that

$$\Omega(u_0) + \int_0^t h(s)\varpi\left(1 + \int_0^s f(\tau)u\exp\left(\int_0^\tau (f(k)+g(k))dk\right)d\tau\right)ds \in dom(\Omega^{-1})$$

Pachpatte gave more general form of Theorem 2.15

Theorem 2.17 Pachpatte (1975b)

Let $u(t)$, $f(t)$, $h(t)$ and $g(t)$ be real-valued nonnegative continuous functions defined on **I**, $\varpi(u)$ be a positive, continuous, monotonic, nondecreasing, subadditive and submultiplicative function for $u > 0$ $\varpi(0) = 0$; the function $y(t) > 0$, $\psi(t) \ge 0$ be nondecreasing in t and continuous on **I**, $\psi(0) = 0$ and suppose further that the inequality

$$\begin{aligned} u(t) &\le y(t) + \int_0^t f(s)u(s)ds \\ &+ \int_0^t f(s)\left(\int_0^s g(\tau)u(\tau)d\tau\right)ds + \psi\left(\int_0^t h(s)\varpi(u(s))ds\right), \quad t \in \mathbf{I} \end{aligned} \tag{2.51}$$

is satisfied for all all $t \in \mathbf{I}$. Then

$$u(t) \leq \Bigg[p(t) + \psi \Bigg(G-1 \Bigg[G \Bigg(\int_0^s h(s)$$

$$\times\psi \left(p(s) \left(1 + \int_0^s f(\tau) \exp \left(\int_0^\tau (f(k) + g(k))\, dk\right) d\tau\right)\right) ds\Bigg)$$

$$+ \int_0 h(s)\psi \left(1 + \int_0^s f(\tau) \exp \left(\int_0^\tau (f(k) + g(k))dk\right) d\tau\right) ds\Bigg]\Bigg)\Bigg] \tag{2.52}$$

$$\times \left[1 + \int_0^t f(s) \exp \left(\int_0^s (f(\tau) + g(\tau))\, d\tau\right) ds\right], \qquad 0 \leq t \leq b.$$

where $G(u)$ is defined as equation (2.29) and G^{-1} is the inverse function of G, and t is in the subinterval [0, b] of $\mathbf{I}$ so that

$$G \left(\int_0^s h(s)\psi \left(p(s) \left(1 + \int_0^s f(\tau) \exp \left(\int_{t_0}^\tau (f(k) + g(k))\, dk\right) d\tau\right)\right) ds\right)$$

$$+ \int_0 h(s)\psi \left(1 + \int_0^s f(\tau) \exp \left(\int_0^\tau (f(k) + g(k))dk\right) d\tau\right) ds \in dom(G^{-1})$$

Theorem 2.18 Pachpatte (1975b)

Let $u(t)$ $g(t)$ be real valued positive continuous function defined on $\mathbf{I}$, $n(t)$ be a positive, monotonic, nondecreasing continuous function defined on $\mathbf{I}$, and $\varpi \in \Psi$, for which the inequality

$$u(t) \leq n(t) + \int_0^t g(s) \left(u(s) + \int_0^s g(\tau)\varpi(u(\tau))d\tau\right) ds, \quad t \in \mathbf{I} \tag{2.53}$$

holds.Then

$$u(t) \leq n(t) \left(1 + \int_0^t g(s)H^{-1} \left(H(1) + \int_0^s g(\tau)d\tau\right) ds\right) \quad t \in [0, b] \tag{2.54}$$

where

$$H(u) = \int_{u_0}^u \frac{ds}{s + \varpi(s)}, \qquad 0 < u_0 \leq u \tag{2.55}$$

and H^{-1} is the inverse of H, and t is in the subinterval $[0, b]$ of $\mathbf{I}$ so that

$$H(1) + \int_0^t g(s)ds \in Dom(H^{-1})$$

Pachpatte applied Theorem 2.18 to establish the following integral inequalities:

Theorem 2.19 Pachpatte (1975b)

Let $u(t)$, $g(t)$ and $h(t)$ be real valued positive continuous function defined on $\mathbf{I}$: $\varpi_1 \in \Psi$ and ϖ is the same function in Theorem 2.16, and suppose further that the inequality.

$$u(t) \leq u_0 + \int_0^t g(s) \left(u(s) + \int_0^s g(\tau)\varpi(u(\tau))d\tau\right) ds + \int_0^t h(s)\varpi(u(s))ds, \quad t \in \mathbf{I} \tag{2.56}$$

holds for all $t \in \mathbf{I}$, where u_0 a positive constant. Then

$$u(t) \leq \Omega^{-1}\left[\Omega(u_0) + \int_0^s h(s)\varpi\left(1+\int_0^s g(\tau)H^{-1}\left(H(1)+\int_0^\tau g(k)dk\right)d\tau\right)ds\right] \cdot \left[1+\int_0^t g(s)H^{-1}\left(H(1)+\int_0^s g(\tau)d\tau\right)ds\right] \quad 0 \leq t \leq b \tag{2.57}$$

Where H and Ω are defined as equations (2.55) and (2.13) respectively. Further more H^{-1} and Ω^{-1} are the inverses of H and Ω respectively and t is the subinterval such of $\mathbf{I}$ that

$$H(1)+\int_0^t g(s)ds \in Dom(h^{-1})$$

and

$$\Omega(u_0)+\int_0^s h(s)\varpi\left(1+\int_0^s g(\tau)H^{-1}\left(H(1)+\int_0^\tau g(k)dk\right)d\tau\right)ds \in Dom(\Omega^{-1})$$

Theorem 2.20 Pachpatte(1975b)

Let $u(t)$, $g(t)$ and $h(t)$ be real valued positive continuous function defined on $\mathbf{I}$: $\varpi_1 \in \Psi$ and ϖ is the same function in Theorem 2.19,the function $y(t) > 0$, $\psi(t) \geq 0$ be nondecreasing in t and continuous on $\mathbf{I}$, $\psi(0) = 0$ and suppose that the inequality

$$u(t) \leq y(t) + \int_0^t g(s)\left(u(s)+\int_0^s g(\tau)\varpi(u(\tau))d\tau\right)ds + \psi\left(\int_0^t h(s)\varpi(u(s))ds\right), \quad t \in \mathbf{I} \tag{2.58}$$

holds for all $t \in \mathbf{I}$. Then

$$u(t) \leq \Big[y(t) + \psi\Big(G^{-1}\Big[G\Big(\int_0^t h(s)\psi\Big(y(s)\Big(1+\int_0^s g(\tau)\times H^{-1}\Big(H(1)+\int_0^\tau g(k)dk\Big)d\tau\Big)\Big)\Big) + \int_0^s h(s)\varpi\Big(1+\int_0^s g(\tau)H^{-1}\Big(H(1)+\int_0^\tau g(k)dk\Big)d\tau\Big)ds\Big]\Big)\Big] \times \left[1+\int_0^t g(s)H^{-1}\left(H(1)+\int_0^s g(\tau)d\tau\right)ds\right] \quad 0 \leq t \leq b \tag{2.59}$$

Where H and Ω are as equations (2.55) and (2.13) respectively. Further more H^{-1} and Ω^{-1} are the inverses of H and Ω respectively. For t is in the subinterval $\mathbf{I}$ such that

$$H(1)+\int_0^t g(s)ds \in Dom(h^{-1})$$

and

$$\Omega(u_0)+\int_0^s h(s)\varpi\left(1+\int_0^s g(\tau)H^{-1}\left(H(1)+\int_0^\tau g(k)dk\right)d\tau\right)ds \in Dom(\Omega^{-1})$$

Pachpatte (1975c) developed the following integral inequalities:

$$u(t) \leq n(t) + \int_0^t f(s)\varpi\left(u(s)+\int_0^s g(\tau)\varpi(u(\tau))d\tau\right)ds, \quad t \in \mathbf{I} \tag{2.60}$$

where $u(t)$, $f(t)$ and $g(t)$ be real nonnegative continuous function defined on $\mathbf{R}_+$,

$n(t)$ be a positive, monotonic, nondecreasing continuous function defined on $\mathbf{R}_+$ and $\varpi \in \Psi$

$$u(t) \leq n(t) + \int_0^t f(s)\varpi\left(u(s) + \int_0^s g(\tau)\varpi(u(\tau))d\tau\right)ds + \int_0^t h(s)\alpha(u(s))ds \quad t \in \mathbf{I} \tag{2.61}$$

where $u(t),\ f(t),\ g(t)$ and $h(t)$ be real nonnegative continuous function defined on $\mathbf{R}_+$, $n(t)$ be a positive, monotonic, nondecreasing continuous function defined on $\mathbf{R}_+$, $\varpi \in \Psi : \alpha(u)$ be a positive, continuous, monotonic, nondecreasing and submultiplicative function for $u > 0$.

$$u(t) \leq n(t) + \int_0^t f(s)\varpi\left(u(s) + \int_0^s g(\tau)\varpi(u(\tau))d\tau\right)ds + h(t)\psi\left(\int_0^t h(s)\alpha(s, u(s))ds\right) \quad t \in \mathbf{I} \tag{2.62}$$

where $u(t),\ f(t),\ g(t),\ h(t)$and $n(t)$ be real nonnegative continuous function defined on $\mathbf{R}_+$, $n(t)$ be a positive, monotonic, nondecreasing continuous function defined on $\mathbf{R}_+$, $\varpi \in \Psi : \alpha(t, u)$ be a nonnegative, continuous, monotonic, nondecreasing in u, $u > 0$,for each fixed $t \in \mathbf{I}$; the function $n(t) > 0$, $\psi(t) > 0$ be nondecreasing in t and continuous on $\mathbf{I}$, $\psi(0) = 0$.

$$u(t) \leq n(t)\varpi^{-1}\left[\int_0^t f(s)\varpi(u(s))ds + \int_0^t f(s)\left(\int_0^s g(\tau)\varpi(u(\tau))d\tau\right)ds\right] \tag{2.63}$$

where $u(t),\ f(t)$ and $g(t)$ be real valued nonnegative continuous functions on $\mathbf{R}_+$: $n(t)$ be a positive, monotonic, nondecreasing continuous function defined on $\mathbf{R}_+$; $\varpi(u)$ be a positive, continuous, monotonic nondecreasing, subadditive and submultiplicative function for $u > 0$, $\varpi(0) = 0$, and ϖ^{-1} denote the inverse function of ϖ.

$$u(t) \leq n(t)\varpi^{-1}\left[\int_0^t f(s)\varpi(u(s))ds + \int_0^t f(s)\left(\int_0^s g(\tau)\varpi(u(\tau))d\tau\right)ds\right] + \int_0^t h(s)\alpha(u(s))ds \quad t \in \mathbf{I} \tag{2.64}$$

$$u(t) \leq n(t)\varpi^{-1}\left[\int_0^t f(s)\varpi(u(s))ds + \int_0^t f(s)\left(\int_0^s g(\tau)\varpi(u(\tau))d\tau\right)ds\right] + h(t)\psi\left(\int_0^s q(s)\alpha(s, u(s))ds\right) \quad t \in \mathbf{I} \tag{2.65}$$

$$u(t) \leq n(t) + \int_0^t f(s)\varpi\left(u(s)\int_0^s f(\tau)\left(\int_0^\tau f(k)\varpi(u(k))dk\right)d\tau\right)ds \tag{2.66}$$

$$u(t) \leq u_0 + \int_0^t f(s)\varpi\left(u(s) + \int_0^s f(\tau)\left(\int_0^\tau f(k)\varpi(u(k))dk\right)d\tau\right)ds + \int_0^t g((s)\alpha(s, u(s))ds \quad t \in \mathbf{I} \tag{2.67}$$

$$u(t) \leq p(t) + \int_0^t f(s)\varpi\left(u(s) + \int_0^s f(\tau)\left(\int_0^\tau f(k)\varpi(u(k))dk\right)d\tau\right)ds$$
$$+h(t)\psi\left(\int_0^t g((s)\alpha(s,u(s))ds\right) \quad t \in \mathbf{I} \qquad (2.68)$$

Theorem 2.21 Pachpatte (1975c)

Let $u(t),\ f(t),$ and $g(t)$ be a real-valued nonnegative continuous function defined on $\mathbf{R}_+$, and $n(t)$ be a positive, monotonic, nondecreasing continuous function defined on $\mathbf{R}_+$, for which the inequality

$$u(t) \leq n(t) + g(t)\left(\int_0^t f(s)u(s)ds\right), \quad t \in \mathbf{I} \qquad (2.69)$$

holds. Then

$$u(t) \leq n(t)\left[1 + g(t)\left(\int_0^t f(s)\exp\left(\int_0^s g(\tau)f(\tau)d\tau\right)ds\right)\right], \quad t \in \mathbf{I} \qquad (2.70)$$

Theorem 2.22 Pachpatte (1975c)

Let $u(t),\ f(t),\ g(t)$ and $h(t)$ be a real-valued nonnegative continuous function defined on $\mathbf{R}_+$, and $n(t)$ be a positive, monotonic, nondecreasing continuous function defined on $\mathbf{R}_+$, let $\varpi(u)$ be a positive continuous,monotonic, nondecreasing and submultiplicative function for $u > 0$, $\varpi(0) = 0$, and suppose further that inequality

$$u(t) \leq u_0 + g(t)\left(\int_0^t f(s)u(s)ds\right) + \int_0^t h(s)\varpi(u(s))ds, \quad t \in \mathbf{I} \qquad (2.71)$$

is satisfied for all $t \in \mathbf{I}$, where u_0 is a positive constant. Then

$$u(t) \leq \Omega^{-1}\left[\Omega(u_0) + \int_0^t h(s)\right.$$
$$\left.\times\varpi\left(1 + g(s)\left(\int_0^s f(\tau)\exp\left(\int_0^\tau g(k)f(k)dk\right)d\tau\right)\right)ds\right] \qquad (2.72)$$
$$\times\left[1 + g(t)\left(\int_0^t f(s)\exp\left(\int_0^s g(\tau)f(\tau)d\tau\right)ds\right)\right], \quad t \in \mathbf{I}$$

where Ω is defined in equation(2.13) and Ω^{-1} is the inverse function of Ω, and t is in the subinterval $[0, b]$ of $\mathbf{I}$ so that

$$\Omega(u_0)+\int_0^t h(s)\times\varpi\left(1 + g(s)\left(\int_0^s f(\tau)\exp\left(\int_0^\tau g(k)f(k)dk\right)d\tau\right)\right)ds \in Dom(\Omega^{-1})$$

Theorem 2.23 Pachpatte (1975c)

Let $u(t),\ f(t),\ g(t)$ and $h(t)$ be a real-valued nonnegative continuous function defined on $\mathbf{R}_+$, and $n(t)$ be a positive, monotonic, nondecreasing continuous function defined on $\mathbf{R}_+$, let $\varpi(u)$ be a positive continuous,monotonic, nondecreasing and submultiplicative and subadditive function for $u > 0$, $\varpi(0) = 0$, let $p(t) > 0$, $\psi(t) \geq 0$ be nondecreasing in t and continuous on $\mathbf{R}_+$, $\psi(0) = 0$: and suppose further that the inequality

$$u(t) \leq p(t) + g(t)\left(\int_0^t f(s)u(s)ds\right) + \psi\left(\int_0^t h(s)\varpi(u(s))ds\right), \qquad (2.73)$$

is satisfied for all $t \in \mathbf{I}$. Then

$$u(t) \leq \left[p(t) + \psi\left(G^{-1}\left[G\left(\int_0^t h(s) \times \varpi\left(p(s)\left[1 + g(s)\left(\int_0^s f(\tau)\exp\left(\int_\tau^s g(k)f(k)dk\right)d\tau\right)\right]\right)ds\right) + \int_0^t h(s)\varpi\left(1 + g(s)\left(\int_0^s f(\tau)\exp\left(\int_\tau^s g(k)f(k)dk\right)d\tau\right)\right)ds\right]ds\right)\right] \times \left[1 + g(t)\left(\int_0^t f(s)\exp\left(\int_0^t g(\tau)f(\tau)d\tau\right)ds\right)\right], \quad t \in \mathbf{I} \tag{2.74}$$

where $G(u)$ is defined in equation(2.29) and G^{-1} is the inverse function of Ω, and $t \in [0, b] \subseteq \mathbf{I}$ so that

$$G\left(\int_0^t h(s) \times \varpi\left(p(s)\left[1 + g(s)\left(\int_0^s f(\tau)\exp\left(\int_\tau^s g(k)f(k)dk\right)d\tau\right)\right]\right)ds\right) + \int_0^t h(s)\varpi\left(1 + g(s)\left(\int_0^s f(\tau)\exp\left(\int_\tau^s g(k)f(k)dk\right)d\tau\right)\right)ds \in Dom(\Omega^{-1})$$

Theorem 2.24 Pachpatte(1975c)

Let $u(t),\ f(t),\ g(t)\ h(t)$ and $q(t)$ be a real-valued nonnegative continuous function defined on $\mathbf{R}_+$, let $\varpi(t, u)$ be a positive, monotonic, continuous, nondecreasing in $u,\ u > 0$ for each fixed $t \in \mathbf{I}$; let the functions $p(t) > 0,\ \psi(t) \geq 0$ be nonincreasing in t and continuous on $\mathbf{R}_+$, $\varpi(0) = 0$; and suppose further that the inequality

$$u(t) \leq p(t) + g(t)\left(\int_0^t f(s)u(s)ds\right) + h(t)\psi\left(\int_0^t h(s)\varpi(u(s))ds\right), \quad t \in \mathbf{I} \tag{2.75}$$

is satisfied for all $t \in \mathbf{I}$. Then

$$u(t) \leq k(t)\left[p(t) + h(t)\psi(r(t))\right], \quad t \in \mathbf{I} \tag{2.76}$$

where

$$k(t) = 1 + g(t)\left(\int_0^t f(s)\exp\left(\int_0 sg(\tau)f(\tau)d\tau\right)ds\right) \tag{2.77}$$

and $r(t)$ is the maximal solution of

$$r'(t) = q(t)\varpi\left(t, k(t)\left[p(t) + h(t)\psi(r(t))\right]\right), \quad r(0) = 0 \tag{2.78}$$

existing on $\mathbf{R}_+$. Pachpatte provided bounds for the following nonlinear inequalities, the detail is in Pachpatte(1975d)

$$u(t) \leq n(t) + \varpi^{-1}\left[\phi\left(\int_0^t f(s)\varpi(u(s))ds\right)\right], \quad t \in \mathbf{I} \tag{2.79}$$

where $\phi \in \Psi$ and defined

$$P(u) = \int_0^t \frac{ds}{1 + \phi(s)}, \quad u \geq u_0 > 0, \tag{2.80}$$

From integral inequality (2.11) the author developed

$$u(t) \leq u_0 + \varpi^{-1}\left[\phi\left(\int_0^t f(s)\varpi(u(s))ds\right)\right] + \int_{t_0}^t g(s)\varpi(u(s))ds, \quad t \in \mathbf{I} \tag{2.81}$$

where u_0 is a positive constant.

$$u(t) \leq p(t) + \varpi^{-1}\left[\phi\left(\int_0^t f(s)\varpi(u(s))ds\right)\right] + \psi\left(\int_0^t g(s)\varpi(u(s))ds\right), \quad t \in \mathbf{I}$$

$$(2.82)$$

where $\psi(t) \geq 0$ and $p(t) > 0,\ \psi(0) = 0$

$$u(t) \leq p(t) + \varpi^{-1}\left[\phi\left(\int_0^t f(s)\varpi(u(s))ds\right)\right] + h(t)\psi\left(\int_0^t g(s)\varpi(u(s))ds\right), \quad (2.83)$$

where $h(t)$ be real-valued positive continuous function defined on $\mathbf{R}_+$ and with ϕ, $\varpi \in \Psi$ defined as

$$Q(u) = \int_{u_0}^{u} \frac{ds}{\varpi(1+\phi(s))}, \quad u \geq u_0 > 0. \quad (2.84)$$

Chandra and Davis(1976) studied the linear integral inequalities such as

$$u(t) \leq a(t) + G(t)\int_{t_0}^{t} H(s)u(s)ds \quad t_0 \leq t \quad (2.85)$$

where $G(t),\ H(t)$ be continuous, nonnegative matrices for $t_0 \leq t$ Pachpatte(1976) gave four main theorems about Gronwall's inequality as follow:

$$u(t) \leq n(t) + g(t)\left(\int_0^t h(s)u^{\alpha}(s)ds\right)^{\frac{1}{\alpha}}, \quad t \in \mathbf{I} \quad (2.86)$$

where $1 \leq \alpha < \infty$

$$u(t) \leq u_0 + g(t)\left(\int_0^t h(s)u^{\alpha}(s)ds\right)^{\frac{1}{\alpha}} + \int_0^t p(s)\varpi(u(s))ds, \quad t \in \mathbf{I} \quad (2.87)$$

where u_0 is positive constant

$$u(t) \leq f(t) + g(t)\left(\int_0^t h(s)u^{\alpha}(s)ds\right)^{\frac{1}{\alpha}} + \phi\left(\int_0^t p(s)\varpi(u(s))ds\right), \quad t \in \mathbf{I} \quad (2.88)$$

$$u(t) \leq f(t) + g(t)\left(\int_0^t h(s)u^{\alpha}(s)ds\right)^{\frac{1}{\alpha}} + q(t)\phi\left(\int_0^t p(s)\varpi(u(s))ds\right). \quad (2.89)$$

Dehongade and Deo(1976) considered the inequality dealing with generalisation of Bihari's integral inequality in the form

$$u(t) \leq f(t) + \sum_{i=1}^{n}\int_0^t h_i(s)\varpi_i(u(s))ds \quad t \in \mathbf{I} \quad (2.90)$$

where

$$\Omega_k(u) = \int_{u_0}^{u} \frac{ds}{\varpi_k(s)}, \quad 0 < u_0, u > 0 \quad (2.91)$$

and $\varpi \in \Psi$. Agarwal and Thandapani(1981) gave the following integral inequalities of the form

$$u(t) \leq p(t) + q(t)\Sigma_{r=1}^{n}E_r(t,u) \quad (2.92)$$

The inequality (2.92) is considered when $p(t)$ is not nondecreasing and when it is nondecreasing.

$$u(t) \leq p(t) + \Sigma_{r=1}^{n}g_r(t)E_r(t,u) \quad (2.93)$$

where $g_i(t) \geq 1(i = 1, 2, ...n)$ and the inequality (2.93) is not nondecreasing and when it is nondecreasing.

$$u(t) \leq p(t) + q(t)\Sigma_{r=1}^{n}Q_r(t,u) \quad (2.94)$$

where

$$Q_r(t,u) = E_r(t,u) + e_r(t,u)$$

Other integral inequalities considered by Agarwa and Thandapani(1981) are given below:

$$u(t) \le \omega(t) + \int_0^t K(t,s)u(s)ds \tag{2.95}$$

where the function $K(t,s)$ is continuous, differentiable with respect to t on $\mathbf{R}_+ \times \mathbf{R}_+$. Further more $K(t,t) \ge 0$, $\frac{\partial K(t,s)}{\partial t} \ge 0$ but $K(t,s)$ or $\frac{\partial K(t,s)}{\partial t}$ is not necessarily separable as considered by Willett(1965) In the next integral inequality, $K(t,s) \le \sum_{r=i}^{n} g_r(t)h_r(s)$ is considered as follow:

$$u(t) \le \omega(t) + \sum_{r=1}^{n} g_r(t) \int_0^t h_r(s)u(s)ds \tag{2.96}$$

where:

(i) $\omega(t)$ is nondecreasing

(ii) $g_r(t) \ge 1$ $(i = 1, 2, ..., n)$ and are nondecreasing for $i \ge 2$.

The authors also considered the following nonlinear generalisation of Gronwall type inequalities of the form

$$u(t) \le p(t)\left[u_0 + \Sigma_{r=1}^{n} E_r^*(t,u)\right] \tag{2.97}$$

where

$$E_r^*(t,u) = \int_0^t f_{r^1}(t_1)u_{r^1}^{k}(t_1) \int_0^t f_{r^2}(t_2)u_{r^2}^{k}(t_2)\ldots\ldots. \int_0^{t_{r-1}} f_{rr}(t_r)u_{rr}^{k}(t_r)dt_r.dt_{r-1}.....dt_1$$

and $k_{ij}(i = 1, 2, ..., n, \quad j = 1, 2, ...n)$ are nonnegative real numbers and $u_0 > 0$. The further consideration of nonlinear generalisation of integral inequality yield the following:

$$u(t) \le p(t) + q(t)\Sigma_{r=1}^{n} E_r(t,u) + \Sigma_{r=1}^{m} g_r(t) \int_0^t h_r(s)\omega_r(u(s))ds. \tag{2.98}$$

The next linear integral inequalities considered is given as:

$$u(t) \le p(t) + q(t)\Sigma_{r=1}^{n} E_r(t,u) + g(t) \int_0^t h_1(t_1) \int_0^t h_2(t_2)\ldots\ldots \int_0^{t_{m-1}} h_m(t_m)\omega(u(t_m))dt_m dt_{m-1}..dt_1. \tag{2.99}$$

Akinyele(1984) gave the following integral inequalities which extended the results of Yeh and shih (1982) and Pachpatte(1975)

Theorem 2.25 Akinyele(1984)

Let $u(t)$, $f(t)$ and $g(t)$ be real nonnegative continuous functions on $\mathbf{R}_+$ and $n(t)$ be a positive ,nondecreasing continuous function on $\mathbf{R}_+$. Suppose σ and $p \in \Psi$ and $q(t) \ge 1$ is a real valued continuous function defined on $\mathbf{R}_+$. Let the functional inequality

$$u(t) \le p(t) + q(t)\left[\int_{t_0}^t f(s)u(\sigma(s))ds + \int_{t_0}^t f(s)\left(\int_{t_0}^s g(x)u(p(x))dx\right)ds\right] \tag{2.100}$$

hold for $t \in \mathbf{R}_+$. with $t \geq t_0$. Then for all $t \in \mathbf{R}_+$ with $t \geq 0$

$$u(t) \leq q(t)n(t) \times \exp\left(\int_{t_0}^{t}\left\{\frac{f(s)q(\sigma(s))n(\sigma(s)) + g(s)q(p(s))n(p(s))}{n(\sigma(s))}\right\}ds\right) \quad (2.101)$$

and

$$q(t)n(t)\left[1 + \int_{t_0}^{t} f(s)q(\sigma(s)) \times \exp\left(\int_{t_0}^{s}\left\{\frac{f(x)q(\sigma(x))n(\sigma(x)) + g(x)q(p(x))n(p(x))}{n(\sigma(x))}\right\}dx\right)ds\right] \quad (2.102)$$

Theorem 2.26 Akinyele(1984)

Let $u(t)$, $f(t)$, $g(t)$, $q(t)$, $\sigma(t)$, and $p(t)$ be as in Theorem 2.25. Let $h(t)$ be a real valued nonnegative continuous defined on $\mathbf{R}_+ \subset \mathbf{R}^n$, and $H(u)$ be a positive , continuous,monotonic, nondecreasing and submultiplicative function for $u > 0$ and $H(0) = 0$. If for $t \in \mathbf{R}_+$ with $t \geq t_0$, and $D_k H(u(t)) \geq 0$ for $k = 2, 3....n$

$$u(t) \leq u_0 + q(t)\left[\int_{t_0}^{t} f(s)u(\sigma(s))ds + \int_{t_0}^{t} f(s)\left(\int_{t_0}^{s} g(x)u(\sigma(x))dx\right)ds\right] + \int_{t_0}^{t} h(s)\varpi(u(p(s))ds \quad (2.103)$$

then

$$u(t) \leq q(t)\exp\left(\int_{t_0}^{t} q(\sigma(s))\Big\{f(x) + g(x)\Big\}dx\right) \times \Big[\Omega^{-1}\Big(\Omega(u_0) + \int_{t_0}^{t} h(s)\varpi\left[q(p(s)) \times \exp\left(\int_{t_0}^{p(s)} q(\sigma(x))\Big\{f(t) + g(x)\Big\}dx\right)\right]ds\Big)\Big] \quad (2.104)$$

and

$$u(t) \leq E(t)\left[1 + \int_{t_0}^{t} f(s)q(\sigma(s)) \times \exp\left(\int_{t_0}^{s} q(\sigma(x))\Big\{f(x) + g(x)\Big\}dx\right)\right] \quad (2.105)$$

where

$$E(t) = q(t)\Omega^{-1}\left[\Omega(u_0) + \int_{t_0}^{t} h(y) \times\varpi\left(q(p(y))\Big\{1 + \int_{t_0}^{p(y)} f(s)q(\sigma(s))A(s)ds\Big\}\right)dy\right]$$

with

$$A(t) = \exp\left(\int_{t_0}^{t} q(\sigma(s))\Big\{f(s) + g(s)\Big\}ds\right)$$

where Ω is defined in equation (2.13)

Dannan(1985) developed the following integral inequalities by using multiplier functions ϕ and $h(t)$

Theorem 2.27 Dannan (1985)

Assume that $u(t)$ and $f(t)$ are positive continuous functions on $\mathbf{R}_+$ $\varpi(u) \in \Psi$ with corresponding multiplier function ϕ and $h(t) > 0$ is monotonic, nondecreasing and continuous on $\mathbf{R}_+$.

If

$$u(t) \leq h(t) + \int_0^t f(s)\varpi(u(s))ds, \ \ t \in I \tag{2.106}$$

then

$$u(t) \leq h(t)\Omega^{-1}\left[\Omega(1) + \int_0^t f(s)\frac{\phi(h(s))}{h(s)}ds\right], \ \ 0 \leq t \leq b \tag{2.107}$$

where Ω is defined as equation(2.13) Ω^{-1} is the inverse of Ω and $(0, b)$ is the subinterval for which

$\Omega(1) + \int_0^t f(s)\frac{\phi(h(s))}{h(s)}ds \in Dom(\Omega^{-1})$.

Corollary 2.2 Dannan (1985)

Let u,f,ϖ,h,Ω all be as in Theorem 2.26 and suppose $b(t)$ is nonnegative, continuous and nondecreasing on $\mathbf{R}_+$. if

$$u(t) \leq h(t) + b(t)\int_0^t f(s)\varpi(u(s))ds, \ \ t \in I \tag{2.108}$$

then

$$u(t) \leq h(t)\Omega^{-1}\left[\Omega(1) + b(t)\int_0^t f(s)\frac{\phi(h(s))}{h(s)}ds\right], \ \ 0 \leq t \leq b. \tag{2.109}$$

where Ω is defined as equation (2.13), Ω^{-1} as inverse of Ω and $[0, t_0]$ is subinterval for which

$\Omega(1) + b(t_0)\int_0^t f(s)\frac{\phi(h(s))}{h(s)}ds \in Dom(\Omega^{-1})$.

Dannan (1985) extended the Theorem 2.26 to the following theorem:

Theorem 2.28 Dannan (1985)

Let $u(t)$, $f(t)$ be positive continuous functions on $\mathbf{R}_+$ and $\varpi(u) \in H$ with corresponding multiplier function ϕ, for which the inequality

$$u(t) \leq u_0 + \int_0^t f(s)\varpi(u(s))ds + \int_0^t g(s)\left(\int_0^s f(\tau)\varpi(u(\tau))d\tau\right)ds, \ \ t \in I \tag{2.110}$$

holds, where $u_0 > 0$ is constant. Then

$$\begin{aligned} &u(t) \leq u_0 A(t)E(t)\Omega^{-1}\left[\Omega(1)\right. \\ &\left.+u_0^{-1}E(t)\int_0^t f(s)\frac{\phi(u_0A(s)E(s))}{A(s)E(s)}ds\right] \ \ \text{for } 0 \leq t \leq b, \end{aligned} \tag{2.111}$$

where

$E(t) \equiv \exp\left(\int_0^t g(s)ds,\right)$

$A(t) \equiv \left(\frac{\int_0^t g(s)ds}{E(t)}\right)$,

$\Omega(u)$is defined in equation (2.13). Ω^{-1} is the inverse of Ω and t is in the subinterval $[0, t_0]$ so that

$\Omega(1) + u_0^{-1}E(t)\int_0^t f(s)\frac{\phi(u_0A(s)E(s))}{A(s)E(s)}ds \in Dom(\Omega^{-1})$.

Theorem 2.29 Dannan (1986)

Let $u(t),\ f(t),\ g(t),\ p(t)$ and $k(t)$ be real valued positive functions defined on $\mathbf{R}_{+}$, let $\varpi(u) \in \Psi$ with corresponding multiplier function ϕ and let $k(t)$ also be a monotonic, nondecreasing function, for which the inequality

$$u(t) \leq k(t) + p(t)\int_0^t f(s)\varpi(u(s))ds + \int_0^t g(s)\varpi(u(s))ds, \quad t \in I. \tag{2.112}$$

Then

$$u(t) \leq k(t)r(t)\Omega\left[\Omega(1) + \int_0^t \frac{1}{k(s)}g(s)\phi(k(s))\phi(r(s))ds\right] \tag{2.113}$$

for $t \in [0, b]$ where

$$r(t) = 1 + p(t)\left[\int_0^t \exp\left(\int_0^t p(\theta)f(\theta)d\theta\right)ds\right], \quad t \in \mathbf{I} \tag{2.114}$$

where Ω is defined as equation (2.13) and Ω^{-1} is the inverse function of Ω, and $t \in [0, b] \subset \mathbf{I}$ so that

$$\Omega(1) + \int_0^t \frac{1}{k(s)}g(s)\phi(k(s))\phi(r(s)) \in Dom(\Omega^{-1})$$

Theorem 2.30 Dannan (1986)

Let $u(t),\ a(t),\ k(t)$ and $h(t)$ be real valued nonnegative continuous functions defined on $I = [0, \beta)$. Let $g(u) \in M$ with corresponding function ϕ on an interval $\mathbf{R}_{+}$ such that $u(J) \subset \mathbf{R}_{+}$ and $a(J) \subset \mathbf{R}_{+}$. Suppose also that the function h is monotonic, nondecreasing on an interval K such $0 \in K$, $h(K) \subset \mathbf{R}_{+}$. Then

$$u(t) \leq a(t) + h\left[\int_0^t k(s)g(u(s))ds\right] \tag{2.115}$$

implies

$$u(t) \leq a(t) + h\left(G^{-1}\left[\int_0^t k(s)ds + G\left(\int_0^t \phi(a(s))ds\right)\right]\right) \tag{2.116}$$

for $0 \leq t < \beta_1$, where $G(u)$ is defined as equation (2.29) for $\beta_1 = min(u_1,\ u_2,\ u_3)$ with

$$u_1 = sup\left\{u \in J : a(t) + h\left(\int_0^t k(s)g(u(s))ds\right) \in \mathbf{R}_{+},\ 0 \leq t \leq u\right\}$$

$$u_2 = sup\left\{u \in J : \int_{u_0}^u k(s)\left[\phi(a(s)) + goh\left(\int_0^t k(\theta)g(u(\theta))d\theta\right)\right]ds \in \mathbf{K}\right\}$$

$$u_3 = sup\left\{u \in J : \int_{u_0}^u k(s)ds + G\left(\int_0^T k(s)\phi(u(s))ds\right) \in G(K),\ \ 0 \leq t \leq T \leq u\right\}$$

Pinto 1990 considered integral inequalities with several nonlinear terms.

Theorem 2.31 Pinto(1990)

Suppose the following two hypotheses

(H_1) The functions $\varpi_i,\ \ i = 1,\ 2,\ 3,\ ...., n$ are continuous and nondecreasing on $\mathbf{R}_{+}$ and positive on $\mathbf{R}_{+}$ such that $\varpi_1 \propto \varpi_2 \propto \varpi_3 \propto ...... \propto \varpi_n$. (H_2) The functions $u,\ \{\lambda\}_{i-1}^n$ are continuous and nonnegative on $\mathbf{I} - [a, b]$ and c is a positive constant.

if

$$u(t) \leq c + \sum_{i=1}^{n} \int_{t_0}^{t} \lambda_i(s)\varpi_i(u(s))ds, \;\; t \in \mathbf{I} \tag{2.117}$$

then

$$u(t) \leq \Omega_n^{-1}\left[\Omega_n(c_{n-1}) + \int_{t_0}^{t} \lambda_n(s)ds\right] \tag{2.118}$$

for $t \in [t_0, b_1]$ where Ω_k is defined in equation (2.91) and Ω_k^{-1} is the inverse of the of Ω_k.

The constants c_k are given by $c_0 = c$ and

$c_k = \Omega_k^{-1}\left[\Omega_k(c_{k-1}) + ||\lambda||_{b_1}\right] \;\; k = 1, 2......., n-1$

The number $b_1 \subset \mathbf{I}$ is the largest number such that

$||\lambda||_{b_1} = \displaystyle\int_{t_0}^{b_1} \lambda_k(s)ds \leq \int_{c_{k-1}}^{\infty} \frac{ds}{\omega_k(s)}, \;\; k = 1, 2, ...n$

Theorem 2.32 Pinto (1990)

Assume that u and $\lambda_i, \;\; i = 1, 2, ..., n$ are continuous and nonnegative functions on $[t_0, b], \varpi_i \in \Psi, \; i = 1, 2.........., n$ with corresponding multiplier functions $r_i \;\; i = 1, 2, ...$ such that $\varpi_1 \propto \varpi_2 \propto \varpi_3 \propto ...... \propto \varpi_n$. and c is a positive constant. if

$$u(t) \leq c + \int_{t_0}^{t} \lambda_0(s)u(s)ds + \sum_{i=1}^{n} \int_{t_0}^{t} \lambda_i(s)\varpi_i(u(s))ds, \;\; t \in \mathbf{I} \tag{2.119}$$

then for all $t \in [t_0, b_1]$

$$u(t) \leq E(t).\Omega_n^{-1}\left[\Omega_n(c_{n-1}) + \int_{t_0}^{t} \lambda_n(s).\frac{r_n(E(s))}{E(s)}ds\right], \tag{2.120}$$

where the notations are the same as in Theorem 2.31 by replacing λ_i by $\lambda_i r_i(E)/E$ with $E(t) = \exp\left(\int_{t_0}^{t} \lambda_0(s)ds\right)$.

Theorem 2.33 Pinto (1990)

Let $u, \; \lambda_i, \; \varpi_i \;\; i = 1, 2, ..., n$ be as in Theorem 2.31 and suppose that $\varpi_i \in \Psi, \; i = 1, 2.........., n$ with corresponding multiplier functions $r_i \;\; i = 1, 2, ...$ and $h > 0$ is a continuous function on $[t_0, b]$. if

$$u(t) \leq h(t) + \sum_{i=1}^{n} \int_{t_0}^{t} \lambda_i(s)\varpi_i(u(s))ds, \;\; t \in [t_0, b] \tag{2.121}$$

then for all $t \in [t_0, b_1]$

$$u(t) \leq h(t)\Omega_n^{-1}\left[\Omega_n(c_{n-1}) + \int_{t_0}^{t} \lambda_n(s)r_n(h(s))ds\right], \tag{2.122}$$

where the notations are the same as in Theorem 2.31 by replacing λ_i by $\lambda_i.r_i(h)$ and $c_0 = 1$.

Corollary 2.3 Pinto (1990) Let $u, \; \lambda_i, \; \varpi_i \;\; i = 1, 2, ..., n$ and h be as in Theorem 2.33 and suppose that $f(t)_i, \;\; i = 1, 2.........., n$ are nonnegative, continuous and nondecreasing functions on $[t_0, b]$. if

$$u(t) \leq h(t) + \sum_{i=1}^{n} f_i(t) \int_{t_0}^{t} \lambda_i(s)\varpi_i(u(s))ds, \;\; t \in [t_0, b] \tag{2.123}$$

then for all $t \in [t_0, b_1]$

$$u(t) \leq h(t)\Omega_n^{-1}\left[\Omega_n(c_{n-1}) + f_n(t)\int_{t_0}^{t}\lambda_n(s)r_n(h(s))ds\right], \tag{2.124}$$

where the notations are the same as in Theorem 2.31 by replacing

λ_i by $f(b_i).\lambda_i.r_i(h)$ $1 \leq i \leq n$ and $c_0 = 1$.

Corollary 2.4 Pinto (1990)

Let $u,\ \lambda_i,\ \varpi_i\ r_i,\ i = 1, 2, ..., n$ be as in Theorem 2.33 and $f(t)_i,\ i = 1, 2.........., n$ as in Corollary 2.5. Suppose that h and f are continuous and positive function on $[t_0, b]$ such that $f \propto h$. if

$$u(t) \leq h(t) + f(t)\sum_{i=1}^{n} f_i(t)\int_{t_0}^{t}\lambda_i(s)\varpi_i(u(s))ds, \quad t \in [t_0, b] \tag{2.125}$$

then for all $t \in [t_0, b_1]$

$$u(t) \leq \frac{h(t)}{f(t)}\Omega_n^{-1}\left[\Omega_n(c_{n-1}) + \int_{t_0}^{t}\lambda_n(s)r_n(f(s))r_n(h(s))/f(s)ds\right], \tag{2.126}$$

where the notations are the same as in Theorem 2.31 by replacing λ_i by $f(b_i).\lambda_i.r_i(f)r_i(h/f)$ $1 = 1, 2....., n$ and $c_0 = 1$. Pinto (1990) established his result for $n = 1$ only

Theorem 2.34 Pinto (1990)

Let $u,\ \lambda_i,\ i = 1, 2, 3,\ \ \varpi_i\ i = 1, 2, 3,$ and c be as in Theorem 2.31. If

$$u(t) \leq c + \int_{t_0}^{t}\lambda_1(s)\varpi_1(u(s))ds + \int_{t_0}^{t}\lambda_2\varpi_2\left(\int_{t_0}^{s}\lambda_3(\tau)\varpi_3(u(\tau))d\tau\right)ds, \quad t \in \mathbf{I} \tag{2.127}$$

then, for $t \in [t_0, b_1]$,

$$u(t) \leq \Omega_3^{-1}\left[\Omega_3(c_2) + \int_{t_0}^{t}\lambda_3(s)ds\right] \tag{2.128}$$

where the notations are the same as in Theorem 2.31. Lipovan (2000) generalised the Gronwall inequality as follow:

Theorem 2.35 Lipovan (2000)

Let $u,\ f \in C(\mathbf{I}, \mathbf{R}_+)$. Suppose $\varpi \in C(\mathbf{R}_+, \mathbf{R}_+)$ be nondecreasing with $\varpi(u) > 0$ on $\mathbf{R}_+$ and $\alpha \in C^1(\mathbf{I}, \mathbf{R}_+$ be nondecreasing with $\alpha(t) \leq t$ on $\mathbf{I}$. If

$$u(t) \leq u_0 + \int_{\alpha(0)}^{\alpha(t)} f(s)\varpi(u(s))ds \quad 0 \leq t \leq b \tag{2.129}$$

where u_0 is a nonnegative constant, then for $0 \leq t < t_1$

$$u(t) \leq \Omega^{-1}\left(\Omega(u_0) + \int_{\alpha(0)}^{\alpha(t)} f(s)ds\right) \tag{2.130}$$

where Ω is defined in equation (2.13) and $t_1 \in (0, b)$ is chosen so that

$$\Omega(u_0) + \int_{\alpha(0)}^{\alpha(t)} f(s)ds \in Dom(\Omega^{-1})$$

for all t lying in the interval $[t_0, t_1]$. Oguntuase (2000) obtained bounds to the linear Gronwall-Bellman-Bihari type integral inequalities for a more general kernel $k(t, s)$ and a product kernel.

Theorem 2.36 Oguntuase(2000)

Let $k(t,s)$ be a good kernel, $u(t)$ is a real valued nonnegative continuous function on $\mathbf{R}_+$ and $g(t)$ be a positive, nondecreasing continuous function on $\mathbf{R}_+$. Suppose that the following inequality.

$$u(t) \le g(t) + \int_{t_0}^{t} k(t,s)u(s)ds \tag{2.131}$$

holds for all $t \in \mathbf{I}$ with $t \ge t_0$, then

$$u(t) \le g(t)\left[1 + \int_{t_0}^{t} k(s,s)\exp\left(\int_{t_0}^{s} k(\delta,\delta)d\delta\right)ds\right] \tag{2.132}$$

Oguntuase obtained bounds to the following integral inequalities

Theorem 2.37 Oguntuase (2001)

Let $u(t)$ and $f(t)$ be nonnegative continuous functions in a real interval $\mathbf{I}$. Suppose that $k(t,s)$ and its partial derivative $k_t(t,s)$ exist and are continuous function for every $t,s \in \mathbf{I}$. Suppose $k(t,s) \ge 0$, $k_t(t,s) \le 0$ and that inequality

$$u(t) \le c + \int_{t_0}^{t} f(s)u(s)ds + \int_{t_0}^{t} f(s)\left(\int_{t_0}^{s} K(s,\tau)u(\tau)d\tau\right)ds \quad t \in \mathbf{I} \tag{2.133}$$

holds where c is a nonnegative constant. Then

$$u(t) \le c\left[1 + \int_{t_0}^{t} f(s)\exp\left(f(\tau) + K(\tau,\tau)d\tau\right)ds\right] \tag{2.134}$$

As a direct consequence of Theorem 2.37, in which $k(t,s) = h(t)g(s)$. The following integral inequalities stated thus;

Corollary 2.5 Oguntuase (2001)

If $h'(t) \le 0$ and $c \ge 0$ is a constant, then

$$u(t) \le c + \int_{t_0}^{t} f(s)u(s)ds + \int_{t_0}^{t} f(s)h(s)\left(\int_{t_0}^{s} g(\tau)u(\tau)d\tau\right)ds \quad t \in \mathbf{I} \tag{2.135}$$

implies

$$u(t) \le c\left[1 + \int_{t_0}^{t} f(s)\exp\left(f(\tau) + h(\tau)g(\tau)d\tau\right)ds\right] \tag{2.136}$$

Theorem 2.38 Oguntuase (2001)

Let $u(t)$ and $f(t)$ be nonnegative continuous functions in a real interval $\mathbf{I}$. Suppose that $k(t,s)$ and its partial derivative $k_t(t,s)$ exist and are continuous functions for every $t,s \in \mathbf{I}$. Suppose $k(t,s) \ge 0$, $k_t(t,s) \le 0$ and that inequality

$$u(t) \le c + \int_{t_0}^{t} f(s)u(s)ds + \int_{t_0}^{t} f(s)\left(\int_{t_0}^{s} K(s,\tau)u^p(\tau)d\tau\right)ds \quad t \in \mathbf{I} \tag{2.137}$$

holds, where $0 \le p < 1$ and $c > 0$ are constants. Then for $t \in \mathbf{I}$,

$$u(t) \le c + \int_{t_0}^{t} f(s)\exp\left(\int_{t_0}^{s} f(\tau)d\tau\right)$$
$$\left[c^{1-p} + (1-p)\int_{t_0}^{t} k(\tau,\tau)\exp\left(-(1-p)\int_{t_0}^{\tau} f(\delta)d\delta\right)^{\frac{1}{1-p}} ds\right] \tag{2.138}$$

Theorem 2.39 Oguntuase (2001) If $h'(t) \leq 0$ and $c \geq 0$ is a constant, then

$$u(t) \leq c + \int_{t_0}^{t} f(s)u(s)ds + \int_{t_0}^{t} f(s)h(s)\left(\int_{t_0}^{s} g(\tau)u^p(\tau)d\tau\right) ds \quad t \in \mathbf{I} \tag{2.139}$$

holds, where $0 \leq p < 1$, $q = 1 - p > 0$ and $c > 0$ are constants. Then for $t \in \mathbf{I}$,

$$u(t) \leq c + \int_{t_0}^{t} f(s) \exp\left(\int_{t_0} f(\tau)d\tau\right)$$
$$\left[c^{1-p} + (1-p)\int_{t_0}^{t} h(\tau)f(\tau)\exp\left(-(1-p)\int_{t_0}^{\delta} f(\delta)\right) d\delta\right]^{\frac{1}{1-p}} \tag{2.140}$$

In 2014 Akhan established new explicit bounds on the following integral inequalities.

Theorem 2.40 Khan(2014)

Let $u(t), f(t)$ and $g(t)$ be nonnegative continuous functions defined for $\mathbf{I} = [0.\infty)$. Let $K(t) > 1$ defined for $K(t) > 1$ and also $k'(t)$ be nonnegative continuous functions defined for $K(t) > 1$ if

$$u(t) \leq k(t) + \int_{0}^{t} f(s)u(s)ds + \int_{0}^{t} f(s)\left(\int_{0}^{s} g(\partial)u(\partial)d\partial\right) ds, \forall t \in (I) \tag{2.141}$$

then

$$u(t) \leq k(t) + k(0)\int_{0}^{t} f(s)\exp\left(k(s) - k(0) + \int_{0}^{s} (f(\partial)\right.$$
$$\left. + g(\partial))\, d\partial\right) ds \quad \forall t \in \mathbf{I}. \tag{2.142}$$

Theorem 2.41 Khan (2014)

Let $u(t)$, $f(t)$ and $g(t)$, $k(t)$ and $k'(t)$ be defined in Theorem 2.40. If

$$u(t) \leq k(t) + \int_{0}^{t} f(s)u(s)ds$$
$$+ \int_{0}^{t} f(s)\left(f(\tau)\left(\int_{0}^{s} g(\partial)u(\partial)d\partial\right) d\tau\right) ds, \forall t \in (I) \tag{2.143}$$

then

$$u(t) \leq k(t) + \int_{0}^{t} f(s)\left[k(s)\right.$$
$$\left. + k(0)\int_{0}^{s} f(\tau)\exp\left(k(\tau) - k(0) + \int_{0}^{r} (f(\partial) + g(\partial))d\partial\right) d\tau\right] ds \quad \forall t \in \mathbf{I}. \tag{2.144}$$

In 2015, Wang obtained a result for the following integral inequality

$$w(u(t)) \leq K + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} f_i(s)\Pi_{j=1}^{m} H_{ij}(u(s))G_{ij}(\max_{s-h\leq\xi\leq s} u(\xi))ds, \tag{2.145}$$

$t_0 \leq t < T$.

In 2019, Hussain, Sadia and Aslam established the following integral inequalities

$$u^p \leq a(t) + b(t)\int_{0}^{t} u^q(s)ds + g(t)\int_{0}^{t} (t-s)^{\alpha-1}u^q(s)ds \tag{2.146}$$

where $a(t), b(t), g(t)$ and $u(t)$ be nonnegative on $\mathbf{I} = [0, T)$, $T \leq +\infty$ $\alpha \in (0, 1)$.

$$u^p \leq a(t) + b(t)\int_{0}^{t} u^q(s)ds + g(t)\int_{0}^{t} (t-s)^{\alpha-1}L(s, u^q(s))ds \tag{2.147}$$

for this $p \geq 1$

Tian and Fan(2020) considered the new nonlinear integral inequalities in the following form:

$$u(t) \leq a(t) + \int_0^{\alpha(t)} b(s) \left(u^m(s) + \int_0^s c(\xi) u^n(\xi) d\xi \right)^p ds \tag{2.148}$$

where m, n, p are nonnegative constants satisfying $0 < m, n \leq 1$, $p > 1$. α is nondecreasing with $\alpha \in C^1(\mathbf{R}_+, \mathbf{R}_+)$, $\alpha(t) \leq t$, $\alpha(0) = 0, u, a, b, c \in C(\mathbf{R}_+, \mathbf{R}_+)$

$$u^q(t) \leq a(t) + \int_0^{\alpha(t)} b(s) \left(u^m(s) + \int_0^s c(\xi) u^n(\xi)^p d\xi \right)^p ds \tag{2.149}$$

with $q \geq m > 0, q \geq n > 0$, $p > 0, \alpha(t)$ is nondecreasing with $\alpha \in C^1(\mathbf{R}_+, \mathbf{R}_+)$ $\alpha(t) \leq t$, $\alpha(0) = 0, u, a, b, c \in C(\mathbf{R}_+, \mathbf{R}_+)$

2.2 Hyers-Ulam and Hyers-Ulam-Rassias Stability of Linear Ordinary Differential Equations

Mathematical models of most dynamic processes in engineering, physical and biological sciences are conveniently expressed in the form of linear and nonlinear ordinary differential equations. In recent years numerous methods have been developed to tackle the qualitative behaviour of solutions of ordinary differential equations. A.M. Lyapunov introduced a method, which was called the second or direct method,which was used to establish stability theorems, this was made known in a memoir published in Russian in 1892. The method gained tremendous popularity among many authors. Years later, precisely in 1940, the stability problem of functional equations with the question concerning stability of group homomorphisms proposed by Ulam(1940)in Ulam(1960) came into existence through a wide-rage talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of important unsolved problems. Among these was the question concerning the stability of homomorphisms. On stability of functional equation, Ulam proposed the following problem: "Give condition in order for a linear mapping near an approximately linear mapping to exist". This problem was also put in the sense: "For what metric group $\mathbf{G}$ is necessarily near to a strict automorphism?" In 1941, Hyers(1941) solved the problem of Ulam for additive functions defined on Banach spaces thus: If $\mathbf{X}$ and $\mathbf{Y}$ are real Banach spaces and $\epsilon \leq 0$. Then for every function $f : \mathbf{X} \to \mathbf{Y}$ satisfying

$$||f(x+y) - f(x) - f(y)|| \leq \epsilon, \quad \text{for all } x, \ y, \in \mathbf{X}, \tag{2.150}$$

there exists a unique additive function $A : \mathbf{X} \to \mathbf{Y}$ with the property

$$\|f(x) - A(x)\| \leq \epsilon \tag{2.151}$$

for all $x \in \mathbf{X}$. The result by Hyers is called the Hyers-Ulam stability of the additive Cauchy equation

$$f(x+y) = f(x) + f(y). \tag{2.152}$$

After a decade Aoki(1950) generalised the result given by Hyers and made the result of Aoki an extension of Hyers'result. Bourgin(1951) worked on the extension of Hyers' result and had a publication tittle " Classes of transformations and bordering transformations". In 1978, Rassias(1978) introduced a new functional inequality called Cauchy-Rassias inequality and also succeeded in extending the result of Hyers, by weakening the condition for the Cauchy differences to unbounded map as follows: "If there exists $\epsilon \geq 0$ and $0 \leq p < 1$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p) \tag{2.153}$$

for all$x, y \in \mathbf{X}$, then there exist a unique additive mapping $A : \mathbf{X} \to \mathbf{Y}$ such that

$$\|f(x) - A(x)\| \leq \frac{2\epsilon}{|2-2^p|} \|x\|^p \qquad \text{for every } x \in \mathbf{X}. \tag{2.154}$$

The result of Rassias was called Hyers-Ulam-Rassias stability. In 1991, Gajda(1991) solved the problem for $1 < p$, the result established stated as thus, let $\mathbf{X}$ and $\mathbf{Y}$ be two (real) norm linear spaces and assumed that $\mathbf{Y}$ is complete. Let $f : \mathbf{X} \to \mathbf{Y}$ be a mapping for which there exist two constants $\epsilon \in [0, \infty)$ and $p \in (\mathbf{R} - \{1\})$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon\,(\|x\|^p + \|y\|^p) \tag{2.155}$$

for all $x, y \in \mathbf{X}$. Then, there exists a unique additive mapping $T : \mathbf{X} \to \mathbf{Y}$ such that

$$\|f(x) - T(x)\| \leq \delta\,\|x\|^p \tag{2.156}$$

for all $x \in \mathbf{X}$, where

$$\delta = \begin{cases} \dfrac{2\epsilon}{2-2^p} & \text{for} \quad p < 1 \\ \dfrac{2\epsilon}{2^p-2} & \text{for} \quad p > 1 \end{cases}$$

for each $x \in \mathbf{X}$ the transformation T such that $x \to f(x)$ is continuous, then the mapping T is linear. The result of Rassias is true for $1 < p$; Gajda gave an example to show that the above result failed to hold for $p = 1$. Rassias and Semrl (1992) gave an example to show that Hyers-Ulam stability does not occur for approximately linear mapping and also investigated the behaviour of such mapping by expressing the results in two directions, in the first result the researchers investigated the behavior of approximately linear mappings between Euclidean spaces and in the second result, they investigated the behavior of mappings that are approximately linear in the sense of inequality. After Hyers, many Mathematicians had extended the stability of functional equations extensively. These researchers include Forti (1995), Jung(1996), Hyers, Isac and Rassias (1998), Lee and Jun(1999), Rassias (2000), Jung(2001), Park (2002), Miura, Takahasi and Choda(2001), Gavruta(1994), Jun and Lee(2004), Park, Cho and Han(2007), Jung(2011) and a host of others. A generalisation of Ulam's problem recently proposed by replacing functional equations with the differential equation;

$$\alpha(f(t), u(t), u'(t), ...u^{(n)}(t)) = 0 \tag{2.157}$$

and stated that (2.157) has Hyers-Ulam stability if for a given $\epsilon > 0$ and a function $u(t)$ such that

$$|\alpha(f(t), u(t), u'(t), ...u^{(n)}(t))| \leq \epsilon \tag{2.158}$$

there exists a solution of $u_0(t)$ of the differential equation (2.157) satisfying

$$|u(t) - u_0(t)| \leq K(\epsilon)$$

and

$$\lim_{\epsilon \to 0} K(\epsilon) = 0$$

If ϵ and $K(\epsilon)$ is replaced by $\varphi(t)$ and $\eta(t)$, where φ, η are appropriate functions not depending on $u(t)$ and $u_0(t)$ explicitly, then the corresponding differential equation has the generalised Hyers-Ulam-Rassias stability.

Obloza(1993) and Obloza(1997) investigated Hyers-Ulams stability of the linear differential equation. Thereafter, Alsina and Ger(1998) handled the Hyers-Ulam stability of the linear differential equation

$$u'(t) = u(t) \tag{2.159}$$

It was stated that If a differentiable function $u(t)$ is a solution of the inequality

$$|u'(t) - u(t)| \leq \epsilon$$

for any $t \in (t_0, \infty)$, then there exits a positive constant c such that

$$|u(t) - c\exp(t)| \leq 3\epsilon$$

for all $t \in (a, \infty)$. The above results by Obloza, Alsina and Ger on linear differential equations were extended by several authors. These authors include: Miura, Takahasi and Choda(2001), Miura (2002) and Takahasi, Miura and Miyajima (2002) generalised the result of Alsina and Ger by considering Hyers-Ulam stability of the differential equation

$$u'(t) = \lambda u(t). \tag{2.160}$$

Jung(2007) investigated the generalised Hyers-Ulam stability or Hyers-Ulam-Rassias stability of the first order linear differential equation(2.160). Miura, Miyajima and Takahasi [Miura*et al*(2003) investigated the Hyers-Ulam stability of equation (2.160) as

$$u'(t) + g(t)u(t) = 0 \tag{2.161}$$

where λ is replaced with a continuous function $g(t)$,. While Jung (2004) proved the equation(2.159)in the form

$$\varphi(t)u'(t) = u(t). \tag{2.162}$$

Thereafter, Takahasi, Takagi, Miura and Miyajima (2004), Miura,Hirasawa, Takahasi (2004) and Jung (2006) discussed the Hyers-Ulam stability of the nonhomogeneous linear differential equation of first order

$$u'(t) + p(t)u(t) + q(t) = 0. \tag{2.163}$$

This is improved result of Jung(2004) and Miura*et al*(2003). Wang, Zhou and Sun(2008) considered the Hyers-Ulam stability of

$$q(t)u'(t) - p(t)u(t) - r(t) = 0 \tag{2.164}$$

which is an extension of equation(2.163). In another development Jung(2005) considered the generalised Hyers-Ulam stability of the following nonhomogeneous linear differential equations.

$$tu'(t) + \alpha u(t) + \beta t^r x_0 = 0, \tag{2.165}$$

the result obtained from equation (2.164) assisted the researcher to investigate further the Hyers-Ulam stability of second order linear differential equation

$$u''(t) + \alpha t u'(t) + \beta u(t) = 0. \tag{2.166}$$

Years later, Onitsuka and Shoji(2017) investigated Hyers-Ulam stability of the first order linear differential equation

$$u'(t) - au(t) = 0 \tag{2.167}$$

where a is a non-zero real number. Li and Shen (2009) investigated the Hyers-Ulam stability of differential equation of second order

$$u''(t) + p(t)u'(t) + r(t) = 0 \tag{2.168}$$

under some special conditions. The group of these researchers remarked that if f is an approximate solution of the equation(2.168), then there exists an exact solution of the equation near f. Gavruta, Jung and Li, (2011)discussed Hyers-Ulam stability of linear differential equation of second order of the form

$$u''(t) + \beta(t)u(t) = 0 \tag{2.169}$$

with the boundary conditions

$u(t_0) = u'(t'_0) = 0.$

or with initial condition

$u(t_0) = u'(t_0) = 0.$

Li and Shen(2010) investigated the Hyers-Ulam stability of the following linear differential equations of second order;

$$u''(t) + \alpha u'(t) + \beta u(t) = 0 \quad (2.170)$$

and

$$u''(t) + \alpha u'(t) + \beta u(t) = f(t). \quad (2.171)$$

by applied the condition that the characteristics equation has two different positive roots. The equation considered by Li and Shen generalised the equation considered by Gavruta, Jung and Li(2011). Xue(2014) made an improvement on the result of Li and Shen(2010) by investigating the Hyers-Ulam stability of equation (2.170) and (2.171), by not considering the nature of their characteristic roots either the roots are real or complex. The results obtained improved and extended the results of Li and Shen. Li(2010) was motivated by the result of the following researchers Takahas, Miura and Miyajim(2002) and Wang,Zhuo and sun (2008) which enabled Li to consider the stability in the sense of Hyers-Ulam stability of second order linear differential equation

$$u''(t) = \lambda^2 u. \quad (2.172)$$

Jung (2005) considered the Hyers-Ulam stability of the second order Euler equation of the form

$$t^2 u''(t) + \alpha t u'(t) + \beta u(t) = 0, \quad (2.173)$$

this equation is sometimes called the Cauchy equation. Jung(2010) solved the nonhomogeneous differential equation of the form

$$y''(t) + 2ty' - 2ny = \sum_{m=0}^{\infty} a_m t^m$$

where n is a nonnegative integer, and apply this result to the proof of a local Hyers-Ulam stability of the differential equation

$$y''(t) + 2ty' - 2ny = 0$$

in a special class of analytic function. In [Javadian *et al*,2010] the stability of an extension of the equation(2.168) was considered as equation

$$u''(t) + p(t)u'(t) + q(t)u(t) = f(t) \quad (2.174)$$

by considering its generalised Hyers-Ulam stability with the condition that there exits a nonzero $u_1 : \mathbf{I} \to u$ in $C^2(\mathbf{I})$ such that

$$u_1'' + p(t)u_1' + q(t)u_1 = 0 \tag{2.175}$$

has its solutions in an open interval. Furthermore, Popa and Rasa(2011) obtained some results on generalised Hyers-Ulam stability of the linear differential equation

$$u'(t) - \lambda(t)u(t) = f(t) \quad t \in \mathbf{I} \tag{2.176}$$

in a Banach spaces. Gavruta, Jun and Li (2011) investigated Hyers-Ulam stability of the linear differential equations

$$u'' + \beta(t)u = 0 \tag{2.177}$$

with boundary conditions and initial conditions. Abdollahpour *et al*(2012) proved the Hyers-Ulam stability of the perfect linear differential equation of the form

$$f(t)u''(t) + f_1(t)u'(t) + f_2(t)u(t) = Q(t) \tag{2.178}$$

by transforming the equation (2.178) to a perfect differential equation written as

$$\frac{d}{dt}[f'(t)u'(t) + (f_1'(t) - f'(t))u(t)] = Q(t) \tag{2.179}$$

after the transformation, Hyers-Ulam stability of the perfect equation obtained. Further development, Li and Huang(2013) proved the Hyers-Ulam stability of linear second order differential equations

$$u''(t) + \alpha u'(t) + \beta u(t) = 0 \tag{2.180}$$

and

$$u''(t) + \alpha u'(t) + \beta u(t) = f(t) \tag{2.181}$$

in complex Banach spaces. Modebei *et al*, (2014) investigated generalised Hyers-Ulam stability of second order ordinary differential nonhomogeneous equation

$$u''(t) + \beta(t)u(t) = f(t) \tag{2.182}$$

with initial condition

$$u(t_0) = u'(t_0) = 0$$

where $u(t) \in C^2[t_0, b], \beta \in C[t_0, b]$. Javadian *et al*(2011) investigated the generalised Hyers-Ulam stability of differential equations of the form

$$u''(t) + p(t)u'(t)q(t)u = f(t), \tag{2.183}$$

the author made the equation different from Li and Huang by introducing the continuous function $p(t)$. Jung, Kim and Rassias(2008) proved the Hyers-Ulam stability of a special type of systems of Euler differential equations of first order. Jung and Lee (2007) investigated Hyers-Ulam-Rassias stability of linear differential of second order In June 2015, Mohapatra(2015) proved the Hyers-Ulam stability and Hyer-Ulam-Aoki-Rassias or simply Hyers-Ulam-Rassias stability of the n-th order ordinary linear differential equation with smooth coefficients on compact and semi-bounded intervals using successive integration by part. Furthermore, the following author Li and Shen(2009) discussed much on Hyers-Ulam stability of nonhomogeneous linear differential equations of second order.

In another development, Tunc and Bicer(2013) discussed the stability in the sense of Hyers-Ulam stability of nonhomogeneous Euler equations of third and fourth order:

$$t^3u'''(t) + \alpha t^2u'''(t) + \beta tu''(t) + \gamma u(t) = F(t) \tag{2.184}$$

and

$$t^4u''''(t) + \alpha_1 t^3u'''(t) + \beta_1 t^2u''(t) + \gamma_1 tu'(t) + \zeta u(t) = G(t) \tag{2.185}$$

using transformation method. Murali and PonmanaSelvan(2018a) Investigated the Hyers-Ulam stability of the homogeneous linear differential equation of third order

$$u'''(t) + \alpha(t)u''(t) + \beta(t)u'(t) + (\gamma(t) - p(t))u(t) = 0 \tag{2.186}$$

with initial conditions

$$u(t_0) = u'(t_0) = u''(t_0) = 0$$

and boundary conditions with the help of Taylor's series Formula. Tripathy and Satapathy(2014) investigated the generalised Hyers-Ulam stability of third order Euler's differential equations of the form

$$t^3u'''(t) + \alpha t^2u''(t) + \beta tu'(t) + \gamma u(t) = 0 \tag{2.187}$$

on any open interval. Murali and Ponmana(2018b) investigated the Hyers-Ulam-Rassias stability of the homogenous and nonhomogeneous linear differential equation

$$u'''(t) + \alpha(t)u(t) = 0 \tag{2.188}$$

and

$$u'''(t) + \alpha(t)u(t) = \varphi(t) \tag{2.189}$$

by approximation method of solution. Abdollahpour *et al*(2012) investigated the Hyers-Ulam stability of the perfect linear differential equation

$$f(t)u''(t) + f_1(t)u'(t) + f_2(t)u(t) = Q(t) \tag{2.190}$$

by setting

$$f_2(t) = f_1'(t) - f''(t). \tag{2.191}$$

which the authors used to transform the equation (2.191) for easy investigation of stability via Hyers-Ulam stability. Abdollahpour and Najati(2011) proved that the

third order differential equation
$u'''(t) + \alpha u''(t) + \beta u'(t) + \gamma u(t) = f(t)$
has the Hyers-Ulam stability. In the following year, Abdollahpour and Najati(2012) investigated the Hyers-Ulam stability of the linear differential equation

$$u'''(t) - f(t)u''(t) + u'(t) - f(t)u(t) = H(t) \tag{2.192}$$

which extended the linear differential equation considered in 2011 by making the following transformations

$$g(t) = u''(t) + u(t), \; F(t) = \exp\left(\int_{t_0}^{t} f(s)ds\right)$$

$$z(t) = \frac{g(b)}{F(b)}F(t) - F(t)\int_{t}^{b} \frac{H(t)}{F(t)}dt$$

for all $t \in [t_0, b]$

Finally, Lee and Jun(2016)considered the generalisation of Hyers-Ulam-Rassias stability of Jensen equation. Li, Zada and Faisal(2016) investigated the Hyers-Ulam stability of nth order linear differential equations with non constant coefficients, the author proved that Hyers-Ulam stability by using open mapping theorem. The author further investigated generalised Hyers-Ulam stability of the same nth order linear differential equations.

2.3 Hyers-Ulam and Hyers-Ulam-Rassias Stability of Nonlinear Ordinary Differential Equations

Large number of authors studied Hyers-Ulam and Hyers-Ulam-Rassias stability of linear differential equations compared to nonlinear differential equations. In this section the results of some few authors who studied the Hyers-Ulam and Hyers-Ulam-Rassias stability of nonlinear differential equations are reviewed. Rus(2009) presented four types of Ulam stability for ordinary differential equations, they are: Hyers-Ulam stability, general Hyers-Ulam stability, Hyers-Ulam-Rassias stability and generalised Hyers-Ulam-Rassias stability. Rus (2009) gave the following the following equation and inequalities:

$$u'(t) = f(t, u(t)) \quad \text{forall } t \in [t_0, b), \tag{2.193}$$

$$|u'(t) - f(t, u(t))| \leq \varphi(t) \quad \text{for all } t \in [t_0, b), \tag{2.194}$$

$$|u'(t) - f(t, u(t))| \leq \epsilon \quad \text{for all } t \in [t_0, b), \tag{2.195}$$

and

$$|u'(t) - f(t, u(t))| \leq \epsilon\varphi(t) \quad \text{for all } t \in [t_0, b). \tag{2.196}$$

Rus(200) gave also the following definitions:

Definition 1.1

The equation (2.193) is Hyers-Ulam stable if there exists a real number $c_f > 0$ such that for each $\epsilon > 0$ and for each solution $u(t) \in C^1([a,b),\mathbf{B})$ of equation (2.193) there exists a solution $u_0(t) \in C^1([a,b),\mathbf{B})$ of equation (2.194) with

$|u(t) - u_0(t)| \leq C_f\epsilon,\ \forall t \in [a,b)$

Definition 1.2

The equation (2.193) is generalised Hyers-Ulam stable if there exists $\theta_f \in C(\mathbf{R},\mathbf{R}), \theta_f(0) = 0$, such that for each solution $u(t) \in C^1([a,b),\mathbf{B}$ of equation (2.195) there exists a solution $u(t) \in C^1([a,b),\mathbf{B})$ of equation (2.193) with

$|u(t) - u_0(t)| \leq \beta_f(\epsilon), \quad \forall t \in [a,b)$

Definition 1.3

The equation (2.193) is Hyers-Ulam-Rassias stable with respect to φ if there exists $C_{f,\varphi} > 0$ such that for each $\epsilon > 0$ and for each solution $u(t) \in C^1([a,b),\mathbf{B})$ of equation (2.196) there exists a solution $u(t) \in C^1([a,b),\mathbf{B})$ of equation (2.193) with

$|u(t) - u_0(t)| \leq C_{f,\varphi}\epsilon\varphi(t), \quad \forall t \in [a,b)$

Definition 1.4

The equation (2.193) is generalised Hyers-Ulam-Rassias stable with respect to φ if there exists $C_{f,\varphi} > 0$ such that for each solution $u(t) \in C^1([a,b),\mathbf{B})$ of equation (2.196) there exists a solution $u_0(t) \in C^1([a,b),\mathbf{B})$ of equation (2.193) with

$|u(t) - u_0(t)| \leq C_{f,\varphi}\varphi(t), \quad \forall t \in [a,b).$

Rus(2010) used these definitions to investigate the stability of the nonlinear differential equation

$$u'(t) = A(u(t)) + f(t,u(t)), \quad t \in \mathbf{I} \subset \mathbf{R}. \tag{2.197}$$

Qarqwani(2012a) investigated the stability of a generalised nonlinear second order differential equation

$$u''(t) - F(t,u(t)) = 0 \tag{2.198}$$

with the initial condition

$u(t_0) = u'(t_0) = 0$

by making use of Gronwall lemma. In addition, the author proved the Hyers-Ulam stability of a special case of equation (2.198) called Emden-Fowler type equation of the form

$$u''(t) - h(t)\,|u|^{\alpha}\, sgnu = 0 \tag{2.199}$$

with initial conditions

$u(t_0) = u'(t_0) = 0.$

Qarawani(2012b), further the investigation of Hyers-Ulam stability of nonlinear differential equation

$$u'' + p(t)u' + q(t)u = h(t)\,|u|^{\beta}\, \mathrm{e}^{(\frac{\beta-1}{2})\int p(t)dt} sgnu \quad \beta \in (0,1) \tag{2.200}$$

with the initial conditions

$u(t_0) = u'(t_0) = 0.$

Qarawani(2013) improved on the earlier result by considering Hyers-Ulam stability of nonlinear differential equation

$$u^n = f(t, u(t), u'(t), u''(t) \cdots u^{(n-1)}) \tag{2.201}$$

with initial conditions

$u(t_0) = u_0, u'(t_0) = u_1 \cdots, u^{(n-1)}(t_0) = u_{n-1} = 0.$

In Qarawani(2014) stability of nonlinear differential equations of first order in the sense of Hyers-Ulam-Rassias stability was considered using equation

$$u' + p(t)u = G(t, u) \tag{2.202}$$

with initial condition

$u(t_0) = u_0 = 0$

. The author also considered the Hyers-Ulam-Rassias stability of Bernoulli's equation

$$u' + p(t)u = q(t)u^n. \tag{2.203}$$

Alqifiary and Jung(2014) used Grownwall inequality to investigate the Hyers-Ulam stability of second order differential equations

$$u''(t) + F(t, u(t)) = 0. \tag{2.204}$$

Huang*et al*(2015) investigated the Hyers-Ulam stability of nonlinear differential equations of the form

$$u^n(t) = F\left(t, u(t), u'(t), \cdots, u^{(n-1)}(t)\right) \tag{2.205}$$

by applying Lipschitz condition and fixed point method. In 2016, Li *et al*(2016) established Hyers-Ulam stability of

$$\sum_{i=0}^{n} \beta_{n-i}(t)u^{(n-i)}(t) = \alpha(t) \tag{2.206}$$

and the generalised Hyers-Ulam stability of

$$u^{(n)}(t) + \sum_{i=0}^{n} Q_{n-i}(t) u^{n-1}(t) = f(t) \tag{2.207}$$

by using basic theory of differential equation. Ravi*et al*(2016) investigated the Hyers-Ulam stability of a general n^{th} order nonlinear differential equation of the form

$$u^{(n)} - F(t, u(t)) = 0 \tag{2.208}$$

with the initial condition

$$u(t_0) = u_0, u'(t_0) = u_1 \cdots, u^{(n-1)}(t_0) = u_{n-1} = 0,$$

the author also proved the Hyers-Ulam stability of the Emden-Fowler type of non-linear differential equation of n^{th} order

$$u^{(n)}(t) - h(t) \left| u(t) \right|^{\alpha} sgnu(t) = 0. \tag{2.209}$$

Recently, Bicer and Tunc(2018) investigated the Hyers-Ulam stability of second order nonlinear differential equation with multiple variable time lags of the form

$$\frac{d^2u}{dt^2} + F(t, u(t))\frac{du}{dt} + H(t, u(t)) = 0. \tag{2.210}$$

by using fixed point theorem.

In this research work, our results on Hyers-Ulam and Hyers-Ulam-Rassias stability will extend all the results of aforementioned authors.

CHAPTER THREE

METHODOLOGY

3.0 Introduction

The origin of integral inequalities to be used here could be traced to the Gronwall inequality which is embraced by many researchers. This also opened the usefulness of integral inequalities to the world of mathematicians. The development of the integral inequalities are considered under the following headings:
Integral inequality with one nonlinear term
Integral inequality with two nonlinear terms
Integral inequality with three nonlinear terms

3.0.1 Integral Inequality with One Nonlinear Term

This section is devoted to consider the development of integral inequalities with two terms of integrals, one containing a linear term and the other a nonlinear term.

Theorem 3.1:
Suppose $u(t)$ and $f(t)$ are nonnegative, and continuous functions on $\mathbf{R}_+$. Suppose that $K(t,s)$ and its partial derivative $K_t(t,s)$ exist and are continuous for every $t,s \in \mathbf{I}$ and $K_t(t,s) \leq 0$. Moreover, let $\varpi \in C(\mathbf{R}_+, \mathbf{R}_+)$ be nondecreasing with $\varpi(u) > 0$ on $\mathbf{R}_+$ for which the inequality

$$u(t) \leq u_0 + L\int_{t_0}^{t} f(s)u(s)ds + \int_{t_0}^{t} f(s)\left(\int_{t_0}^{s} K(s,\tau)\varpi(u(\tau))d\tau\right)ds \tag{3.1}$$

holds, where u_0 and L are positive constants then,

$$u(t) \leq \Omega^{-1}\left(\Omega(u_0) + L\int_{t_0}^{t} f(s)K(s,s)\left(u_0Q(s) + \int_{t_0}^{s} Q(\alpha)d\alpha\right)ds\right), \tag{3.2}$$

where,

$$Q(t) = \exp L\int_{t_0}^{t} f(s)ds, \tag{3.3}$$

$\Omega(u)$ is defined as equation(2.13) and $t_1 \in [t_0, \infty)$ is chosen so that

$$\left(\Omega(u_0) + L\int_{t_0}^{t} f(s)K(s,s)\left(u_0Q(s) + \int_{t_0}^{s} Q(\alpha)d\alpha\right)ds\right) \in Dom(\Omega^{-1})$$

for all t lying in the subinterval $[t_0, t_1]$ of $\mathbf{I}$. Ω^{-1} is the inverse of the function Ω,. Note Ω^{-1} is also a nondecreasing function.

Proof:

Since $u_0 > 0$ and denote the r.h.s of inequality (3.1) by $z(t)$;

$$z(t) = u_0 + L\int_{t_0}^{t} f(s)u(s)ds + \int_{t_0}^{t} f(s)\left(\int_{t_0}^{s} K(s,\tau)\varpi(u(\tau))d\tau\right)ds. \tag{3.4}$$

From equation (3.4) and inequality (3.1) we have

$$u(t) \le z(t),\ z(t_0) = u_0 \tag{3.5}$$

Differentiate (3.4) with respect to t yields

$$z'(t) = Lf(t)u(t) + f(t)\int_{t_0}^{t} K(t,\tau)\varpi(u(\tau))d\tau \tag{3.6}$$

Using (3.5) in (3.6) the result is

$$z'(t) \le Lf(t)\left(u(t) + f(t)\int_{t_0}^{t} K(t,\tau)\varpi(u(\tau))d\tau\right).$$

It is clear that

$$z'(t) \le Lf(t)M(t), \tag{3.7}$$

where

$$M(t) = z(t) + \int_{t_0}^{t} K(t,\tau)\varpi(z(\tau))d\tau. \tag{3.8}$$

Differentiate (3.8) we have

$$M'(t) = z'(t) + K(t,t)\varpi(z(t)) + \int_{t_0}^{t} \frac{\partial}{\partial t}K(t,\tau)\varpi(z(\tau))d\tau.$$

Since $K(t,s)$ and its partial derivative $K_t(t,s)$ exist and are continuous with $K_t(t,s) \le 0$, the result is

$$M'(t) \le z'(t) + P(t), \tag{3.9}$$

where

$$P(t) = K(t,t)\varpi(z(t)),\ t \in [t_0, \infty). \tag{3.10}$$

Let $T \in [t_0, \infty)$ be any arbitrary number such that

$$M'(t) \le Lf(t)M(t) + P(T) \text{ forall } t_0 \le t \le T \tag{3.11}$$

Solving the first order differential inequality (3.11)by using integrating factor defined as

$$\mu(t) = \exp(-L\int_{t_0}^{t} f(s)ds), \tag{3.12}$$

the result yields

$$M(T) \le u_0 \exp(L\int_{t_0}^{T} f(s)ds) + P(T)\left(\int_{t_0}^{T}\left(\exp(L\int_{t_0}^{T} f(\tau)d\tau)\right)ds\right). \tag{3.13}$$

Substituting $M(T)$ in equation (3.7) yields

$$z'(T) \le Lf(T)\left(u_0 \exp(L\int_{t_0}^{T} f(s)ds) + P(T)\int_{t_0}^{T}\left(\exp(L\int_{t_0}^{T} f(\tau)d\tau)\right)ds\right) \tag{3.14}$$

Let $P(T) > 1$, then
$$z'(T) \le LP(T)f(T)\left(u_0 \exp(L\int_{t_0}^{t} f(s)ds) + \int_{t_0}^{t}\left(\exp(L\int_{t_0}^{t} f(\tau)d\tau)\right)ds\right) \quad (3.15)$$
By equation (2.13), letting $z_0 = z(t_0)$, $t = T$, then
$$\frac{d\Omega(z(t))}{dt} \le Lf(t)K(t,t)\left(u_0 Q(t) + \int_{t_0}^{t} Q(s)ds\right)$$
$Q(t)$ is defined in equation (3.3), integrating equation(3.15) from t_0 to t we get
$$z(t) \le \Omega^{-1}\left(\Omega(u_0) + L\int_{t_0}^{t} f(s)K(s,s)\left(u_0 Q(s) + \int_{t_0}^{s} Q(\alpha)d\alpha\right)ds\right)$$
Since $u(t) \le z(t)$, we arrive at the result (3.4).

Theorem 3.2:

Let all the conditions of Theorem 3.1 remain valid and let $n(t)$ be nonnegative, nondecreasing, monotonic continuous function on $\mathbf{R}_+$, then the inequality
$$u(t) \le n(t) + L\int_{t_0}^{t} f(s)u(s)ds + \int_{t_0}^{t} f(s)\left(\int_{t_0}^{s} K(s,\tau)\varpi(u(\tau))d\tau\right)ds \quad (3.16)$$
holds, L a positive constant, then
$$u(t) \le n(t)\Omega^{-1}\left(\Omega(1) + L\int_{t_0}^{t} f(s)K(s,s)\left(Q(s) + \int_{t_0}^{s} Q(\alpha)d\alpha\right)ds,\right) \quad (3.17)$$
where $\Omega(u)$ is defined in (2.13), $t_1 \in [t_0, \infty)$ is chosen so that
$$\left(\Omega(1) + L\int_{t_0}^{t} f(s)K(s,s)\left(Q(s) + \int_{t_0}^{s} Q(\alpha)d\alpha\right)ds\right) \in Dom(\Omega^{-1}),$$
for all $t \in [t_0, t_1] \subset \mathbf{I}$. Ω^{-1} a nondecreasing function and the inverse of the function Ω.

Proof:

Since $n(t)$ is positive, monotonic, nondecreasing continuous function on $\mathbf{R}_+$.By putting $z(t) = \frac{u(t)}{n(t)}$ in equation (3.16) we have
$$\frac{u(t)}{n(t)} \le 1 + L\int_{t_0}^{t} f(s)\frac{u(s)}{n(s)}ds + \int_{t_0}^{t} f(s)\left(\int_{t_0}^{s} K(s,\tau)\varpi\left(\frac{u(\tau)}{n(\tau)}\right)d\tau\right)ds \quad (3.18)$$
Let $z(t) = \dfrac{u(t)}{n(t)}$, equation (3.18) is written as
$$z(t) \le 1 + L\int_{t_0}^{t} f(s)z(s)ds + \int_{t_0}^{t} f(s)\left(\int_{t_0}^{s} K(s,\tau)\varpi(z(\tau))d\tau\right)ds \quad (3.19)$$
By applying the Theorem 3.1 we obtain
$$z(t) \le \Omega^{-1}\left(\Omega(1) + L\int_{t_0}^{t} f(s)K(s,s)\left(Q(s) + \int_{t_0}^{s} Q(\alpha)d\alpha\right)ds\right).$$
Substituting $z(t)$ in the above equation, then, we arrive at inequality (3.19)

Remark 3.1:

The results obtained, extended the result obtained by Oguntuase (2001) and Pachpatte (1975a).

Consequence of Theorems 3.1 and Theorem 3.2 is given as thus:

Theorem 3.3:

Let $K(t,s) = r(t)p(s)$, where $r(t)$ and $p(s)$ are continuous on $\mathbf{R}_+$, and $r'(t) \le$

0, $p'(t) \leq 0$ and $u_0 > 0$ a constant. Then, the inequality (3.3) in Theorem 3.1 reduce to

$$u(t) \leq u_0 + L\int_{t_0}^{t} f(s)u(s)ds + \int_{t_0}^{t} f(s)r(s)\left(\int_{t_0}^{s} p(\tau)\varpi(u(\tau))d\tau\right)ds \tag{3.20}$$

by Theorem 3.2, we have

$$u(t) \leq \Omega^{-1}\left[\Omega(u_0) + L\int_{t_0}^{t} f(s)r(s)p(s)\left[u_0Q(s) + \int_{t_0}^{s} Q(\tau)d\tau\right]ds\right], \tag{3.21}$$

if we choose $t_1 \in \mathbf{I}$ so that

$$\Omega(u_0) + L\int_{t_0}^{t} f(s)r(s)p(s)\left[u_0Q(s) + \int_{t_0}^{s} Q(\tau)d\tau\right]ds \in Dom(\Omega^{-1}) \tag{3.22}$$

for all t lying in the interval $[t_0, t_1] \subset \mathbf{I}$, where Ω^{-1} be the inverse of Ω. Where $\Omega(u)$ and $Q(t)$ are defined in (2.13) and (3.3) respectively.

Proof:

Define the function $v(t)$ as

$$v(t) = u_0 + L\int_{t_0}^{t} f(s)u(s)ds + \int_{t_0}^{t} f(s)r(s)\left(\int_{t_0}^{s} p(\tau)\varpi(u(\tau))d\tau\right)ds \tag{3.23}$$

It is clear that from equation (3.23) we have

$$u(t) \leq v(t), \quad v(t_0) = u_0 \tag{3.24}$$

Differentiate (3.23) we obtain

$$v'(t) = Lf(t)v(t) + f(t)r(t)\int_{t_0}^{t} p(\tau)\varpi(v(\tau))d\tau.$$

Using inequality (3.24) we arrive at

$$v'(t) \leq Lf(t)\left[v(t) + r(t)\int_{t_0}^{t} p(\tau)\varpi(v(\tau))\right]d\tau \tag{3.25}$$

Inequality (3.25) reduces to

$$v'(t) \leq Lf(t)M(t), \tag{3.26}$$

where $M(t)$ is defined as

$$M(t) = v(t) + r(t)p(t)\int_{t_0}^{t} \varpi(v(\tau))d\tau. \tag{3.27}$$

Differentiate (3.27) respect to t we nave

$$M'(t) = v'(t) + r'(t)p(t)\int_{t_0}^{t} \varpi(v(\tau))d\tau + p(t)r(t)\int_{t_0}^{t} \varpi(v(\tau))d\tau$$
$$+r(t)p(t)\varpi(v(\tau)).$$

Since $r'(t) \leq 0$, and $p'(t) \leq 0$ we get

$$M'(t) = v'(t) + r(t)p(t)\varpi(v(t)). \tag{3.28}$$

Since $M(t)$ is nondecreasing and nonnegative function on $\mathbf{R}_+$,without loss of generality, from inequality (3.27) we obtain

$$v(t) \leq M(t) \tag{3.29}$$

Using (3.29) in (3.28) to have

$$M'(t) \leq Lf(t)M(t) + R(t) \tag{3.30}$$

where

$$R(t) = r(t)p(t)\varpi(v(t)) \tag{3.31}$$

Now, let $T \in \mathbf{I}$ be any arbitrary number let $R(T) > 1$, then equation (3.30) becomes

$$M'(t) \leq Lf(t)M(t) + R(T) \tag{3.32}$$

Using integrating factor defined in equation (3.12),we get

$$M(T) \leq u_0 \exp(L\int_{t_0}^{T} f(s)ds) + R(T)\left(\int_{t_0}^{T}\left(\exp(L\int_{t_0}^{T} f(\tau)d\tau)\right)ds\right). \tag{3.33}$$

Substituting $M(t)$ in (3.30), we obtain

$$v'(T) \leq Lf(T)\left(u_0 \exp(L\int_{t_0}^{T} f(s)ds) + R(T)\int_{t_0}^{T}\left(\exp(L\int_{t_0}^{T} f(\tau)d\tau)\right)ds\right) \tag{3.34}$$

Let $R(T) > 1$, then

$$v'(t) \leq LR(t)f(t)\left(u_0 \exp(L\int_{t_0}^{t} f(s)ds) + \int_{t_0}^{t}\left(\exp(L\int_{t_0}^{t} f(\tau)d\tau)\right)ds.\right) \tag{3.35}$$

If $t = T$, equation (3.35) becomes

$$v'(t) \leq LR(t)f(t)\left[u_0 Q(t) + \int_{t_0}^{t} Q(s)ds\right] \tag{3.36}$$

By equation(3.31) with equation(2.13), we obtain

$$\frac{d\Omega(v(t))}{dt} \leq Lf(t)r(t)g(t)\left[u_0 Q(t) + \int_{t_0}^{t} Q(s)ds\right] \tag{3.37}$$

Integrating (3.37) from t_0 to t,

$$v(t) \leq \Omega^{-1}\left[\Omega(u_0) + \int_{t_0}^{t} f(s)r(s)p(s)\left[u_0 Q(s) + \int_{t_0}^{s} Q(\tau)d\tau\right]ds\right] \tag{3.38}$$

Using this in (3.24), we arrive at the inequality (3.21).

Theorem 3.4:

Let all the conditions of Theorem 3.1 remain valid and let $n(t)$ be nonnegative, nondecreasing, monotonic continuous function on $\mathbf{R}_+$, the inequality

$$u(t) \leq n(t) + L\int_{t_0}^{t} f(s)u(s)ds + \int_{t_0}^{t} f(s)\left(\int_{t_0}^{s} K(s,\tau)\varpi(u(\tau))d\tau\right)ds \tag{3.39}$$

holds, L is a positive constant, then

$$u(t) \leq n(t)\Omega^{-1}\left(\Omega(1) + L\int_{t_0}^{t} f(s)K(s,s)\left(Q(s) + \int_{t_0}^{s} Q(\alpha)d\alpha\right)ds\right) \tag{3.40}$$

where $\Omega(u)$ is defined in (2.13), $t_1 \in [t_0, \infty)$ is chosen so that

$$\left(\Omega(1) + L\int_{t_0}^{t} f(s)K(s,s)\left(Q(s) + \int_{t_0}^{s} Q(\alpha)d\alpha\right)ds\right) \in Dom(\Omega^{-1})$$

for all $t \in [t_0, t_1] \subset \mathbf{I}$. Ω^{-1} a nondecreasing function and the inverse of the function Ω.

Proof:

Since $n(t)$ is positive, monotonic, nondecreasing continuous function on $\mathbf{R}_+$. Then equation (3.11) become

$$\frac{u(t)}{n(t)} \leq 1 + L\int_{t_0}^{t} f(s)\frac{u(s)}{n(s)}ds + \int_{t_0}^{t} f(s)\left(\int_{t_0}^{s} K(s,\tau)\varpi\left(\frac{u(\tau)}{n(\tau)}\right)d\tau\right)ds \tag{3.41}$$

Let $z(t) = \dfrac{u(t)}{n(t)}$, equation (3.18) reduce to

$$z(t) \le 1 + L\int_{t_0}^{t} f(s)z(s)ds + \int_{t_0}^{t} f(s)\left(\int_{t_0}^{s} K(s,\tau)\varpi(z(\tau))d\tau\right)ds \tag{3.42}$$

By applying Theorem 3.1 we have

$$z(t) \le \Omega^{-1}\left(\Omega(1) + L\int_{t_0}^{t} f(s)K(s,s)\left(Q(s) + \int_{t_0}^{s} Q(\alpha)d\alpha\right)ds\right)$$

Substituting $z(t)$ in the above equation, then, we arrive at inequality (3.21)

Remark 3.2:

If $r(t) = 1$, $\varpi(u) = u$, and $L = 1$ in Theorem 3.1, the result obtained is the same as in Pachpatte Pachpatte(1975a). Furthermore, these results also extended Pachpatte(1975a) and Oguntuase(2001). Let $\Re(u)$ be defined as

$$\Re(u) = \int_{r_0}^{r} \frac{ds}{2s + \varpi(s)}. \tag{3.43}$$

Theorem 3.5:

Let $u(t)$, $r(t)$ be nonnegative, nondecreasing and continuous functions on $C(\mathbf{I}, \mathbf{R}_+)$ and $r'(t) \le 0$. Suppose

$$u(t) \le u_0 + N\int_{t_0}^{t} f(s)u(s)ds + M\int_{t_0}^{t} f(s)r(s)\left(\int_{t_0}^{s} g(\tau)\varpi(u(\tau))d\tau + \int_{t_0}^{s} g(\tau)(u(\tau))d\tau\right)ds, \tag{3.44}$$

holds, where N, M are positive constants, then,

$$u(t) \le \Re^{-1}\left(\Re(u_0) + C\int_{t_0}^{t} f(s)r(s)g(s)ds\right) \tag{3.45}$$

for $C = NM > 0$, where $u_0 > 0$ and $\Re(u)$ is given as (3.43). For $t_1 \in \mathbf{I}$ chosen such that

$$\left(\Re(u_0) + C\int_{t_0}^{t} f(s)r(s)g(s)ds\right) \in Dom(\Re^{-1}),$$

for all t lying in the interval $[t_0, t_1] \in \mathbf{I}$. $\Re^{-1}$ is the inverse of $\Re$.

Proof Define

$$z(t) = u_0 + N\int_{t_0}^{t} f(s)u(s)ds + M\int_{t_0}^{t} f(s)r(s)\left(\int_{t_0}^{s} g(\tau)\varpi(u(\tau))d\tau + \int_{t_0}^{s} g(\tau)(u(\tau))d\tau\right)ds, \tag{3.46}$$

then

$$u(t) \le z(t),\ z(t) = u_0 \tag{3.47}$$

Differentiating (3.46) to get

$$z(t) = Nf(t)u(t) + Mf(t)r(t)\left(\int_{t_0}^{t} g(\tau)\varpi(u(\tau))d\tau + \int_{t_0}^{t} g(\tau)(u(\tau))d\tau\right),$$

Applying equation(3.47) to have

$$z'(t) \le Nf(t)(z(t) + Mf(t)r(t)\left(\int_{t_0}^{t} g(\tau)\left(\varpi(z(\tau)) + z(\tau)\right)d\tau\right).$$

Simplify further to obtain

$$z'(t) \le Cf(t)\left(z(t) + r(t)\left(\int_{t_0}^{t} g(\tau)\left(\varpi(z(\tau)) + z(\tau)\right)d\tau\right)\right), \tag{3.48}$$

where $C = NM$. Inequality (3.48) reduce to

$$z'(t) \le Cf(t)m(t) \tag{3.49}$$

where

$$m(t) = z(t) + r(t)\left(\int_{t_0}^{t} g(\tau)\left(\varpi(z(\tau)) + z(\tau)\right)d\tau\right). \tag{3.50}$$

Without loss of generality

$$z(t) \le m(t) \tag{3.51}$$

Differentiating the equation (3.50)

$$m.(t) = z'(t) + r'(t)\left(\int_{t_0}^{t} g(\tau)\left(\varpi(z(\tau)) + z(\tau)\right)d\tau\right) + r(t)(g(t)\varpi(z(t) + z(t)).$$

Put $r'(t) \le 0$, by inequalities (3.49) and (3.51), we have

$$m'(t) \le Cf(t)m(t) + r(t)g(t)\left(\varpi(m(\tau)) + m(t)\right) \tag{3.52}$$

Let $f(t) > 1,\ r(t) > 1$ and $g(t) > 1$, to have

$$\frac{m'(t)}{2m(t) + \varpi(m(\tau))} \le Cf(t)r(t)g(t) \tag{3.53}$$

By equation(3.43), we arrive at

$$m(t) \le \Omega^{-1}\left(\Omega(u_0) + C\int_{t_0}^{t} f(s)r(s)g(s)ds\right) \tag{3.54}$$

Using inequality (3.51)

$$z(t) \le \Omega^{-1}\left(\Omega(u_0) + C\int_{t_0}^{t} f(s)r(s)g(s)ds\right) \tag{3.55}$$

By inequality (3.47), we arrive at the result.

Let u_0 in Theorem 3.5 be replaced with function $L(t)$ a nondecreasing, nonnegative, monotonic continuous function on $\mathbf{R}_+$. Then the next result follows:

Theorem 3.6:

Let $u(t),\ r(t),\ L(t)$ be nonnegative, nondecreasing and continuous functions on $C(\mathbf{I}, \mathbf{R}_+)$ and $r'(t) \le 0$. Suppose $\varpi \in \Psi$. If

$$u(t) \le L(t) + N\int_{t_0}^{t} f(s)u(s)ds + M\int_{t_0}^{t} f(s)r(s)\left(\int_{t_0}^{s} g(\tau)\varpi(u(\tau))d\tau + \int_{t_0}^{s} g(\tau)(u(\tau))d\tau\right)ds, \tag{3.56}$$

holds, then the estimate of $u(t)$ is given as

$$u(t) \le L(t)\Omega^{-1}\left(\Omega(1) + C\int_{t_0}^{t} f(s)r(s)g(s)ds\right) \tag{3.57}$$

where $L(t)$, nondecreasing, nonnegative monotonic function on $\mathbf{R}_+$ and the $\Re(u)$ is defined in (3.43)

$t_1 \in \mathbf{I}$ chosen such that

$$\left(\Re(1) + C\int_{t_0}^{t} f(s)r(s)g(s)ds\right) \in Dom(\Re^{-1}).$$

for all t lying in the interval $[t_0, t_1] \in \mathbf{I}$. $\Re^{-1}$ is the inverse of $\Re$.

Proof:

Since $L(t)$ is nondecreasing, nonnegative, monotonic function and $\varpi \in \Psi$, then inequality (3.56) reduce to

$$\frac{u(t)}{L(t)} \leq 1 + N\int_{t_0}^{t} f(s)\left(\frac{u(s)}{L(s)}\right) ds + M\int_{t_0}^{t} f(s)r(s)\left(\int_{t_0}^{s} g(\tau)\varpi\left(\frac{u(\tau)}{L(\tau)}\right) d\tau + \int_{t_0}^{s} g(\tau)\left(\frac{u(\tau)}{L(\tau)}\right) d\tau\right) ds. \tag{3.58}$$

Let inequality (3.58) be

$$P(t) \leq 1 + N\int_{t_0}^{t} f(s)P(s)ds + M\int_{t_0}^{t} f(s)r(s)\left(\int_{t_0}^{s} g(\tau)\varpi(P(\tau))d\tau + \int_{t_0}^{s} g(\tau)P(\tau)d\tau\right) ds. \tag{3.59}$$

Where

$$\frac{u(t)}{L(t)} = P(t). \tag{3.60}$$

The inequality (3.59) satisfies the same condition with the inequality (3.44). Then by applying Theorem 3.5, we obtain

$$P(t) \leq \Re^{-1}\left(\Re(1) + \int_{t_0}^{t} f(s)r(s)g(s)ds\right) \tag{3.61}$$

Replacing $P(t)$ with (3.60), we arrive at the result of the theorem.

3.0.2 Integral Inequality with Two Nonlinear Terms

This section begins with the consideration of integral inequalities derived from Dehongade and Deo (1973) results.

Theorem 3.7:

Suppose $u(t), r(t), h(t) \in C(\mathbf{I}, \mathbf{R}_+)$ and $\varpi(u), \beta(u) \in \Psi$ be nonnegative, monotonic, nondecreasing, continuous and $\omega(u)$ be a submultiplicative function for $u > 0$. Let

$$u(t) \leq E + T\int_{t_0}^{t} r(s)\beta(u(s))ds + L\int_{t_0}^{t} h(s)\varpi(u(s))ds \tag{3.62}$$

for E, T and L are positive constants, then

$$u(t) \leq \Omega^{-1}\left(\Omega(E) + L\int_{t_0}^{t} h(s)\varpi\left(F^{-1}\left(F(1) + B(s)\right)\right) ds\right) F^{-1}\left(F(1) + B(t)\right), \tag{3.63}$$

where $\beta(u) \neq \varpi(u)$,

$$B(t) = T\int_{t_0}^{t} r(s)ds, \tag{3.64}$$

Ω is defined in equation (2.13) and $F(u)$ is defined in equation (2.40) Where F^{-1}, Ω^{-1} are the inverses of F, Ω respectively and t is in the subinterval $(0, b) \in \mathbf{I}$,
$F(1) + B(t) \in Dom(F^{-1})$
and
$\Omega(E) + L\int_{t_0}^{t} h(s)\varpi\left(F^{-1}\left(F(1) + B(s)\right)\right) ds \in Dom(\Omega^{-1})$

Proof:

Define

$$q(t) = E + L\int_{t_0}^{t} h(s)\varpi(u(s))ds \quad t \in \mathbf{I}. \tag{3.65}$$

We write inequality (3.62) as

$$u(t) \leq q(t) + T\int_{t_0}^{t} r(s)\beta(u(s))ds \quad t \in \mathbf{I}. \tag{3.66}$$

$q(t)$ is nonnegative,nondecreasing, monotonic and continuous function on $\mathbf{R}_+$. By Theorem 2.9 and inequality (2.40), we obtain

$$u(t) \leq q(t)F^{-1}\left(F(1) + B(t)\right), \quad t \in \mathbf{I}. \tag{3.67}$$

Since ϖ is submultiplicative, we have

$$u(t) \leq \varpi(q(t)F^{-1}\left(F(1) + B(t)\right)), \quad t \in \mathbf{I}.$$

It clear that

$$u(t) \leq \varpi(q(t))\varpi\left(F^{-1}\left(F(1) + B(t)\right)\right), \quad t \in \mathbf{I}.$$

It follows that

$$\frac{\varpi(u(t))Lh(t)}{\varpi(q(t))} \leq Lh(t)\varpi\left(F^{-1}\left(F(1) + B(t)\right)\right) \tag{3.68}$$

By(2.13) we obtain

$$q(t) \leq \Omega^{-1}\left(\Omega(E) + L\int_{t_0}^{t} h(s)\varpi\left(F^{-1}(F(1) + B(s))\right) ds\right) \tag{3.69}$$

Substituting for $q(t)$ in (3.67), this concludes the proof.

corollary 3.1

Suppose $\rho(t)$ is a nonnegative, monotonic, nondecreasing continuous function on $\mathbf{R}_+$. Let

$$u(t) \leq \rho(t) + T\int_{t_0}^{t} r(s)\beta(u(s))ds + L\int_{t_0}^{t} h(s)\varpi(u(s))ds \tag{3.70}$$

for T and L are positive constants, then

$$\begin{aligned} u(t) \leq \rho(t)\Omega^{-1}\Bigg(\Omega(1) + L\int_{t_0}^{t} h(s)\varpi\big(F^{-1}\big(F(1) \\ +B(s)\big)\big) ds\Bigg) F^{-1}\left(F(1) + B(t)\right) \quad t \in \mathbf{I} \end{aligned} \tag{3.71}$$

where $B(t)$, $\Omega(u)$ and $F(u)$ are defined as in (3.64), (2.13) and (2.40) respectively.

Proof:

Since $\rho(t)$ is nonnegative, monotonic, nondecreasing on $\mathbf{R}_+$, with ϖ, $\beta \in \Psi$ and then, we write (3.70) as

$$\frac{u(t)}{\rho(t)} \leq 1 + T\int_{t_0}^{t} r(s)\beta\left(\frac{u(s)}{\rho(s)}\right)ds + L\int_{t_0}^{t} h(s)\varpi\left(\frac{u(s)}{\rho(s)}\right)ds \tag{3.72}$$

Setting

$$\frac{u(t)}{\rho(t)} = z(t) \tag{3.73}$$

Using(3.83) in (3.72), we obtain

$$z(t) \leq 1 + T\int_{t_0}^{t} r(s)z(s)ds + L\int_{t_0}^{t} h(s)\varpi(z(s))ds \tag{3.74}$$

Applying Theorem 3.7 to inequality (3.74) for = 1, we arrive at

$$z(t) \leq \Omega^{-1}\left(\Omega(1) + L\int_{t_0}^{t} h(s)\varpi\left(F^{-1}\left(F(1) + B(s)\right)\right)ds\right) \\ F^{-1}\left(F(1) + B(t)\right) \quad t \in \mathbf{I} \tag{3.75}$$

Substituting for $z(t)$ in inequality (3.75),this concludes the proof.

In the next theorems, we consider three terms of integrals where only one is linear and the other two are nonlinear.

Theorem 3.8:

Let $u(t), r(t), g(t), l(t) : \mathbf{I} \to \mathbf{R}_+$ be continuous and $\varpi(u), \beta(u) \in \Psi$ be nondecreasing, nonnegative and $u > 0$, $\beta(u)$ a submultiplicative. If

$$u(t) \leq u_0 + A\int_{t_0}^{t} r(s)u(s)ds + B\int_{t_0}^{t} g(s)\varpi(u(s))ds + C\int_{t_0}^{t} l(s)\beta(u(s))ds \quad t \in \mathbf{I} \tag{3.76}$$

holds. Then

$$u(t) \leq F^{-1}\left(F(u_0) + C\int_{t_0}^{t} l(s)\beta(\Lambda(s))Q(s)ds\right)\Lambda(t)Q(t) \tag{3.77}$$

where

$$\Lambda(t) = \Omega^{-1}\left(\Omega(1) + B\int_{t_0}^{t} g(s)\varpi\left(Q(s)\right)ds\right) \tag{3.78}$$

$Q(t)$ is defined in (3.5), $\Omega(r)$ is defined as (2.13) and $F(u)$ is defined in equation (2.40),

$$F\left(\Omega^{-1}\left(\Omega(1) + B\int_{t_0}^{t} g(\alpha)\varpi\left(Q(\alpha)\right)d\alpha\right)Q(s)\right) \in Dom(F^{-1})$$

and

$$\left(\Omega(1) + B\int_{t_0}^{s} g(\alpha)\varpi\left(Q(\alpha)\right)d\alpha\right) \in Dom(\Omega^{-1})$$

Proof:

Define

$$n(t) = u_0 + C\int_{t_0}^{t} l(s)\beta(u(s))ds \tag{3.79}$$

we re-write (3.76) as

$$u(t) \leq n(t) + A\int_{t_0}^{t} r(s)u(s)ds + B\int_{t_0}^{t} g(s)\varpi(u(s))ds \tag{3.80}$$

For $n(t)$ is a nondecreasing, nonnegative and monotonic function.

It follows that

$$\frac{u(t)}{n(t)} \leq 1 + A\int_{t_0}^{t} r(s)\left(\frac{u(s)}{n(s)}\right)ds + B\int_{t_0}^{t} g(s)\varpi\left(\frac{u(s)}{n(s)}\right)ds \tag{3.81}$$

By Theorem 2.8, we obtain

$$u(t) \leq n(t)\Omega^{-1}\left(\Omega(1) + B\int_{t_0}^{t} g(s)\varpi\left(Q(s)\right)ds\right)Q(t) \tag{3.82}$$

Since β is submultiplicative, we obtain

$$\beta(u(t)) \leq \beta(n(t))\beta\left(\Omega^{-1}\left(\Omega(1) + B\int_{t_0}^{t} g(s)\varpi\left(Q(s)\right)ds\right)Q(t)\right) \tag{3.83}$$

By applying the equation(2.40),we obtain

$$\begin{aligned} n(t) \leq F^{-1}\Bigg(F(u_0)) + C\int_{t_0}^{t} l(s)\beta\Big(\Omega^{-1}\Big(\Omega(1)+ \\ B\int_{t_0}^{s} g(\alpha)\varpi\left(Q(\alpha)\right)d\alpha\Big)Q(s)\Big)ds\Bigg) \end{aligned} \tag{3.84}$$

Replacing $n(t)$ in equation (3.82) with (3.84).

This conclues the proof.

Theorem 3.9:

Suppose $u(t),\ r(t),\ g(t),\ l(t) \in \mathbf{R}_+$ are continuous function. Further more,$\chi(t)$ be nondecreasing, monotonic, continuous function on $\mathbf{R}_+$ and $\varpi(u), \beta(u)$ belong to the class Ψ for $u > 0$. Let $\beta(u)$ be a submultiplicative. The inequality

$$u(t) \leq \chi(t) + A\int_{t_0}^{t} r(s)u(s)ds + B\int_{t_0}^{t} g(s)\varpi(u(s))ds + C\int_{t_0}^{t} l(s)\beta(u(s))ds \quad t \in \mathbf{I} \tag{3.85}$$

holds. Then

$$u(t) \leq \chi(t)H^{-1}\left(H(1) + C\int_{t_0}^{t} l(s)\beta\left(\Lambda(s)Q(s)\right)ds\right)\Lambda(t)Q(t) \tag{3.86}$$

Where $\Lambda(t)$ and $Q(t)$ are defined in equations (3.78)and (3.5) respectively, also $\Omega(u)$ and $F(r)$ are defined in (2.13) and (2.40) respectively. where

$$F(1) + C\int_{t_0}^{t} l(s)\beta\left(\Lambda(s)Q(s)\right)ds \in Dom(F^{-1})$$

and

$$\left(\Omega(1) + B\int_{t_0}^{s} g(\alpha)\varpi\left(Q(\alpha)\right)d\alpha\right) \in Dom(\Omega^{-1})$$

Proof:

Since $\chi(t)$ is nondecreasing,nonnegative monotonic function and $\beta.\varpi \in \Psi$ from (3.85), we have

$$\begin{aligned} \frac{u(t)}{\chi(t)} \leq 1 + A\int_{t_0}^{t} r(s)\frac{u(t)}{\chi(t)}ds + B\int_{t_0}^{t} g(s)\varpi\left(\frac{u(t)}{\chi(t)}\right)ds+ \\ C\int_{t_0}^{t} l(s)\beta\left(\frac{u(t)}{\chi(t)}\right)ds \quad t \in \mathbf{I} \end{aligned} \tag{3.87}$$

Let

$$\frac{u(t)}{\chi(t)} = \eta(t) \tag{3.88}$$

Inequality (3.85) is written as

$$\eta(t) \leq 1 + A\int_{t_0}^{t} r(s)\eta(s)ds + B\int_{t_0}^{t} g(s)\varpi\left(\eta(s)\right)ds + C\int_{t_0}^{t} l(s)\beta\left(\eta(s)\right)ds \quad t \in \mathbf{I} \tag{3.89}$$

Application of Theorem 3.8 on inequality (3.87) gives

$$\eta(t) \leq F^{-1}\left(F(1) + C\int_{t_0}^{t} l(s)\beta\left(\Lambda(s)Q(s)\right)ds\right)\Lambda(t)Q(t) \tag{3.90}$$

Replacing the $\eta(t)$ in equation(3.88). This concludes the proof of the theorem.

Theorem 3.10:

Suppose that $u(t)$, $f(t)$ and b(t) are nonnegative and continuous functions on $\mathbf{R}_+$. Let $K(t,s)$ and its partial derivative $K_t(t,s)$ exists and be continuous for every $t, s \in \mathbf{I}$ and $K_t(t,s) \leq 0$. Moreover, let $\varpi, \beta \in C(\mathbf{R}_+, \mathbf{R}_+)$ be nondecreasing with $\varpi(u)$, $\beta(u) > 0$, and $\beta, \varpi \in \psi$, are submultiplicative. The inequality

$$\begin{aligned} u(t) \leq u_0 + L\int_{t_0}^{t} f(s)u(s)ds + \int_{t_0}^{t} f(s)\left(\int_{t_0}^{s} K(s,\tau)\varpi(u(\tau))d\tau\right)ds \\ + \int_{t_0}^{t} b(s)\beta(u(s))ds \end{aligned} \tag{3.91}$$

holds, for u_0 and L are positive constants then,

$$u(t) \leq F^{-1}\left(F(u_0) + \int_{t_0}^{t} b(s)\beta(N(s))ds\right)N(t) \tag{3.92}$$

where

$$N(t) = \Omega^{-1}\left(\Omega(u_0) + L\int_{t_0}^{t} f(s)K(s,s)\left(u_0Q(s) + \int_{t_0}^{s} Q(\alpha)d\alpha\right)ds\right) \tag{3.93}$$

$Q(t)$ is defined by (3.5), $\Omega(u)$ and $F(u)$ is defined by (2.13) and(2.40) respectively. Choosing $t_1 \in [t_0, \infty)$ so that

$$F(u_0) + \int_{t_0}^{t} b(s)\beta(N(s))ds \in Dom(F^{-1})$$

and

$$\left(\Omega(u_0) + L\int_{t_0}^{t} f(s)K(s,s)\left(u_0Q(s) + \int_{t_0}^{s} Q(\alpha)d\alpha\right)ds\right) \in Dom(\Omega^{-1})$$

for all t lying in the subinterval $[t_0, t_1]$ of $\mathbf{I}$. Ω^{-1} and F^{-1} are the inverses of the functions Ω and F respectively.

Proof:

Define

$$n(t) = u_0 + \int_{t_0}^{t} b(s)\beta(s)ds \tag{3.94}$$

where $n(t_0) = u_0$, then we have

$$u(t) \leq n(t) + L\int_{t_0}^{t} f(s)u(s)ds + \int_{t_0}^{t} f(s)\left(\int_{t_0}^{s} K(s,\tau)\varpi(u(\tau))d\tau\right)ds \tag{3.95}$$

Applying Theorem 3.8 to inequality (3.95), it yields

$$u(t) \leq n(t)N(t) \tag{3.96}$$

Since β is submultiplicative, it follows that

$$\frac{b(t)\beta(u(t))}{\beta(n(t))} \leq b(t)\beta(N(t)) \tag{3.97}$$

By equation (2.40), we get

$$n(t) \leq F^{-1}\left(F(n(t_0)) + \int_{t_0}^{t} b(s)\beta(N(s))ds\right) \tag{3.98}$$

Substituting this into (3.96), we arrive at the result.

The next theorem is given with $u_0 = p(t)$ a nondecreasing, nonnegative, monotonic continuous function on $\mathbf{R}_+$,.

Theorem 3.11:

Let all the conditions of Theorem 3.10 remained valid. Let $\beta,\ \varpi \in \Psi$, the inequality

$$\begin{aligned} u(t) \leq p(t) + L\int_{t_0}^{t} f(s)u(s)ds + \int_{t_0}^{t} f(s)\left(\int_{t_0}^{s} K(s,\tau)\varpi(u(\tau))d\tau\right)ds \\ + \int_{t_0}^{t} b(s)\beta(u(s))ds \end{aligned} \tag{3.99}$$

holds, then

$$u(t) \leq p(t)F^{-1}\left(F(1) + \int_{t_0}^{t} b(s)\beta(N(s)ds\right)N(t) \tag{3.100}$$

where $Q(t)$ and $N(t)$ are defined in equations (3.5) and (3.93) respectively, $\Omega(r)$ is by (2.13) and $F(u)$ is defined by (2.40). Choosing $t_1 \in [t_0, \infty)$ so that

$$F(1) + \int_{t_0}^{t} b(s)\beta(N(s)ds \in Dom(F^{-1})$$

and

$$\left(\Omega(1) + L\int_{t_0}^{t} f(s)K(s,s)\left(Q(s) + \int_{t_0}^{s} Q(\alpha)d\alpha\right)ds\right) \in Dom(\Omega^{-1})$$

for all t lying in the subinterval $[t_0, t_1]$ of $\mathbf{I}$. Ω^{-1} and F^{-1} are the inverses of the functions Ω and F respectively.

Proof:

Since $p(t)$ is nondecreasing, monotonic and nonnegative and $\beta,\ \varpi\Psi$ then

$$\begin{aligned} \frac{u(t)}{p(t)} \leq 1 + \int_{t_0}^{t} f(s)\frac{u(s)}{p(s)}ds \\ + \int_{t_0}^{t} f(s)K(s,s)\left(\int_{t_0}^{s} g(\tau)\varpi(\frac{u(\tau)}{p(\tau)})d\tau\right)ds + \int_{t_0}^{t} b(s)\beta(\frac{u(s)}{p(s)})ds \end{aligned} \tag{3.101}$$

By application of Theorem 3.10, let $\omega(t) = \dfrac{u(t)}{p(t)}$. we obtain

$$\omega(t) \leq F^{-1}\left(F(1) + \int_{t_0}^{t} b(s)\beta(N(s))ds\right)N(t) \tag{3.102}$$

Substituting $\omega(t)$, we arrive at the result (3.100).

Let $K(t,s) = h(t)q(s)$ where the functions $h(t)$ and $b(s)$ are continuous on $\mathbf{R}_+$, we have the following results.

Theorem 3.12:

Let $u(t), b(t), q(t), h(t), f(t) : \mathbf{I} \to \mathbf{R}_+$ and $h'(t) \leq 0$ and Let $\varpi, \beta \in C(\mathbf{R}_+, \mathbf{R}_+)$ be nondecreasing and monotonic function with $\beta,\ \varpi$ belong to class Ψ and let $\beta(u)$ be a submultiplicative. If

$$u(t) \leq u_0 + L\int_{t_0}^{t} f(s)u(s)ds + \int_{t_0}^{t} f(s)h(s)\left(\int_{t_0}^{s} q(\tau)\varpi(u(\tau))d\tau\right)ds + \int_{t_0}^{t} b(s)\beta(u(s))ds \tag{3.103}$$

holds. Then,

$$u(t) \leq M(t)F^{-1}\left(F(u_0) + \int_{t_0}^{t} b(s)\beta(M(s))ds\right). \tag{3.104}$$

$\Omega(u)$ is defined in (2.13) and F is defined in (2.40).

Defined

$$M(t) = \Omega^{-1}\left(\Omega(1) + L\int_{t_0}^{t} f(s)h(s)q(s)\left(Q(s) + \int_{t_0}^{s} Q(\alpha)d\alpha\right)ds\right) \tag{3.105}$$

and $Q(t)$ in equation(3.5). Where F^{-1} and Ω^{-1} are the inverse functions of F and Ω respectively and t is the subinterval $[t_0, t_1] \subset \mathbf{I}$ such that

$$F(u_0) + \int_{t_0}^{t} b(s)\beta(M(s))ds \in Dom(F^{-1})$$

and

$$\Omega(1) + L\int_{t_0}^{t} f(s)h(s)q(s)\left(Q(s) + \int_{t_0}^{s} Q(\alpha)d\alpha\right)ds \in (Dom\Omega^{-1})$$

Proof:

Define function $C(t)$ as

$$C(t) = u_0 + \int_{t_0}^{t} b(s)\beta(u(s))ds \tag{3.106}$$

Using this in (3.103), we get

$$u(t) \leq C(t) + L\int_{t_0}^{t} f(s)u(s)ds + \int_{t_0}^{t} f(s)\left(\int_{t_0}^{s} q(\tau)\varpi(u(\tau))d\tau\right)ds \tag{3.107}$$

Applying Theorem 3.2 to inequality (3.107), we have

$$u(t) \leq C(t)M(t) \tag{3.108}$$

Since β is submultiplicative, it follows that

$$\beta(u(t)) \leq \beta(C(t))\beta(M(s)) \tag{3.109}$$

By equation (2.40), we have the following

$$C(t) \leq F^{-1}\left(F(u_0) + \int_{t_0}^{t} b(s)\beta(M(s))ds\right). \tag{3.110}$$

By (3.108) and (3.110), we arrive at equation(3.104).

Let the function $\gamma(t) \in C(\mathbf{I}, \mathbf{R}_+)$ be a monotonic,

nondecreasing and nonnegative. The following theorem is given as thus:

Theorem 3.13:

Let $u(t), f(t), h(t), q(t), \gamma(t) \in C(\mathbf{I}, \mathbf{R}_+)$ remain the same as in Theorem 3.12. Let

$$u(t) \leq \gamma(t) + L\int_{t_0}^{t} f(s)u(s)ds + \int_{t_0}^{t} f(s)h(s)\left(\int_{t_0}^{s} q(\tau)\varpi(u(\tau))d\tau\right)ds + \int_{t_0}^{t} b(s)\beta(u(s))ds \tag{3.111}$$

By Theorem 3.12, it Implies that

$$u(t) \leq \gamma(t)M(t)F^{-1}\left(F(1) + \int_{t_0}^{t} b(s)\beta(M(s))ds\right) \tag{3.112}$$

where $Q(t)$ and $M(t)$ are defined in equations (3.5) and (3.105)respectively, $\Omega(u)$ and $F(u)$ are defined in (2.13) and (2.40) respectively. for F^{-1} and Ω^{-1} are the inverse functions of F and Ω respectively and t is the subinterval $[t_0, t_1] \subset \mathbf{I}$ such that

$$F(1) + \int_{t_0}^{t} b(s)\beta(M(s))ds \in Dom(F^{-1})$$

and

$$\Omega(1) + \int_{t_0}^{t}\left(f(s)h(s)g(s)\exp\left(\int_{t_0}^{s} f(\tau)d\tau\right) + \int_{t_0}^{s} Q(\alpha)d\alpha\right)ds \in (Dom\Omega^{-1})$$

Proof:

Since $\gamma(t)$ is nondecreasing, monotonic,nonnegative function on $\mathbf{R}_+$ and $\beta,\ \varpi \in \Psi$. Then

$$r(t) \leq 1 + \int_{t_0}^{t} f(s)r(s)ds + \int_{a}^{t} f(s)h(s)\left(\int_{t_0}^{s} q(\tau)\varpi(r(\tau))d\tau\right)ds + \int_{t_0}^{t} b(s)\beta(r(s))ds, \tag{3.113}$$

where $r(t) = \dfrac{u(t)}{\gamma(t)}$. Therefore, by application of Theorem 3.12 we obtain

$$r(t) \leq M(t)F^{-1}\left(F(1) + \int_{t_0}^{t} b(s)\beta(M(s))ds\right) \tag{3.114}$$

By replacing $r(t)$ in (3.114) concludes the proof.

The following integrals are the extensions of Theorem 3.3.

Theorem 3.14:

Let $u(t), r(t) \in C(\mathbf{I}, \mathbf{R}_+)$ and $r'(t) \leq 0$. If

$$u(t) \leq u_0 + N\int_{t_0}^{t} f(s)u(s)ds + M\int_{t_0}^{t} f(s)r(s)\left(\int_{t_0}^{s} g(\tau)\varpi(u(\tau))d\tau + \int_{t_0}^{s} g(\tau)(u(\tau))d\tau\right)ds + \int_{t_0}^{t} b(s)\beta(u(s))ds \tag{3.115}$$

holds, where M and N are positive constants, then

$$u(t) \leq F^{-1}\left(F(u_0) + \int_{t_0}^{t} b(s)\beta(W(s)ds\right)W(t) \tag{3.116}$$

Define

$$W(t) = \Re^{-1}\left(\Re(1) + C\int_{t_0}^{t} f(s)r(s)g(s)ds\right) \tag{3.117}$$

where $u_0 > 0$ and the definition of $\Re(u)$ is given in (3.43) and $t_1 \in \mathbf{I}$ is chosen such that

$$\left(\Re(u_0) + \int_{t_0}^{t} f(s)r(s)g(s)ds\right) \in Dom(\Re^{-1}).$$

and

$$F(u_0) + \int_{t_0}^{t} b(s)\beta(W(s))ds \in Dom(F^{-1})$$

for all t lying in the interval $[t_0, t_1] \in \mathbf{I}$. $\Re^{-1},\ H^{-1}$ are the inverses of $\Re$ and F.

Proof:

Define

$$n(t) = u_0 + \int_{t_0}^{t} b(s)\beta(u(s))ds. \tag{3.118}$$

Inequality (3.115) is written as

$$\begin{aligned} u(t) \le n(t) + N\int_{t_0}^{t} f(s)u(s)ds + M\int_{t_0}^{t} f(s)r(s)\left(\int_{t_0}^{s} g(\tau)\varpi(u(\tau))d\tau\right. \\ \left. + \int_{t_0}^{s} g(\tau)(u(\tau))d\tau\right) ds, \end{aligned} \tag{3.119}$$

where $n(t)$ is nondecreasing, nonnegative, monotonic continuous function on $\mathbf{R}_+$.

By Theorem 3.3, we obtain

$$u(t) \le n(t)W(t) \tag{3.120}$$

By submultiplicative property of β, we have

$$u(t) \le \beta(n(t))\beta\left(\Re^{-1}\left(\Re(1) + C\int_{t_0}^{t} f(s)r(s)g(s)ds\right)\right) \tag{3.121}$$

By equation(3.43),we arrive at

$$n(t) \le F^{-1}\left(F(n(t_0)) + \int_{t_0}^{t} b(s)\beta(W(s))ds\right). \tag{3.122}$$

Substituting this in (3.120), we have the result. This concludes the proof.

Let $m(t)$ be nonnegative, nondecreasing and monotonic continuous function on $\mathbf{R}_+$,

Theorem 3.15:

Let $u(t), r(t), m(t) \in C(\mathbf{I}, \mathbf{R}_+)$ and $r'(t) \le 0$. If

$$\begin{aligned} u(t) \le m(t) + N\int_{t_0}^{t} f(s)u(s)ds + \int_{t_0}^{t} f(s)r(s)\left(\int_{t_0}^{s} g(\tau)\varpi(u(\tau))d\tau\right. \\ \left. + \int_{t_0}^{s} g(\tau)(u(\tau))d\tau\right) ds + \int_{t_0}^{t} b(s)\beta(u(s))ds \end{aligned} \tag{3.123}$$

holds, then

$$u(t) \le m(t)F^{-1}\left(F(1) + \int_{t_0}^{t} b(s)\beta(W(s))ds\right) W(t) \tag{3.124}$$

where $\Re(u)$ and $W(t)$ are defined in equations (3.50) and (3.127) respectively, $t_1 \in \mathbf{I}$ is chosen such that

$$\left(\Re(u_0) + \int_{t_0}^{t} f(s)r(s)g(s)ds\right) \in Dom(\Re^{-1}).$$

and

$F(u_0) + \int_{t_0}^{t} b(s)\beta(W(s))ds \in Dom(F^{-1})$

for all t lying in the interval $[t_0, t_1] \in \mathbf{I}$. $\Re^{-1},\ F^{-1}$ are the inverses of $\Re$ and F.

Proof:

Dividing both sides of (3.123) by $m(t)$, we get

$$z(t) \leq 1 + \int_{t_0}^{t} f(s)r(s)\left(\int_{t_0}^{s} g(\tau)\varpi(z(\tau))d\tau + \int_{t_0}^{s} g(\tau)z(\tau)d\tau\right)ds + \int_{t_0}^{t} b(s)\beta(z(s))ds \tag{3.125}$$

Where

$$z(t) = \frac{u(t)}{m(t)}, \tag{3.126}$$

By applying the Theorem 3.14 gives

$$z(t) \leq F^{-1}\left(F(1) + \int_{t_0}^{t} b(s)\beta(W(s))ds\right)W(t) \tag{3.127}$$

Substituting for $z(t)$ by using equation(3.126)

3.0.3 Integral Inequality with Three Nonlinear Terms

In this section, we consider the development of integral inequalities with three nonlinear terms which are the special case of integral inequalities developed by Dehongade and Deo (1976), Agarwal and Thandapain (1981) and Pinto (1990). Most of the results in the previous sections are needed in this section.

Theorem 3.16:

Let $u(t), r(t), h(t), g(t) \in C(\mathbf{I}, \mathbf{R}_+)$ and $\omega, f, \gamma \in \Psi$ be nonnegative, monotonic, nondecreasing continuous functions. Let γ be a submultiplicative for $u > 0$.

If

$$u(t) \leq K + A\int_{t_0}^{t} r(s)\beta(u(s))ds + B\int_{t_0}^{t} h(s)\varpi(u(s))ds + L\int_{t_0}^{t} g(s)\gamma(u(s))ds \tag{3.128}$$

for K, A, B and L are positive constants and $t \in \mathbf{I}$, then

$$u(t) \leq \Upsilon^{-1}\left[\Upsilon(K) + L\int_{t_0}^{t} g(s)\gamma\left[\Omega^{-1}\left(\Omega(1) + B\int_{t_0}^{s} h(\alpha)\varpi\left(T(\alpha)\right)d\alpha\right)T(s)\right]ds\right] \Omega^{-1}\left(\Omega(1) + B\int_{t_0}^{t} h(s)\varpi\left(T(s)\right)ds\right)T(t) \tag{3.129}$$

where $T(t)$ is defined as

$$T(t) = F^{-1}\left(F(1) + A\int_{t_0}^{t} r(s)ds\right), \tag{3.130}$$

Ω, and F are defined in (2.13),(2.40) .

$$\Upsilon(r) = \int_{t_0}^{t} \frac{ds}{\gamma(s)} \quad 0 < r_0 \leq r \tag{3.131}$$

and H^{-1}, Ω^{-1} and Υ^{-1} are the inverses of H, Ω, G respectively t is in the subinterval $(0, b) \in (I)$. So that

$$G(K) + L\int_{t_0}^{t} g(s)\gamma \left[\Omega^{-1}\left(\Omega(1) + B\int_{t_0}^{s} h(\alpha)\varpi\left(T(\alpha)\right)d\alpha\right)T(s)\right]ds \in Dom(\Upsilon^{-1}$$

Proof:

Define

$$n(t) = K + L\int_{t_0}^{t} g(s)\gamma(u(s))ds \tag{3.132}$$

we re-write (3.130) as

$$u(t) \leq n(t) + A\int_{t_0}^{t} r(s)\beta(u(s))ds + B\int_{t_0}^{t} h(s)\varpi(u(s))ds \tag{3.133}$$

Since, $n(t)$ is monotonic, nondecreasing on $\mathbf{R}_+$

Applying Corollary 3.1, we obtain

$$u(t) \leq n(t)\Omega^{-1}\left(\Omega(1) + B\int_{t_0}^{t} h(s)\varpi\left(T(s)\right)ds\right)T(t) \tag{3.134}$$

Hence,

$$\gamma(u(t)) \leq \gamma\left[n(t)\Omega^{-1}\left(\Omega(1) + B\int_{t_0}^{t} h(s)\varpi\left(T(s)\right)ds\right)T(t)\right] \tag{3.135}$$

By submultiplicative property of $\gamma(u)$, we get

$$\frac{\gamma(u(t))}{\gamma(n(t))} \leq \gamma\left[\Omega^{-1}\left(\Omega(1) + B\int_{t_0}^{t} h(s)\varpi\left(T(s)\right)ds\right)T(t)\right] \tag{3.136}$$

By equation (3.131), we arrive at

$$\begin{gathered} n(t) \leq \Upsilon^{-1}\left[\Upsilon(n(t_0))\right. \\ \left. +L\int_{t_0}^{t} g(s)\gamma\left[\Omega^{-1}\left(\Omega(1) + B\int_{t_0}^{s} h(\alpha)\varpi\left(T(\alpha)\right)d\alpha\right)T(s)\right]ds\right] \end{gathered} \tag{3.137}$$

Substituting for $n(t)$ in (3.134), we arrive at the result

Theorem 3.17:

Let $u(t), r(t), h(t), g(t), \beta(t)$ and $b(t)$ be as in Theorem 3.16 and $\varpi(u), f(u), \gamma(u) \in \psi$ be nonnegative,monotonic, nondecreasing continuous functions. Let $\gamma(u)$ be a submultiplicative for $u > 0$. If

$$\begin{gathered} u(t) \leq \beta(t) + A\int_{t_0}^{t} r(s)\beta(u(s))ds + B\int_{t_0}^{t} h(s)\varpi(u(s))ds + \\ L\int_{t_0}^{t} g(s)\gamma(u(s))ds \end{gathered} \tag{3.138}$$

for K, A, B and L are positive constants, then

$$\begin{gathered} u(t) \leq \beta(t)\Upsilon^{-1}\left[\Upsilon(1) + L\int_{t_0}^{t} g(s)\gamma\left[\Omega^{-1}\right.\right. \\ \left.\left.\left(\Omega(1) + B\int_{t_0}^{s} h(\alpha)\varpi\left(T(\alpha)\right)d\alpha\right)T(s)\right]ds\right] \\ \Omega^{-1}\left(\Omega(1) + B\int_{t_0}^{t} h(s)\varpi\left(T(s)\right)ds\right)T(t), \end{gathered} \tag{3.139}$$

where Ω, Υ, and F are defined in (2.13),(3.131) and (2.40)respectively,

for F^{-1}, Ω^{-1} and Υ^{-1} are the inverses of F, Ω and Υ respectively such that t is in the subinterval $(0,b) \in (R_+)$. So that

$$\Upsilon(1) + L\int_{t_0}^{t} g(s)\gamma\left[\Omega^{-1}\left(\Omega(1) + B\int_{t_0}^{s} h(\alpha)\varpi\left(T(\alpha)\right)d\alpha\right)T(s)\right]ds \in Dom(\Upsilon^{-1})$$

Proof:

Since $\beta(t)$ a monotonic, nondecreasing and nonnegative continuous function on $\mathbf{R}_+$, then Inequality (3.139) reduce to

$$\begin{aligned}\frac{u(t)}{\beta(t)} \leq 1 + A\int_{t_0}^{t} r(s)f(\frac{u(s)}{\beta(s)})ds + B\int_{t_0}^{t} h(s)\varpi(\frac{u(s)}{\beta(s)})ds \\ +L\int_{t_0}^{t} g(s)\gamma(\frac{u(s)}{\beta(s)})ds\end{aligned} \tag{3.140}$$

It follows that

$$\begin{aligned}z(t) \leq 1 + A\int_{t_0}^{t} r(s)f(z(s))ds + B\int_{t_0}^{t} h(s)\varpi(z(s))ds \\ +L\int_{t_0}^{t} g(s)\gamma(z(s))ds\end{aligned} \tag{3.141}$$

Where

$$\frac{u(t)}{\beta t} = z(t).$$

Applying the Theorem 3.16 and putting $K = 1$, we arrive at

$$\begin{aligned}z(t) \leq \Upsilon^{-1}\left[G(1) + L\int_{t_0}^{t} g(s)\gamma\left[\Omega^{-1}\left(\Omega(1) + B\int_{t_0}^{s} h(\alpha)\varpi\left(T(\alpha)\right)d\alpha\right)T(s)\right]ds\right] \\ \Omega^{-1}\left(\Omega(1) + B\int_{t_0}^{t} h(s)\varpi\left(T(s)\right)ds\right)T(t)\end{aligned} \tag{3.142}$$

Now if $z(t)$ is replaced in (3.142), we arrive at (3.139).

The next chapter consists of application of integral inequalities.

CHAPTER FOUR

MAIN RESULTS

4.1 Hyers-Ulam Stability of Perturbed and Nonperturbed Nonlinear second Order Differential Equations

4.1.1 Introduction

Here, we provide results which satisfy the objectives in chapter 1 of this work. Some researchers refer to Hyers-Ulam stability as a special case of Hyers-Ulam-Rassias stability. This shows that Hyers-Ulam-Rassias stability is more advanced than Hyers-Ulam stability in scope. The major tools used are the inequalities developed in chapter three. The method will be to reduce all perturbed and non-perturbed nonlinear second order DE to equivalent integral forms.

4.1.2 Hyers-Ulam Stability of $u''(t) + f(t, u(t)) = P(t, u(t))$

We begin this section by considering Hyers-Ulam stability of a perturbed non-linear second order differential equation is given as

$$u''(t) + f(t, u(t)) = P(t, u(t)) \tag{4.1}$$

with initial conditions $u(t_0) = u'(t_0) = 0$, where $P(t, 0) = 0,\ f,\ P \in C(\mathbf{I} \times \mathbf{R}, \mathbf{R})$.

Definition 4.1:

Equation (4.1) is Hyers-Ulam stable with initial data if for every $\epsilon > 0$, constant $K > 0$ and $t \in \mathbf{I}$ sufficiently large there exists a solution $u(t) \in C^2(\mathbf{I}, \mathbf{R}_+)$ satisfying

$$|u'' + f(t, u(t)) - P(t, u(t))| \leq \epsilon \tag{4.2}$$

such that

$$|u(t) - u_0(t)| \leq K\epsilon,$$

where $u_0(t) \in C^2(\mathbf{I}, \mathbf{R}_+)$ is the solution of nonlinear differential equation(4.1) and K is the Hyers-Ulam constant.

Theorem 4.1:

Let

$$|P(t, u(t))| \leq \alpha(t)\beta(|u(t)|) \tag{4.3}$$

and

$$|f(t,u(t))| \leq h(t)\varpi(|u(t)|) \tag{4.4}$$

where $\beta(u)$, $\varpi(u)$ are nonnegative, nondecreasing, continuous functions, suppose β, $\varpi \in \Psi$, and the functions $\alpha(t)$, $h(t)$ are continuous functions on $\mathbf{R}_+$, then, equation (4.1) is stable in the sense of Hyers-Ulam, if the following additional conditions hold:

C_1 $|f(t,u(t))| \geq 1$ for all $t \geq t_0$.

C_2 $\dfrac{f'(t,u(t))u(t)}{f(t,u(t))} = q(t,u(t),u'(t))$,

are satisfied.

Proof:

Multiplying inequality (4.2) by $|u'(t)|$ to get

$$-\epsilon|u'(t)| \leq u'(t)u''(t) + f(t,u(t))u'(t) - P(t,u(t))u'(t) \leq \epsilon|u'(t)| \tag{4.5}$$

for all $t \geq t_0$. Integrating each term from t_0 to t and applying Lemma 1.1 , we obtain

$$\begin{aligned}-\epsilon\int_{t_0}^{t}|u'(s)|ds \leq \frac{1}{2}u'(t)^2 + \int_{t_0}^{t} f(s,u(s))u'(s)ds - \int_{t_0}^{t} P(s,u(s))u'(s)ds \\ \leq \epsilon\int_{t_0}^{t}|u'(s)|ds.\end{aligned} \tag{4.6}$$

For $t \geq t_0$, let

$\int_{t_0}^{\infty}|u'(s)|ds \leq L$, equation (4.6) becomes

$$-\epsilon L \leq \frac{1}{2}u'(t)^2 + \int_{t_0}^{t} f(s,u(s))u'(s)ds - \int_{t_0}^{t} P(s,u(s))u'(s)ds \leq \epsilon L \tag{4.7}$$

Integrating by part, let $f_u(t,u(t)) \leq 0$. we get

$$\begin{aligned}f(t,u(t))u(t) \leq \epsilon L - \frac{1}{2}u'(t)^2 + \int_{t_0}^{t}\frac{f'(s,u(s))u(s)}{f(s,u(s))}f(s,u(s))ds \\ + \int_{t_0}^{t} P(s,u(s))u'(s)ds\end{aligned} \tag{4.8}$$

Application of condition c_2 yields

$$\begin{aligned}f(t,u(t))u(t) \leq \epsilon L - \frac{1}{2}u'(t)^2 + \int_{t_0}^{t} q(t,u(s),u'(t))f(s,u(s))ds \\ + \int_{t_0}^{t} P(s,u(s))u'(s)ds.\end{aligned} \tag{4.9}$$

Taking the absolute value, using condition c_1 and Theorem 1.1 ,there exists $\xi \in [t_0,t]$ such that

$$\begin{aligned}|u(t)| \leq \epsilon L + \frac{1}{2}|u'(t)|^2 + |q(\xi,u(\xi),u'(\xi)|\int_{t_0}^{t}|f(s,u(s))|ds \\ +|u(t)|\int_{t_0}^{t}|P(s,u(s))|ds.\end{aligned} \tag{4.10}$$

Setting $|u'(t)| \leq \lambda,$ where $\lambda > 0$, by hypothesis of the Theorem 4.1 and $\epsilon L > 0$, we get

$$\frac{|u(t)|}{\epsilon\left(L+\frac{1}{2}\lambda^2\right)} \leq 1 + |q(\xi, u(\xi), u'(\xi))| \int_{t_0}^{t} \phi(s)\varpi\left(\frac{|u(s)|}{\epsilon\left(L+\frac{1}{2}\lambda^2\right)}\right) ds + \lambda \int_{t_0}^{t} |\alpha(s)\beta\left(\frac{|u(s)|}{\epsilon\left(L+\frac{1}{2}\lambda^2\right)}\right) ds. \tag{4.11}$$

Using the Theorem 3.7, we obtain

$$|u(t)| \leq \epsilon\left(L+\frac{1}{2}\lambda^2\right)\Omega^{-1}\left(\Omega(1)+\lambda\int_{t_0}^{t}\alpha(s)\varpi\left(F^{-1}\left(F(1)+|q((\xi),u(\xi),u'(\xi))|\int_{t_0}^{s}\phi(\delta)d\delta\right)\right)ds\right)F^{-1}\left(F(1)+|q((\xi),u(\xi),u'(\xi))|\int_{t_0}^{t}\phi(s)ds\right). \tag{4.12}$$

Setting $\lim_{t\to\infty}\int_{t_0}^{t}\alpha(s)ds = m < \infty$, $\lim_{t\to\infty}\int_{t_0}^{t}\phi(s)ds = n < \infty$. Using this in equation(4.12)

$$|u(t)| \leq \epsilon\left(L+\frac{1}{2}\lambda^2\right)\Omega^{-1}\left(\Omega(1)+m\lambda\varpi\left(F^{-1}\left(F(1) +n|q(\xi,u(\xi),u'(\xi))|\right)\right)\right)F^{-1}\left(F(1)+n|q((\xi),u(\xi),u'(\xi))|\right). \tag{4.13}$$

Therefore,

$$|u(t)-u_0(t)| \leq |u(t)| \leq \epsilon\left(L+\frac{1}{2}\lambda^2\right)\Omega^{-1}\left(\Omega(1)+m\lambda\varpi\left(F^{-1}\left(F(1) +n|q((\xi),u(\xi),u'(\xi))|\right)\right)\right)F^{-1}\left(F(1)+n|q((\xi),u(\xi),u'(\xi))|\right), \tag{4.14}$$

where

$$K = \left(L+\frac{1}{2}\lambda^2\right)\Omega^{-1}\left(\Omega(1)+m\lambda\varpi\left(F^{-1}\left(F(1) +n|q((\xi),u(\xi),u'(\xi))|\right)\right)\right)F^{-1}\left(F(1)+n|q((\xi),u(\xi),u(\xi),u'(\xi))|\right). \tag{4.15}$$

Thus (4.15) reduce to

$|u(t)-u_0(t)| \leq K\epsilon.$

Hence, equation(4.1) is Hyers-Ulam stable.

This ends the proof.

Special case of equation (4.1) is given as

$$u''(t) + c(t)f(u(t)) + u(t) = hu^n(t). \tag{4.16}$$

Theorem 4.2:

Suppose $u(t)$ is twice differentiable continuous function on $\mathbf{R}_+$. Let $h(t), \alpha(t) \in C(\mathbf{R}_+)$ and suppose $f(u)$ belongs to class Ψ. Then equation (4.16) with initial condition $u(t_0) = u'(t_0) = 0$ is said to be Hyers-Ulam stable with Hyers-Ulam constant

$$K = \left(L + L\eta h(\xi) + \frac{1}{2}\lambda^2\right)\Omega^{-1}\left(\Omega(1)+m\lambda\right). \tag{4.17}$$

Proof:

From (4.2), we have

$$-\epsilon \leq u''(t) + c(t)f(u(t)) + u(t) - hu^n(t) \leq \epsilon. \tag{4.18}$$

Multiplying through by $u'(t)$,

$$-\epsilon \leq u''(t)u,(t) + c(t)f(u(t))u'(t) + u(t)u'(t) - u'(t)hu^n(t) \leq \epsilon.$$

Integrating from t_0 to t, using initial conditions, we obtain

$$\begin{aligned}\frac{1}{2}(u'(t))^2 + u'(t)\int_{t_0}^{t} c(s)f(u(s))ds + \mathbb{G}(u(t) - u^n(t)\int_{t_0}^{t} h(s)u'(s)ds \\ \leq \epsilon \int_{t_0}^{t} u'(s)ds.\end{aligned} \tag{4.19}$$

where

$$\mathbb{G}(u(t)) = \int_{u(t_0)}^{u(t)} sds \tag{4.20}$$

By Theorem 1.1, there exists $t_0 \leq \xi \leq t$ such that

$$\begin{aligned}\frac{1}{2}(u'(t))^2 + u'(t)\int_{t_0}^{t} c(s)f(|u(s)|)ds + \mathbb{G}(u(t)) - u^n(t)h(\xi)\int_{t_0}^{t} u'(s)ds \\ \leq \epsilon \int_{t_0}^{t} u'(s)ds.\end{aligned} \tag{4.21}$$

Re-arrange the inequality (4.21) to have

$$\begin{aligned}\mathbb{G}(u(t)) \leq \epsilon \int_{t_0}^{t} u'(s)ds - \frac{1}{2}(u'(t))^2 - u'(t)\int_{t_0}^{t} c(s)f(u(s))ds \\ + u^n(t)h(\xi)\int_{t_0}^{t} u'(s)ds,\end{aligned}$$

Taking the absolute value to obtain

$$\begin{aligned}|\mathbb{G}(u(t))| \leq \epsilon \int_{t_0}^{t} |u'(s)|ds + \frac{1}{2}|(u'(t))|^2 + |u'(t)|\int_{t_0}^{t} c(s)f(|u(s)|)ds \\ + u^n(t)h(\xi)\int_{t_0}^{t} |u'(s)|ds,\end{aligned}$$

setting $\int_{t_0}^{t} |u'(s)|ds \leq L,$ where $L > 0$, it clear that

$$|\mathbb{G}(u(t))| \leq \epsilon L + \frac{1}{2}u'(t)^2 + |u'(t)|\int_{t_0}^{t} c(s)f(|u(s)|)ds + Lu^n(t)|h(\xi)| \tag{4.22}$$

Setting $|u(t)| \leq |\mathbb{G}(u(t))|$, $|u^n(t)| \leq \eta$ and

$|u'(t)| \leq \lambda,$ where $\lambda > 0$, we have

$$|u(t)| \leq \epsilon L + L\eta|h(\xi)| + \frac{1}{2}\lambda^2 + \lambda \int_{t_0}^{t} c(s)f(|u(s)|)ds. \tag{4.23}$$

Let $\epsilon\left(L + L\eta|h(\xi)| + \frac{1}{2}\lambda^2\right) = E$, we obtain

$$\frac{|u(t)|}{E} \leq 1 + \lambda \int_{t_0}^{t} c(s)f\left(\frac{|u(s)|}{E}\right)ds. \tag{4.24}$$

By applying Lemma 2.4 we get

$$|u(t)| \leq E\Omega^{-1}\left(\Omega(1) + \lambda \int_{t_0}^{t} c(s)ds\right), \quad t_0 \leq t \tag{4.25}$$

Let $\lim_{t_0\to\infty}\displaystyle\int_{t_0}^{t} c(s)ds = m < \infty,\ \text{m} > 0$, we have

$$|u(t)| \leq E\Omega^{-1}\left(\Omega(1) + m\lambda\right), \quad t_0 \leq t \tag{4.26}$$

Replacing E we get

$$|u(t)| \leq \epsilon\left(L + L\eta h(\xi) + \frac{1}{2}\lambda^2\right)\Omega^{-1}\left(\Omega(1) + m\lambda\right), \quad t_0 \leq t \tag{4.27}$$

Therefore, equation(4.27) is in the form

$|u(t) - u(t_0)| \leq |u(t)| \leq K\epsilon.$

where K is given as

$$K = \left(L + L\eta h(\xi) + \frac{1}{2}\lambda^2\right)\Omega^{-1}\left(\Omega(1) + m\lambda\right).$$

4.1.3 Hyers-Ulam Stability of $u''(t) + f(t, u(t), u'(t)) = P(t, u(t))$

Now we consider the Hyers-Ulam stability of

$$u''(t) + f(t, u(t), u'(t)) = P(t, u(t)) \tag{4.28}$$

with initial conditions $u(t_0) = u'(t_0) = 0$, where $u(t) \in C^2(\mathbf{R}_+, \mathbf{R}_+)$, $\mathbf{I} = [0, \infty)$, $P(t, 0) = 0$, $P \in C(\mathbf{I} \times \mathbf{R}, \mathbf{R})$, $f \in C(\mathbf{R}_+ \times \mathbf{R}^2, \mathbf{R})$,

Definition 4.2:

Equation (4.28) is Hyers-Ulam stable if for every $\epsilon > 0$, constant $K > 0$ called Hyers-Ulam constant and $t \in \mathbf{I}$ sufficiently large there exists a solution $u(t) \in C^2(\mathbf{I}, \mathbf{R}_+)$ satisfying

$$|u'' + f(t, u(t), u'(t)) - P(t, u(t))| \leq \epsilon \tag{4.29}$$

such that

$|u(t) - u_0(t)| \leq \epsilon K.$

where $u_0(t) \in C^2(\mathbf{I}, \mathbf{R}_+)$ is the solution of nonlinear differential equation(4.28) with initial condition $u(t_0) = u'(t_0) = 0$.

Theorem 4.3:

Suppose the function $u(t) \leq f(t, u(t), u'(t))u(t)$ and $|f(t, u(t), u'(t))| \leq h(t)\varpi(|u(t)|)|u'(t)|$, $\displaystyle\int_{t_0}^{t} |u'(s)|ds \leq L$, where constant $L > 0$, for $h(t), \alpha(t)$ positive, nondecreasing continuous functions on $\mathbf{R}_+$ and $\varpi(u)$ belongs to class Ψ. Then, equation (4.29) is stable in the sense of Hyers-Ulam and Hyers-Ulam constant is given as

$$K = L(1 + \frac{1}{2}\lambda^2 + \alpha(\rho)M)\left(\Omega^{-1}\left(\Omega(1) + d\lambda|q(\xi, u(\xi), u'(\xi), u''(\xi))|\right)\right). \tag{4.30}$$

Proof:

Multiplying (4.29) by $|u'(t)|$ we get

$$-\epsilon|u'(t)| \leq u'(t)u''(t) + f(t, u(t), u'(t))u'(t) - P(t, u(t))u'(t) \leq \epsilon|u'(t)| \tag{4.31}$$

for all $t \geq t_0$. Integrating from t_0 to t, we obtain

$$
\begin{aligned}
-\epsilon \int_{t_0}^{t} |u'(s)|ds \leq \frac{1}{2}u'(t)^2 + \int_{t_0}^{t} f(s,u(s),u'(s))u'(s)ds \\
- \int_{t_0}^{t} P(s,u(s))u'(s)ds \leq \epsilon \int_{t_0}^{t} |u'(s)|ds
\end{aligned}
\tag{4.32}
$$

for any $t \geq t_0$.

Integrating by part and by hypothesis of the Theorem 4.3, we get

$$
\begin{aligned}
f(t,u(t),u'(t))u(t) \leq \epsilon L - \frac{1}{2}u'(t)^2 + \int_{t_0}^{t} f'(s,u(s),u'(s))u(s)ds \\
+ \int_{t_0}^{t} P(s,u(s))u'(s)ds,
\end{aligned}
\tag{4.33}
$$

for

$$
f'_u(s,u(s),u'(s))u'(s) + f'_{u'}(s,u(s),u'(s))u''(s) \leq 0,
\tag{4.34}
$$

By hypothesis of the Theorem 4.3 we get

$$
\begin{aligned}
u(t) \leq \epsilon L - \frac{1}{2}u'(t)^2 + \int_{t_0}^{t} q(s,u(s),u'(s),u''(s))f(s,u(s),u'(s))ds \\
+ \int_{t_0}^{t} P(s,u(s))u'(s)ds,
\end{aligned}
\tag{4.35}
$$

where

$$
\frac{f'(t,u(t),u'(t))u(t)}{f(t,u(t),u'(t))} = q(t,u(t),u'(t),u''(t)),
\tag{4.36}
$$

for $q(t,u(t),u'(t),u''(t))$ a continuous function on $\mathbf{I} \times \mathbf{R}^3$.

By Theorem 1.1, there exits points $\xi, \rho \in [t_0, t]$ such that

$$
\begin{aligned}
|u(t)| \leq \epsilon L + \frac{1}{2}|u'(t)|^2 + |q(\xi,u(\xi),u'(\xi),u''(\xi))| \int_{t_0}^{t} |f(s,u(s),u'(s))|ds \\
+|P(\rho,u(\rho))| \int_{t_0}^{t} |u'(s)|ds
\end{aligned}
\tag{4.37}
$$

Setting $|u'(t)| \leq \lambda$ and $|P(\rho,u(\rho))| \leq \alpha(\rho)|u(t)|$ it follows that

$$
\begin{aligned}
|u(t)| &\leq \epsilon L + \frac{1}{2}\lambda^2 + L\alpha(\rho)M + |q(\xi,u(\xi),u'(\xi))| \int_{t_0}^{t} h(s)\omega(|u(s)|)|u'(s)|ds \\
&\leq \epsilon L(1 + \frac{1}{2}\lambda^2 + \alpha(\rho)M) + |q(\xi,u(\xi),u'(\xi))| \int_{t_0}^{t} h(s)\omega(|u(s)|)|u'(s)|ds \\
|u(\rho)| &\leq M
\end{aligned}
$$

Suppose that $\frac{u(t)}{B} \leq u'(t) = z(t)$, where $B = \epsilon L + \frac{1}{2}\lambda^2 + L\alpha(\rho)M$,, Therefore,

$$
|z(t)| \leq 1 + |q((\xi),u(\xi),u'(\xi),u''(\xi))| \int_{t_0}^{t} h(s)\varpi(|z(s)|)|z(s)|ds.
\tag{4.38}
$$

By applying Lemma 2.4, where $\varpi(|z(t)) \leq \varpi(|z(t)|)|z(t)|$ and $M = |q((\xi),u(\xi),u'(\xi),u''(\xi))|$, we get

$$
|z(t)| \leq \Omega^{-1}\left(\Omega(1) + |q((\xi),u(\xi),u'(\xi),u''(\xi))| \int_{t_0}^{t} h(s)ds\right).
\tag{4.39}
$$

By replacing the $z(t)$ we obtain

$$\begin{aligned}\frac{|u(t)|}{B} &\leq \Omega^{-1}\left(\Omega(1)+|q((\xi),u(\xi),u'(\xi),u''(\xi))|\int_{t_0}^{t}h(s)ds\right) \text{ for } t\geq t_0\\ &\leq \Omega^{-1}\left(\Omega(1)+|q(\xi,u(\xi),u'(\xi),u(\xi))|s\right) \ as\ t\to\infty\end{aligned}$$

provided $\lim_{t_0\to\infty}\int_{t_0}^{t}h(s)ds = d<\infty,\ d>0$

Hence,

$|u(t)-u_0(t)|\leq|u(t)|\leq K\epsilon$

Therefore,

$$K = L(1+\frac{1}{2}\lambda^2+\alpha(\rho)M)$$

$\left(\Omega^{-1}\left(\Omega(1)+d|q(\xi,u(\xi),u'(\xi),u''(\xi))|\right)\right).$

Let $|P(t,u(t))|\leq A|u(t)|$ where constant $A>0$. Then, the following results are established.

Corollary 4.1:

Let $|P(t,u(t))|\leq A|u(t)|,\quad P(t,0)=0$, where $A>0$. Let the function $f(t,u(t),u'(t))$ be continuous and satisfies the same conditions as in Theorem 3.17. Then, equation (4.28) is Hyers-Ulam stable if inequality (4.29) is satisfied with Hyers-Ulam constant

$$K = L(1+AM)\left(\Omega^{-1}\left(\Omega(1)+\lambda d|q(\xi,u(\xi),u'(\xi),u''(\xi))|\right)\right). \tag{4.40}$$

Provided $\int_{t_0}^{\infty}h(s)ds = d<\infty,\ d>0$

Proof:

Let the proof begins from equation (4.33) and by applying the Theorem 1.1 there exits points $\xi\in[t_0,t]$ such that

$$\begin{aligned}f(t,u(t),u'(t))u(t)\leq \epsilon L+q(\xi,u(\xi),u'(\xi),u''(\xi))\int_{t_0}^{t}f(s,u(s),u'(s))ds\\ +\int_{t_0}^{t}P(s,u(s))u'(s)ds\end{aligned} \tag{4.41}$$

Taking the absolute value and Let $|f(t,u(t),u'(t))||u(t)|\geq|u(t)|$, by the hypothesis of the Corollary we have

$$\begin{aligned}|u(t)|\leq \epsilon L+|q(\xi,u(\xi),u'(\xi),u''(\xi))|\int_{t_0}^{t}|f(s,u(s),u'(s))|ds\\ +A\int_{t_0}^{t}|u'(s)||u(s)|ds\end{aligned} \tag{4.42}$$

By Theorem 1.1 there exits points $\rho\in[t_0,t]$ such that

$$\begin{aligned}|u(t)|\leq \epsilon L+|q(\xi,u(\xi),u'(\xi),u''(\xi))|\int_{t_0}^{t}|f(s,u(s),u'(s))|ds\\ +A|u(\rho))|\int_{t_0}^{t}|u'(s)|ds\end{aligned} \tag{4.43}$$

We have that

$$\begin{aligned}
|u(t)| &\leq \epsilon L + LA|u(\rho)| + |q(\xi, u(\xi), u'(\xi), u''(\xi))||u'(s)| \int_{t_0}^{t} h(s)\varpi(|u(s)|)ds \\
|u(t)| &\leq \epsilon L + LAM + |q(\xi, u(\xi), u'(\xi), u''(\xi))||u'(s)| \int_{t_0}^{t} h(s)\varpi(|u(s)|)ds \\
&\leq \epsilon L(1 + AM) + |q(\xi, u(\xi), u'(\xi), u''(\xi))||u'(s)| \int_{t_0}^{t} h(s)\varpi(|u(s)|)|u'(s)|ds \\
|u(\rho)| &\leq M \\
|u'(t)| &\leq \lambda
\end{aligned}$$

Hence,

$$\frac{|u(t)|}{B} \leq 1 + |q(\xi, u(\xi), u'(\xi), u''(\xi))|\lambda \int_{t_0}^{t} h(s)\varpi\left(\frac{|u(s)|}{B}\right) ds \tag{4.44}$$

Where $B = \epsilon L + LAM$, setting $z =$ R.H.S of (4.44)

Therefore,

$$z(t) \leq 1 + |q((\xi), u(\xi), u'(\xi), u''(\xi))|\lambda \int_{t_0}^{t} h(s)\varpi(z(s))ds \tag{4.45}$$

$0 < \omega(z(t)) \leq \varpi(v(t))$

for $v(t) =$ R.H.S of (4.45)

$$\frac{v'(t)}{\varpi(v(t))} \leq |q(\xi, u(\xi), u'(\xi), u''(\xi))|\lambda h(t)$$

$$\frac{d\Omega(v(t))}{dt} \leq |q((\xi), u(\xi), u'(\xi), u''(\xi))|\lambda h(t)$$

Integrating

$$\Omega(v(t)) - \Omega(v(t_0)) \leq |q((\xi), u(\xi), u'(\xi), u(\xi))|\lambda \int_{t_0}^{t} h(s)ds$$

since $v(t_0) = 1$ and $\Omega^{-1}(u)$ is an increasing function also we have

$$z(t) \leq v(t) \leq \Omega^{-1}\left(\Omega(1) + |q((\xi), u(\xi), u'(\xi), u''(\xi))|\lambda \int_{t_0}^{t} h(s)ds\right)$$

$$\begin{aligned}
\frac{|u(t)|}{B} &\leq \Omega^{-1}\left(\Omega(1) + |q((\xi), u(\xi), u'(\xi), u''(\xi))| \int_{t_0}^{t} h(s)ds\right) \quad \text{for } t \geq t_0 \\
&\leq \Omega^{-1}\left(\Omega(1) + |q(\xi, u(\xi), u'(\xi), u''(\xi))|\lambda d\right) \; as\; t \to \infty
\end{aligned}$$

provided $\lim_{t_0 \to \infty} \int_{t_0}^{t} h(s)ds \leq d < \infty$

that is

$|u(t)| \leq \epsilon K$,

where,

$K = L(1 + AM)\left(\Omega^{-1}\left(\Omega(1) + |q(\xi, u(\xi), u'(\xi), u''(\xi))|\lambda d\right)\right)$ forall $t \geq t_0$.

Corollary 4.2:

Let all the conditions of Theorem 4.3 remained valid. Suppose

$$|f(t, u(t), u'(t))| \leq h(t)|u'(t)|\left(\varpi(|u(t)|) + 2|u(t)|\right) \tag{4.46}$$

and

$|P(t,u(t))| \leq \alpha(t)|u'(t)|^n$

where ϖ belongs to the class Ψ and $h, \alpha \in C(\mathbf{I}, \mathbf{R}_+)$. Then, equation (4.28) is stable in the sense of Hyers-Ulam stability with Hyers-Ulam constant

$$K = L(1+\alpha(\xi)\delta^n)\left(\Omega^{-1}\left(\Omega(1) + |q(\xi,u(\xi),u'(\xi))|\lambda d\right)\right). \quad (4.47)$$

Proof:

Using (4.33) with the Theorem 1.1, there exits points $\xi, \rho \in [t_0, t]$ such that

$$\begin{aligned} f(t,u(t),u'(t))u(t) \leq \epsilon L + q(\xi,u(\xi),u'(\xi),u''(\xi)) \int_{t_0}^{t} f(s,u(s),u'(s))ds \\ +P(\rho,u(\rho)) \int_{t_0}^{t} u'(s)ds. \end{aligned} \quad (4.48)$$

Taking the absolute value, using the hypothesis in Corollary 4.1 together with hypothesis in Theorem 4.3 and setting $|f(t,u(t),u'(t))||u(t)| \geq |u(t)|$, $|f(t,u(t),u'(t))||u(t)| \geq |u(t)|$,

$$\frac{|u(t)|}{B} \leq 1 + |q(\xi,u(\xi),u'(\xi),u''(\xi))|\lambda \int_{t_0}^{t} h(s)\left(\varpi\left(\frac{|u(s)|}{B}\right) + 2\frac{|u(s)|}{B}\right)ds, \quad (4.49)$$

where $B = \epsilon L\left(1+\alpha(\rho)M^n\right)$, setting $z =$ L.H.S of (4.49).

Therefore,

$$z(t) \leq 1 + |q((\xi),u(\xi),u'(\xi),u''(\xi))|\lambda \int_{t_0}^{t} h(s)\left(\varpi(z(s)) + 2z(s)\right)ds \quad (4.50)$$

$$0 < \varpi(z(t)) \leq \omega(v(t)) \quad (4.51)$$

for $v(t)=$ R.H.S of (4.51)

$$\frac{v'(t)}{\varpi(v(t)) + 2v(t)} \leq |q(\xi,u(\xi),u'(\xi),u(\xi))|\lambda h(t) \quad (4.52)$$

Using Lemma 2.4 by defining $\Re(u)$ as in equation (3.43) we obtain

$$\begin{aligned} \frac{|u(t)|}{B} &\leq \Re^{-1}\left(\Re(1) + |q((\xi),u(\xi),u'(\xi),u''(\xi))|\lambda \int_{t_0}^{t} h(s)ds\right) \text{ for } t \geq t_0 \\ &\leq \Re^{-1}\left(\Re(1) + |q(\xi,u(\xi),u'(\xi),u''(\xi))|\lambda d\right) \text{ as } t \to \infty \end{aligned}$$

provided $\lim_{t_0 \to \infty} \int_{t_0}^{t} h(s)ds \leq d < \infty$ that is

$|u(t)| \leq \epsilon K$

Furthermore, we have

$|u(t) - u(t_0)| \leq |u(t)| \leq \epsilon K$

Where,

$K = L(1+\alpha(\rho)\delta^n)\left(\Re^{-1}\left(\Re(1) + |q(\xi,u(\xi)u'(\xi),u''(\xi))|\lambda m\right)\right)$ forall $t \geq t_0$.

Corollary 4.3:

Let $|P(t,u(t))| \leq \alpha(t)|u(t)|^n$, $P(t,0) = 0$, where $\alpha(t)$ nonnegative, nondecreasing continuous function and $n \in \mathbf{N}$. Let the function $f(t,u(t),u'(t))$ be expressed as in Theorem 4.3. Then, equation (4.28) is Hyers-Ulam stable with Hyers-Ulam constant given as

$K = (1+\alpha(\rho)|u(\rho)|^n)\Omega^{-1}\left(\Omega(1)+|q(\xi,u(\xi),u'(\xi),u''(\xi))|\lambda m\right).$

Provided $\int_{t_0}^{\infty} h(s)ds = d < \infty \; d > 0$

Proof:

From equation (4.29), we obtain

$u(t) \leq L\epsilon + \int_{t_0}^{t} q(s,u(s),u'(s))f(s,u(s),u'(s))ds + \int_{t_0}^{t} P(s,u(s))u'(s)ds$

Applying the Theorem 1.1, that is there exist $\xi, \rho \in [t_0, t]$ such that

$$|u(t)| \leq \epsilon L + |q(\xi,u(\xi),u'(\xi),u''(\xi))| \int_{t_0}^{t} |f(s,u(s),u'(s))|ds + |P(\rho,u(\rho))| \int_{t_0}^{t} |u'(s)|ds. \tag{4.53}$$

Using the hypothesis of this Corollary and Theorem 4.3, with application of Theorem 1.1 there exists $\eta \in [t_0, t]$ such that

$$|u(t)| \leq \epsilon L + |q(\xi,u(\xi),u'(\xi),u''(\xi))||u'(t)||\varpi(\eta)| \int_{t_0}^{t} h(s)ds + L\alpha(\rho)|u(\rho)|^n. \tag{4.54}$$

It follows that

$|u(t)| \leq L\epsilon(1+\alpha(\rho)|u(\rho)|^n) + |q(\xi,u(\xi),u'(\xi),u''(\xi))||u'(t)| \int_{t_0}^{t} h(s)\varpi(|u(s)|)ds$

By application of Lemma 2.4 and some hypothesis of this Corollary, we arrived at

$|u(t)| \leq \epsilon L(1+\alpha(\rho)|u(\rho)|^n)\Omega^{-1}\left(\Omega(1)+|q(\xi,u(\xi),u'(\xi),u''(\xi))|\lambda m\right)$

where $|u'(t)| \leq \lambda$ and Hyers-Ulam constant is given as

$K = (1+\alpha(\rho)|u(\rho)|^n)\Omega^{-1}\left(\Omega(1)+|q(\xi,u(\xi),u'(\xi),u''(\xi))|\lambda m\right)$

This concludes the proof.

In the next theorem, we consider the Hyers-Ulam stability of special case of equation(4.28) in the form

$$u''(t) + 2f(t)\alpha(u(t))u'(t) + u(t) + P(t,u(t)) = 0. \tag{4.55}$$

with initial conditions $u(t_0) = u'(t_0) = 0$.

Theorem 4.4:

Equation (4.55) is Hyers-Ulam stable if there exists constants $K \geq 0, \; \epsilon > 0$ and the solution $u(t) \in C^2(\mathbf{I}, \mathbf{R}_+)$ of

$$|u''(t) + 2f(t)\alpha(u(t))u'(t) + u(t) + P(t,u(t))| \leq \epsilon \tag{4.56}$$

such that

$|u(t) - u_0(t)| \leq K\epsilon$

for $u_0(t) \in C^2(\mathbf{I}, \mathbf{R}_+)$ is a solution of equation (4.55), where

$|P(t,u(t))| \leq \alpha(t)\varpi(|u(t)|)$ and K is H-U constant given as

$$K = (L + \frac{1}{2}\lambda^2)\Omega^{-1}\left(\Omega(1) + \lambda j\omega\left(F^{-1}\left(F(1)+2\lambda^2 l\right)\right)\right) F^{-}\left(F(1)+2\lambda^2 j\right) \tag{4.57}$$

Proof:

From (4.56), we obtain

$$-\epsilon \leq u''(t) + 2f(t)\alpha(u(t))u'(t) + u(t) - P(t,u(t)) \leq \epsilon \tag{4.58}$$

Multiplying through by $u'(t)$, we get

$$-u'(t)\epsilon \leq u''(t)u'(t) + 2f(t)\alpha(u(t))u'(t)u'(t) + u(t)u'(t)$$
$$-P(t,u(t))u'(t) \leq u'(t)\epsilon$$

Integrating from t_0 to t,

$$\int_{t_0}^{t} u'(s)\epsilon ds \leq \int_{t_0}^{t} u''(s)u'(s)ds + \int_{t_0}^{t} 2f(s)\alpha(u(s))u'(s)u'(s)ds$$
$$+\int_{t_0}^{t} u(s)u'(s)ds - \int_{t_0}^{t} P(s,u(s))u'(s)ds$$

Using equation (4.20) to have

$$\int_{t_0}^{t} u'(s)\epsilon ds \leq \int_{t_0}^{t} u''(s)u'(s)ds + \int_{t_0}^{t} 2f(s)\alpha(u(s))u'(s)u'(s)ds$$
$$+\int_{t_0}^{t} \frac{d}{ds}G(u(s))ds - \int_{t_0}^{t} P(s,u(s))u'(s)ds$$

Evaluate the integration by applying the initial conditions and using the hypothesis of Theorem 4.4 we have

$$|G(u(t))| \leq L\epsilon + \frac{1}{2}(u'(t))^2 + 2\int_{t_0}^{t} f(s)\alpha(|u(s)|)(|u'(s|))^2 ds$$
$$+\int_{t_0}^{t} |P(s,u(s))||u'(s)|ds. \tag{4.59}$$

Using the hypothesis of the Theorem 4.4, let $|u'(t)| \leq \lambda, \ \ \lambda > 0$, and $\mathbb{G}(|u(t)|) \geq |u(t)|$

$$\frac{|u(t)|}{A} \leq 1 + 2\lambda^2 \int_{t_0}^{t} f(s)\alpha(\frac{|u(s)|}{A})ds + \lambda\int_{t_0}^{t} \alpha(s)\omega(\frac{|u(s)|}{A})ds, \tag{4.60}$$

for $A = (L + \frac{1}{2}\lambda^2)\epsilon$

Applying Theorem 3.7, we obtain

$$|u(t)| \leq (L + \frac{1}{2}\lambda^2)\epsilon\Omega^{-1}\left(\Omega(1) + \lambda\int_{t_0}^{t} \alpha(s)\omega\left(F^{-1}\left(F(1) + 2\lambda^2\int_{t_0}^{s} f(\delta)d\delta\right)\right)ds\right)$$
$$F^{-1}\left(F(1) + 2\lambda^2\int_{t_0}^{t} f(s)ds\right)$$

Let $\lim_{t\to\infty}\int_{t_0}^{t} f(s)ds = l < \infty$, $\lim_{t\to\infty}\int_{t_0}^{t} \alpha(s)ds = j < \infty$. for $l, j > 0$ it follows that

$$|u(t)| \leq \epsilon(L + \frac{1}{2}\lambda^2)\Omega^{-1}\left(\Omega(1) + \lambda j\varpi\left(F^{-1}\left(F(1) + 2\lambda^2 l\right)\right)\right)$$
$$F^{-1}\left(F(1) + 2\lambda^2 j\right) \tag{4.61}$$

$$|u(t) - u_0(t)| \leq |u(t)| \leq \epsilon(L + \frac{1}{2}\lambda^2)\Omega^{-1}\left(\Omega(1) + \lambda j\omega\left(F^{-1}\left(F(1) + 2\lambda^2 l\right)\right)\right)$$
$$F^{-1}\left(F(1) + 2\lambda^2 j\right) \tag{4.62}$$

Hence, Hyers-Ulam constant is given as

$$K = (L + \frac{1}{2}\lambda^2)\Omega^{-1}\left(\Omega(1) + \lambda j\omega\left(F^{-1}\left(F(1) + 2\lambda^2 l\right)\right)\right)F^{-1}\left(F(1) + 2\lambda^2 j\right)$$

.

4.1.4 Hyers-Ulam Stability Nonlinear Differential Equation with Forcing Term

In this subsection, equation

$$u''(t) + f(t, u(t), u'(t)) = P(t, u(t), u'(t)) \quad (4.63)$$

initial condition $u(t_0) = u'(t_0) = 0$ is considered, where $P, f \in C(\mathbf{I} \times \mathbf{R}^2)$ is going to be considered.

Definition 4.3:

The equation (4.63) with the initial conditions $u(t_0) = u'(t_0) = 0$ has Hyers-Ulam stability if there exists a positive constant K with following property. For every $\epsilon > 0$ $u \in C^2(\mathbf{R}_+)$, if

$$|u''(t) + f(t, u(t), u'(t)) - P(t, u(t), u'(t))| \leq \epsilon \quad (4.64)$$

then, there exists a solution $u_0(t) \in C^2(\mathbf{R}_+)$ of the equation (4.63), such that $|u(t) - u_0(t)| \leq K\epsilon$

Theorem 4.5:

The equation (4.63) with its initial value is said to be Hyers-Ulam stable if

$$|f(t, u(t), u'(t))| \leq f(t)|u(t)| + f(t)h(t)\left(\int_{t_0}^{t} g(s)\varpi(|u(s)|)ds\right), \quad (4.65)$$

where Ψ and $h, f, g \in C(\mathbf{I}, \mathbf{R}_+)$. with Hyers-Ulam constant

$$K = L(1 + \frac{1}{2}\lambda^2 + |q(\xi, u(\xi), u'(\xi), u''(\xi))| + |p(u(\rho), u'(\rho))|)\Omega^{-1}\left(\Omega(1) + nQ\right) \quad (4.66)$$

Proof:

Multiplying (4.64) by $|u'(t)|$, to get

$$|u'(t)|\epsilon \leq |u'(t)|u''(t) + f(t, u(t), u'(t))|u'(t)| - |u'(t)|P(t, u(t), u'(t))| \leq |u'(t)|\epsilon.$$

Integrating from t to t_0 to obtain

$$\begin{aligned} -\epsilon\int_{t_0}^{t}|u'(s)|ds \leq \frac{1}{2}u'(t)^2 + \int_{t_0}^{t} f(s, u(s), u'(s))u'(s)ds \\ - \int_{t_0}^{t} P(u(s), u'(s))u'(s)ds \leq \epsilon\int_{t_0}^{t}|u'(s)|ds \end{aligned} \quad (4.67)$$

Integrating by part and using initial condition and let $f(t, u(t), u'(t))u(t) \geq u(t)$,

$$\begin{aligned} u(t) \leq \epsilon L + \frac{1}{2}u'(t)^2 + \int_{t_0}^{t} q(s, u(s), u'(s), u''(s))f(s, u(s), u'(s))ds \\ + \int_{t_0}^{t} P(u(s), u'(s))u'(s)ds \end{aligned} \quad (4.68)$$

where
$\dfrac{f'(s,u(s),u'(s))u(s)}{f(s,u,u')} = q(t,u,u',u'')$
By Theorem 1.1 there exist points $\xi, \rho \in (t_0, t)$ such that

$$\begin{aligned} u(t) \le \epsilon L + \frac{1}{2}u'(t)^2 + q((\xi),u(\xi),u'(\xi),u''(\xi)) \int_{t_0}^{t} f(s,u(s),u'(s))ds \\ +P(u(\rho),u'(\rho)) \int_{t_0}^{t} u'(s)ds \end{aligned} \tag{4.69}$$

Taking the absolute value, setting $|u'(t)| \le \lambda$ and by the hypothesis of this Theorem 4.5

$$\le \mathbb{B} + \int_{t_0}^{t} f(s)|u(s)|ds + \int_{t_0}^{t} f(s)h(s)\left(\int_{t_0}^{s} g(\tau)\varpi(|u(\tau)|)d\tau\right)ds, \tag{4.70}$$

where
$\mathbb{B} = \epsilon L(1 + \frac{1}{2}\lambda^2 + |q(\xi,u(\xi),u'(\xi),u''(\xi))| + |p(u(\rho),u'(\rho))|)$

$$\begin{aligned} \frac{|u(t)|}{\mathbb{B}} \le 1 + \int_{t_0}^{t} f(s)\frac{|u(s)|}{\mathbb{B}}ds + \int_{t_0}^{t} f(s)h(s) \\ \left(\int_{t_0}^{s} g(\tau)\varpi\left(\frac{|u(\tau)|}{\mathbb{B}}\right)d\tau\right)ds \end{aligned} \tag{4.71}$$

By Theorem 3.10, we have

$$\frac{|u(t)|}{\mathbb{B}} \le \Omega^{-1}\left(\Omega(1) + \int_{t_0}^{t} f(s)h(s)g(s)\left(Q(s) + \int_{t_0}^{s} Q(\tau)d\tau\right)ds\right), \tag{4.72}$$

where $Q(t)$ is defined in equation (3.3).
Setting $\lim_{t_0\to\infty} \int_{t_0}^{t} f(s)ds \le l < \infty$, $Q = \exp l$ as $t \to \infty$, and
$\lim_{t\to\infty} \int_{t_0}^{t} sf(s)h(s)g(s)ds \le n < \infty$, then

$$\frac{|u(t)|}{G} \le \Omega^{-1}(\Omega(1) + Qn) \tag{4.73}$$

Hence,

$|u(t)| \le \epsilon K$
$K = L(1 + \frac{1}{2}\lambda^2 + |q(\xi,u(\xi),u'(\xi),u''(\xi))| + |p(u(\rho),u'(\rho))|)\Omega^{-1}(\Omega(1) + nQ)$
Therefore,

$|u(t) - u_0(t)| \le |u(t)| \le K\epsilon.$

This completes the proof.

In the subsequent theorem, we consider Hyers-Ulam Stability of a perturbed Lienard equation

$$u'' + c(t)f(u(t))u'(t) + \alpha(t)\phi(u(t)) = P(t,u(t),u'(t)) \tag{4.74}$$

with conditions $u(t_0) = u'(t_0) = 0$, where $f, \phi \in C(\mathbf{R}_+, \mathbf{R}_+)$, $\alpha, c, \in C(\mathbf{I}, \mathbf{R}_+)$.

The following definitions are presented here to assist in the proof of our results.

Definition 4.4:

Equation (4.74) is Hyers-Ulam stable, if there exists a constants $K > 0$ and $\epsilon > 0$

such that for $u(t) \in C^2(\mathbf{I}, \mathbf{R}_+)$, satisfying

$$|u'' + c(t)f(u(t))u'(t) + \alpha(t)\phi(u(t)) - P(t, u(t), u'(t))| \leq \epsilon \tag{4.75}$$

there exists a solution $u_0(t) \in C^2(\mathbf{I}, \mathbf{R}_+)$ of the equation (4.74), such that

$|u(t) - u_0(t)| \leq K\epsilon.$

Where K is called Hyers-Ulam constant.

Theorem 4.6:

Let the functions f, c, ϕ, α and P be continuous. Suppose that

$$\lim_{t\to\infty} \int_{t_0}^{t} c(s)ds = m < \infty \tag{4.76}$$

and

$$\lim_{t\to\infty} \int_{t_0}^{t} \beta(s)ds = n =< \infty \tag{4.77}$$

then equation(4.74) is Hyers-Ulam stable with the Hyers-Ulam constant

$$K = \left(\frac{L}{\alpha(\xi)} + \frac{\lambda^2}{2\alpha(\xi)}\right)\Omega^{-1}\left(\Omega(1) + \frac{\lambda^2}{\alpha(\xi)}n\varpi\left(F^{-1}\left(F(1) + \frac{\lambda^2}{\alpha(\xi)}m\right)\right)\right) \\ F^{-1}\left(F(1) + \frac{\lambda^2}{\alpha(\xi)}m\right) \tag{4.78}$$

with Ω defined in (2.13)

Proof:

From the inequality (4.75) and by multiplying by $u'(t)$,

$$-\epsilon u'(t) \leq u''(t)u'(t) + c(t)f(u(t))(u'(t))^2 + \alpha(t)\phi(u(t))u'(t) \\ -P(t, u(t), u'(t))u'(t) \leq \epsilon u'(t) \tag{4.79}$$

Integrating (4.79) from t_0 to t, we have

$$\frac{1}{2}(u'(s))^2 + \int_{t_0}^{t} c(s)f(u(s))(u'(s))^2 ds + \int_{t_0}^{t} \alpha(t)\phi(u(t))u'(s)ds \\ - \int_{t_0}^{t} P(s, u(s), u'(s))u'(s)ds \leq \epsilon \int_{t_0}^{t} u'(s)ds \tag{4.80}$$

By Theorem 1.1 there exist $\xi \in [t_0, t]$ such that

$$\frac{1}{2}(u'(s))^2 + \int_{t_0}^{t} c(s)f(u(s))(u'(s))^2 ds + \alpha(\xi)\int_{t_0}^{t} \phi(u(t))u'(s)ds \\ - \int_{t_0}^{t} P(s, u(s), u'(s))u'(s)ds \leq \epsilon \int_{t_0}^{t} u'(s)ds \tag{4.81}$$

Using

$$\Phi(u(t)) = \int_{u(t_0)}^{u(t)} \phi(u(s))ds \tag{4.82}$$

Equation (4.81) becomes

$$\frac{1}{2}(u'(s))^2 + \int_{t_0}^{t} c(s)f(u(s))(u'(s))^2 ds + \alpha(\xi)\int_{t_0}^{t} \frac{d}{ds}\Phi(s)ds \\ - \int_{t_0}^{t} P(s, u(s), u'(s))u'(s)ds \leq \epsilon \int_{t_0}^{t} u'(s)ds$$

Integrating, taking the absolute value of both sides and setting
$\int_{t_0}^{t} |u'(s)|ds \leq L$,
$|u'(t)| \leq \lambda$ and $|P(t, u(t), u'(t))| \leq \beta(t)\rho(u)|u'(t)|$

$$|\phi(u(t))| \leq \epsilon \left(\frac{L}{\lambda^2\alpha(\xi)} + \frac{\lambda^2}{2\alpha(\xi)} \right) + \frac{\lambda^2}{\alpha(\xi)} \int_{t_0}^{t} c(s)f(|u(s))|ds + \frac{\lambda^2}{\alpha(\xi)} \int_{t_0}^{t} \beta(s)\rho(u(s))ds \tag{4.83}$$

Let $|\phi(u(t)| \geq |u(t)|$, then

$$|u(t)| \leq S + \frac{\lambda^2}{\alpha(\xi)} \int_{t_0}^{t} c(s)f(|u(s))|ds + \frac{\lambda^2}{\alpha(\xi)} \int_{t_0}^{t} \beta(s)\rho(u(s))ds \tag{4.84}$$

where

$$S = \epsilon \left(\frac{L}{\lambda^2\alpha(\xi)} + \frac{\lambda^2}{2\alpha(\xi)} \right)$$

By Theorem 3.10 we have

$$\frac{|u(t)|}{S} \leq \Omega^{-1} \left(\Omega(1) + \frac{\lambda^2}{\alpha(\xi)} \int_{t_0}^{t} \beta(s)\varpi \left(F^{-1} \left(F(1) + \frac{\lambda^2}{\alpha(\xi)} \int_{t_0}^{s} c(\delta)d\delta \right) \right) ds \right) F^{-1} \left(F(1) + \frac{\lambda^2}{\alpha(\xi)} \int_{t_0}^{t} c(s)ds \right) \tag{4.85}$$

Using (4.76) and (4.77), by substituting the value of S

$$|u(t)| \leq \epsilon \left(\frac{L}{\alpha(\xi)} + \frac{\lambda^2}{2\alpha(\xi)} \right) \Omega^{-1} \left(\Omega(1) + \frac{\lambda^2}{\alpha(\xi)} n\varpi \left(F^{-1} \left((1) + \frac{\lambda^2}{\alpha(\xi)} m \right) \right) \right) F^{-1} \left(F(1) + \frac{\lambda^2}{\alpha(\xi)} m \right) \tag{4.86}$$

Hence,

$$K = \left(\frac{L}{\alpha(\xi)} + \frac{\lambda^2}{2\alpha(\xi)} \right) \Omega^{-1} \left(\Omega(1) + \frac{\lambda^2}{\alpha(\xi)} n\varpi \left(F^{-1} \left(F(1) + \frac{\lambda^2}{\alpha(\xi)} m \right) \right) \right) F^{-1} \left(F(1) + \frac{\lambda^2}{\alpha(\xi)} m \right)$$

Therefore,

$|u(t) - u_0(t)| \leq |u(t)| \leq K\epsilon.$

Now, let us examining

$$u'' + c(t)f(u(t))u'(t) + Bg(u(t)) = P(t, u(t), u'(t)) \tag{4.87}$$

with initial value $u(t_0) = u'(t_0) = 0$, for $f \in C(\mathbf{R}_+, \mathbf{R}_+)$, $g \in C(\mathbf{R}_+, \mathbf{R}_+)$, $c \in C(\mathbf{I}, \mathbf{R}_+)$, for $\mathbf{R}_+ = [t_0, \infty)$, $\mathbf{I} = (t_0, b)(b \leq \infty)$, $P \in C(\mathbf{I} \times \mathbf{R}_+^{\mathbf{2}}, \mathbf{R}_+)$ and constant $B > 0$.

Theorem 4.7:

Let the functions f, c, g and P be continuous functions on $C(\mathbf{R}_+)$. Equation (4.87) is said to be stable in the sense of Hyers-Ulam stability if $g(u(t))$ a nonnegative, continuous function on $(\mathbf{R}_+)$, constant $B > 0$, $|P(t, u(t), u'(t))| \leq A|u(t)||u'(t)|^n$, for $n \in \mathbf{Z}_+$, $A > 0$ is a constant and inequality

$$|u'' + c(t)f(u(t))u'(t) + Bg(u(t)) - P(t, u(t), u'(t))| \leq \epsilon \tag{4.88}$$

is satisfied, with Hyers-Ulam constant

$$K = \frac{1}{B}\left(L + LA\lambda^n|u(\xi)|\right)\Omega^{-1}\left(\Omega(1) + \frac{\lambda^2}{B}m\right) \tag{4.89}$$

Proof:

From inequality

$$-\epsilon \leq u'' + c(t)f(u(t))u'(t) + Bg(u(t)) - P(t, u(t), u'(t)) \leq \epsilon,$$

where B a positive constant. Define $G(u(t))$ as

$$G(u(t)) = \int_{u(t_0)}^{u(t)} g(s)ds \tag{4.90}$$

Now integrating from t_0 to t

$$\begin{aligned}&\int_{t_0}^{t} c(s)f(u(s))(u'(s))^2 ds + B\int_{t_0}^{t}\frac{d}{ds}G(u(s))ds\\&\quad - \int_{t_0}^{t} P(s, u(s), u'(s))u'(s)ds \leq \epsilon\int_{t_0}^{t} u'(s)ds\end{aligned} \tag{4.91}$$

Integrating, by hypothesis of the Theorem 4.7 and setting, $|G(u(t)| \geq |u(t)|$, $\int_{t_0}^{t} |u'(s)|ds \leq L$ for constant $L > 0$.

$$|u(t)| \leq \frac{1}{B}\epsilon L + \frac{1}{B}\int_{t_0}^{t} c(s)f(|u(s)|)(|u'(s)|)^2 ds + \frac{1}{B}A\int_{t_0}^{t} |u(s))||u'(s)|^{n+1}ds \tag{4.92}$$

By Theorem 1.1, for $t_0 < \xi < t$, and $|u'(t)| \leq \lambda$, for $\lambda > 0$ we have

$$|u(t)| \leq \frac{1}{B}\epsilon L + \frac{1}{\delta}LA\lambda^n|u(\xi)| + \frac{\lambda^2}{B}\int_{t_0}^{t} c(s)f(|u(s)|)ds \tag{4.93}$$

Setting

$$D = \frac{1}{B}\epsilon\left(L + LA\lambda^n|u(\xi)|\right) \tag{4.94}$$

Since $f \in \Psi$, we obtain

$$z(t) \leq 1 + \frac{\lambda^2}{B}\int_{t_0}^{t} c(s)f(z(s))ds, \tag{4.95}$$

for $\dfrac{|u(t)|}{D} = z(t)$

Let $\omega(z(t)) = f(z(t))$ By Lemma 2.4 and using (4.76)

$$|u(t)| \leq \epsilon\frac{1}{B}\left(L + LA\lambda^n|u(\xi)|\right)\Omega^{-1}\left(\Omega(1) + \frac{\lambda^2}{B}m\right) \tag{4.96}$$

Hence,

$$K = \frac{1}{B}\left(L + LA\lambda^n|u(\xi)|\right)\Omega^{-1}\left(\Omega(1) + \frac{\lambda^2}{B}m\right)$$

Therefore,

$$|u(t) - u_0(t)| \leq |u(t)| \leq K\epsilon$$

This ends the proof of the Theorem.

4.1.5 Hyers-Ulam Stability of Nonperturbed Differential Equation

In this section we first consider the Hyers-Ulam stability of the nonlinear second order differential equation

$$u''(t) + f(t, u(t)) = 0 \tag{4.97}$$

with initial conditions $u(t_0) = u'(t_0) = 0$ where $P(t, u(t)) = 0$.

Definition 4.5:

The equation(4.97) with its initial conditions has Hyers-Ulam stability if there exists a positive constant K such that for every solution $u(t) \in C^2(\mathbf{R}_+)$ satisfying

$$|u''(t) + f(t, u(t))| \leq \epsilon \tag{4.98}$$

for $\epsilon > 0$, then, there exists a solution $u_0(t) \in C^2(\mathbf{R}_+)$ of the equation (4.97) such that

$$|u(t) - u_0(t)| \leq K\epsilon$$

Theorem 4.8:

Suppose inequality (4.98) is satisfied where $\varpi \in \Psi$ and Ω is defined in(2.13), equation(4.97) is said to be Hyers-Ulam stable with Hyers-Ulam constant

$$K = \left(L + \frac{1}{2}\lambda^2\right) \left(\Omega^{-1}\left(\Omega(1) + |g(\xi, u(\xi), u'(\xi))|m\right)\right) \tag{4.99}$$

. **Proof:**

Multiplying (4.98) by $|u'(t)|$ we obtain

$$-\epsilon|u'(t)| \leq u'(t)u''(t) + f(t, u(t))u'(t) \leq \epsilon|u'(t)| \tag{4.100}$$

Integrating each term from t_0 to t, then,

$$-\epsilon \int_{t_0}^{t} |u'(s)|ds \leq \frac{1}{2}u'(t)^2 + \int_{t_0}^{t} f(s, u(s))u'(s)ds \leq \epsilon \int_{t_0}^{t} |u'(s)|ds, \tag{4.101}$$

for any $t \geq t_0$. Integrating by part, setting $\int_{t_0}^{\infty} |u'(s)|ds \leq L$ for $L > 0$ and $f_u(t, u(t)) \leq 0$.

$$\begin{gathered} f(t, u(t))u(t) \leq \epsilon L + \frac{1}{2}u'(t)^2 \\ + \int_{t_0}^{t} g(s, u(s), u'(s))f(s, u(s))u(s)ds \quad \text{for} \quad t \geq t_0, \end{gathered} \tag{4.102}$$

where

$$\frac{f'(t, u(t))u(t)}{f(t, u(t))} = g(t, u(t), u'(t)) \tag{4.103}$$

By Theorem 1.1,there exits $\xi \in [t_0, t]$ such that

$$|u(t)| \leq \epsilon L + \frac{1}{2}\lambda^2 + |g(\xi, u(\xi), u'(\xi))| \int_{t_0}^{t} |f(s, u(s))| \tag{4.104}$$

for $|u'(t)| \leq \lambda$, for $\lambda > 0$, and $|f(t, u(t))||u(t)| \geq |u(t)|$,

if $\mathbb{F} = \epsilon(L + \frac{1}{2}\lambda^2) > 0$, we have

$$\frac{|u(t)|}{\mathbb{F}} \leq 1 + |g((\xi), u(\xi), u'(\xi))| \int_{t_0}^{t} \phi(s)\varpi\left(\frac{|u(s)|}{\mathbb{F}}\right) ds \quad t \geq t_0 \tag{4.105}$$

Setting $v(t) =$ R.H.S ,(4.105) and since ϖ is nondecreasing,then

$$\begin{aligned} 0 < \varpi\left(\frac{|u(t)|}{\mathbb{F}}\right) &\leq \varpi(v(t)) \\ v'(t) &= |g((\xi), u(\xi), u'(\xi))|\phi(t)\varpi\left(\frac{|u(t)|}{\mathbb{F}}\right) \\ &\leq |g(\xi), u(\xi), u'(\xi))|\phi(t)\varpi(v(t)) \end{aligned}$$

then,

$$\frac{v'(t)}{\varpi(v(t))} \leq |g(\xi, u(\xi), u'(\xi))|\phi(t) \tag{4.106}$$

Using equation(2.13), we obtain

$$\frac{d\Omega(v(t))}{dt} \leq |g(\xi, u(\xi), u'(\xi))|\phi(t) \tag{4.107}$$

Integrating from t_0 to t gives

$$v(t) \leq \Omega^{-1}\left(\Omega(1) + |g(\xi, u(\xi), u'(\xi))| \int_{t_0}^{t} \phi(s)ds\right). \tag{4.108}$$

Setting $\lim_{t\to\infty} \int_{t_0}^{t} \phi(s)ds \leq m < \infty$, this leads to

$$\frac{|u(t)|}{\mathbb{F}} \leq \Omega^{-1}\left(\Omega(1) + |g(\xi, u(\xi), u'(\xi))|m\right). \tag{4.109}$$

Substituting $\mathbb{F}$

$$|u(t)| \leq \epsilon(L + \frac{1}{2}\lambda^2\left(\Omega^{-1}\left(\Omega(1) + |g(\xi, u(\xi), u'(\xi))|m\right)\right) \quad \text{forall} \quad t \geq t_0. \tag{4.110}$$

Hence,

$$|u(t) - u_0(t)| \leq K\epsilon$$

Where

$$K = (L + \frac{1}{2}\lambda^2\left(\Omega^{-1}\left(\Omega(1) + |g(\xi, u(\xi), u'(\xi))|m\right)\right).$$

We consider Hyers-Ulam stability of equation

$$u''(t) + f(t, u(t), u'(t)) = 0 \tag{4.111}$$

with the initial conditions $u(t_0) = u'(t_0) = 0$.

Definition 4.6:

Nonlinear differential equation (4.111) together with its initial conditions is Hyers-Ulam stable if there exits any solution $u(t) \in C^2(\mathbf{R}_+)$ satisfying

$$|u''(t) + f(t, u(t), u'(t))| \leq \epsilon \tag{4.112}$$

such that

$$|u(t) - u_0(t)| \leq K\epsilon$$

for $\epsilon > 0$, $K > 0$ and $u_0(t) \in C^2(\mathbf{R}_+)$ is any solution satisfying equation(4.111)

Theorem 4.9:

Suppose

$$|f(t, u(t), u(t))| \leq h(t)\varpi(|u(t)|)|u'(t)|$$

$$u(t) \leq f(t, u(t), u'(t))u(t),$$

where $f(t,u(t),u'(t)) > 1$ and
$\dfrac{f'(t,u(t),u'(t))u(t)}{f(t,u(t),u'(t))} = g(t,u(t),u'(t),u''(t)),$
h a nonnegative, nondecreasing, continuous function on $\mathbf{R}_+$ and g a positive continuous function on $\mathbf{I}\times{\mathbf{R}_+}^3$ and $\varpi\in\Psi$, is continuous, nondecreasing in u. Equation(4.111) is Hyers-Ulam stable with Hyers-Ulam constant

$$K = (L+\frac{1}{2}\lambda^2)\left(\Omega^{-1}\left(\Omega(1) \quad +|g(\xi,u(\xi),u'(\xi),u''(\xi))|s\right)\right) \text{ for all } t\geq t_0. \tag{4.113}$$

Proof:

Multiplying(4.112) by $|u'(t)|$

$$-\epsilon|u'(t)| \leq u'(t)u''(t) + f(t,u(t),u'(t))u'(t) \leq \epsilon|u'(t)|\ \forall t\geq t_0 \tag{4.114}$$

Integrating each term from t_0 to t, then,

$$-\epsilon\int_{t_0}^{t}|u(s)|ds \leq \frac{1}{2}u'(t)^2 + \int_{t_0}^{t} f(s,u(s),u'(s))u'(s)ds \leq \epsilon\int_{t_0}^{t}|u'(s)|ds\ \forall t\geq t_0 \tag{4.115}$$

Integrating by part, let

$f_u(t,u(t),u'(t)) + f_{u'}(t,u(t),u'(t)) \leq 0$

and $\displaystyle\int_{t_0}^{\infty}|u'(s)|ds \leq L$

$$f(s,u(s),u'(s))u(t) \leq \epsilon L + \frac{1}{2}u'(t)^2 + \int_{t_0}^{t}\frac{f'(s,u(s),u'(s))u(s)}{f(s,u(s),u'(s))}f(s,u(s),u'(s))ds \tag{4.116}$$

By hypothesis of the Theorem 4.9

$$u(t) \leq \epsilon L + \frac{1}{2}u'(t)^2 + \int_{t_0}^{t} g(s,u(s),u'(s),u''(s))f(s,u(s),u'(s))ds \tag{4.117}$$

By generalised Mean value Theorem 1.1, there exists $\xi\in[t_0,t]$ such that

$$u(t) \leq \epsilon L + \frac{1}{2}u'(t)^2 + g(\xi,u(\xi),u'(\xi),u''(\xi))\int_{t_0}^{t} f(s,u(s),u'(s)ds \tag{4.118}$$

Taking the absolute value and letting

$|u'(t)|\leq\lambda$, for $\lambda>0$ together with the hypothesis in the theorem

$$\frac{|u(t)|}{\mathbb{T}} \leq 1 + |g(\xi,u(\xi),u'(\xi),u''(\xi))|\int_{t_0}^{t} h(s)\varpi(\frac{|u(s)|}{\mathbb{T}})|u'(s)|ds \tag{4.119}$$

for $\mathbb{T} = \epsilon\left(L+\frac{1}{2}\lambda^2\right)$.

Setting $z(t) = \dfrac{|u(t)|}{\mathbb{T}}$, then equation(4.119) becomes

$$z(t) \leq 1 + |g(\xi,u(\xi),u'(\xi),u''(\xi))|\int_{t_0}^{t} h(s)\varpi(z(s))z(s)ds. \tag{4.120}$$

Setting $v(t)=$ R.H.S of equation (4.120) since ϖ is nondecreasing and $h(t)$ is positive continuous function, we have

$$\begin{aligned} 0 < \varpi(z(t)) &\leq \varpi(v(t)) \\ v'(t) &= |g(\xi, u(\xi), u'(\xi), u''(\xi))| h(t) \varpi(z(t)) z(t) \\ &\leq |g(\xi, u(\xi), u'(\xi), u''(\xi))| h(t) \varpi(v(t)) v(t) \end{aligned}$$

$$\frac{v'(t)}{\varpi(v(t))v(t)} \leq |g(\xi, u(\xi), u'(\xi), u''(\xi))| h(t) \tag{4.121}$$

By applying equation(2.13) we get

$$\frac{d\Omega(v(t))}{dt} \leq |g((\xi), u(\xi), u'(\xi), u''(\xi))| h(t) \tag{4.122}$$

Integrating from t_0 to t, and since $v(t_0) = 1$ we get

$$z(t) \leq v(t) \leq \Omega^{-1} \left(\Omega(1) + |g((\xi), u(\xi), u'(\xi) u''(\xi))| \int_{t_0}^{t} h(s) ds \right) \tag{4.123}$$

Finally from (4.123) and substituting the value of T

$$|u(t)| \leq \epsilon (L + \frac{1}{2}\lambda^2) \left(\Omega^{-1} \left(\Omega(1) + |g(\xi, u(\xi), u'(\xi), u''(\xi))| s \right) \right) \quad \forall\, t \geq t_0, \tag{4.124}$$

provided $\lim_{t_0 \to \infty} \int_{t_0}^{t} h(s) ds \leq s < \infty$

then

$K = (L + \frac{1}{2}\lambda^2) \left(\Omega^{-1} \left(\Omega(1) \qquad + |g(\xi, u(\xi), u'(\xi), u''(\xi))| s \right) \right)$ for all $t \geq t_0$.

Hence,

$|u(t)| \leq K\epsilon$

Therefore,

$|u(t) - u_0(t)| \leq K\epsilon.$

The next theorem deals with the consideration of the Hyers-Ulam stability of equation

$$u''(t) + a(t)u'(t) + \frac{1}{b(t)} u(t) + g(t) f(u(t)) = 0 \tag{4.125}$$

where $a, b, g, \in C(\mathbf{I}, \mathbf{R}_+)$. and $f \in C(\mathbf{R}_+, \mathbf{R}_+)$ with $f \in \psi$

Theorem 4.10:

Suppose that $u(t) \in C^2(\mathbf{I}, \mathbf{R}_+)$ satisfies the differential inequality:

$$|u''(t) + a(t)u'(t) + \frac{1}{b(t)} u(t) + g(t) f(u(t))| \leq \epsilon \tag{4.126}$$

Then, if

(i) $\int_{t_0}^{t} \frac{1}{b(s)} ds \leq p$ for $p > 0$, and all $t \in \mathbf{R}_+$

(ii) $\int_{t_0}^{t} \left(\frac{a(s)}{b(s)} - 1 \right) |u'(s)| ds \leq m.$ for $m \geq 0$

(iii) $|u'(t)| \leq \lambda$ for $\lambda \geq 0$.

(iv) $\lim_{t \to \infty} \int_{t_0}^{t} \frac{1}{b^2(s)} ds = l < \infty$ for $l > 0$

$\lim_{t\to\infty} \int_{t_0}^{t} \frac{g(s)}{b(s)} ds = j < \infty$ for $j > 0$

are satisfied, equation (4.126) is Hyers-Ulam stable with the Hyers-Ulam constant K defined as

$$K = (|E(t_0)| + \lambda + p + m)\,\Omega^{-1}\,(\Omega(1) + j\varpi(\exp l))\exp l \tag{4.127}$$

Proof:

Inequality (4.126), implies that

$$u''(t) + a(t)u'(t) + \frac{1}{b(t)}u(t) + g(t)f(u(t)) \leq \epsilon \tag{4.128}$$

Define

$$E(t) = \frac{u'(t)}{b(t)} + u(t), \quad u(t) \neq 0, \quad b(0) \neq 0 \tag{4.129}$$

It is clear that

$$E(t) = E(t_0) + \int_{t_0}^{t} \frac{d}{ds}\left(\frac{u'(s)}{b(s)} + u(s)\right) ds, \tag{4.130}$$

for

$$E(t_0) = \frac{u'(t_0)}{b(t_0)} + u(t_0) \tag{4.131}$$

$$E(t) = E(t_0) + \int_{t_0}^{t} \left(u'(s) + \frac{u''(s)}{b(s)} - \frac{db(s)}{ds}\frac{u'(s)}{b^2(s)}\right) ds \tag{4.132}$$

Since $b(t)$ is an increasing function, $\dfrac{db(t)}{dt} \geq 0$, then

$$E(t) \leq E(t_0) + \int_{t_0}^{t} \left(u'(s) + \frac{u''(s)}{b(s)}\right) ds \tag{4.133}$$

Substituting for $u''(t)$ in (4.133), using(4.125), we have

$$E(t) \leq E(t_0) - \int_{t_0}^{t} \left(\left(\frac{a(s)}{b(s)} - 1\right) u'(s) + \frac{1}{b^2(s)}u(s) + \frac{g(s)}{b(s)}f(u(s)) - \frac{\epsilon}{b(s)}\right) ds \tag{4.134}$$

Replacing $E(t)$ in (4.134) with (4.129) and taking the absolute value

$$|u(t)| \leq |E(t_0)| + \frac{|u'(t)|}{b(t)} + \int_{t_0}^{t} \left(\left(\frac{a(s)}{b(s)} - 1\right) |u'(s)| + \frac{1}{b^2(s)}|u(s)| + \frac{g(s)}{b(s)}f(|u(s|)) + \frac{\epsilon}{b(s)}\right) ds \tag{4.135}$$

Using conditions (i-iii) with $\dfrac{1}{b(t)} \leq 1$, we get

$$|u(t)| \leq |E(t_0)| + \lambda + \epsilon p + m + \int_{t_0}^{t} \frac{1}{b^2(s)}|u(s)|ds + \int_{t_0}^{t} \frac{g(s)}{b(s)}f(|u(s)|)ds \tag{4.136}$$

Setting

$$\frac{1}{b^2(t)} = \alpha(t), \quad \frac{g(t)}{b(t)} = \gamma(t) \tag{4.137}$$

and using equation (4.137) in (4.136), we have

$$|u(t)| \leq |E(t_0)| + \lambda + \epsilon p + m + \int_{t_0}^{t} \alpha(s)|u(s)|ds \quad + \int_{t_0}^{t} \gamma(s)f(|u(s)|)ds \tag{4.138}$$

with

$f(|u(t)|) = \varpi(|u(t)|)$ and $\epsilon \geq 1$

(4.138) becomes

$$|u(t)| \leq C + \int_{t_0}^{t} \alpha(s)|u(s)|ds + \int_{t_0}^{t} \gamma(s)\varpi(|u(s)|)ds \tag{4.139}$$

where

$$C = \epsilon\left(|E(t_0)| + \lambda + p + m\right) \tag{4.140}$$

thus, we have

$$\frac{|u(t)|}{C} \leq 1 + \int_{t_0}^{t} \alpha(s)\frac{|u(s)|}{C}ds + \int_{t_0}^{t} \gamma(s)\omega(\frac{|u(s)|}{C})ds, \tag{4.141}$$

by applying Theorem 2.12 yields

$$\begin{aligned} |u(t)| \leq C\Omega^{-1}\left(\Omega(1) + \int_{t_0}^{t} \gamma(s)\omega\left(\exp\int_{t_0}^{s} \alpha(\delta)d\delta\right)ds\right) \\ \left(\exp\int_{t_0}^{t} \alpha(s)ds\right) \end{aligned} \tag{4.142}$$

By the condition (iv), we obtain

$$|u(t)| \leq C\exp l\Omega^{-1}\left(\Omega(1) + j\omega(\exp l)\right) \tag{4.143}$$

Substituting for C in (4.142) we obtain

$$\begin{aligned} |u(t) - u_0(t)| \leq |u(t)| \leq \epsilon\left(|E(t_0)| + \lambda + p + m\right)\exp l \\ \Omega^{-1}\left(\Omega(1) + j\varpi(\exp l)\right) \end{aligned} \tag{4.144}$$

Therefore, equation (4.125) is Hyers-Ulam stable with the Hyers-Ulam constant

$K = \left(|E(t_0)| + \lambda + p + m\right)\Omega^{-1}\left(\Omega(1) + j\omega(\exp l)\right)\exp l$

This concludes the proof.

Next, we consider the Hyers-Ulam stability of a second order differential equation which is nonlinear in both $u(t)$ and $u'(t)$

$$u''(t) + \phi(t)g(u(t))h(u'(t)) = 0, \tag{4.145}$$

together with initial conditions $u(t_0) = u'(t_0) = 0$, where $h, \phi \in C(\mathbf{I}, \mathbf{R}_+)$, $g \in C(\mathbf{R}_+, \mathbf{R}_+)$ and $h(u') > 0$.

Theorem 4.11:

Let $u(t) \in C^2(\mathbf{I}, \mathbf{R}_+)$ satisfies the differential inequality

$$|u''(t) + \phi(t)g(u(t))h(u'(t))| \leq \epsilon \tag{4.146}$$

for all $t \in \mathbf{I}$ and for some $\epsilon > 0$, then there exists a solution $u_0(t) \in C^2(\mathbf{I}, \mathbf{R}_+)$ of equation(4.145)such that

$|u(t) - u_0(t)| \leq K\epsilon,$

for

$$K = \frac{1}{\delta\lambda}P\Omega^{-1}\left(\Omega(1) + \frac{1}{\delta\lambda}l\right) \tag{4.147}$$

with $\mathbb{A} \in \psi$, provided the following conditions are satisfied:

i $A(u(t)) = \int_{u(t_0)}^{u(t)} g(s)ds < \infty$

ii $P(u(t)) = \int_{u(t_0)}^{u(t)} \frac{s}{h(s)} ds < \infty$

iii $\lim_{t\to\infty} \int_{t_0}^{t} |\phi'(s)| ds = l < \infty$

iv $\alpha(t) \geq \delta$, where $\delta > 0$

v $\int_{t_0}^{t} \frac{|u'(s)|}{|h(u'(s))|} ds \leq Q$

Proof:

From inequality (4.146), it follows that

$$\frac{u''(t)u'(t)}{h(u'(t))} + \phi(t)g(u(t))u'(t) \leq \frac{u'(t)\epsilon}{h(u'(t))}, \tag{4.148}$$

by conditions (i) and (ii), we get

$$\frac{d}{dt}P(u'(t)) + \phi(t)\frac{d}{dt}G(u(t)) \leq \frac{u'(t)\epsilon}{h(u'(t))} \quad \text{forall} \quad t \geq t_0 \tag{4.149}$$

Integrating by part from t_0 to t

$$P(u'(t)) - \phi(t)G(u(t)) + \int_{t_0}^{t} \phi'(s)G(u(s))ds \leq \epsilon \int_{t_0}^{t} \frac{u'(s)}{h(u'(s))} ds \tag{4.150}$$

Taking the absolute value of both sides, setting

$$|P(u'(t)) - \phi(t)A(u(t))| \geq \alpha(t)|u(t)||u'(t)| \tag{4.151}$$

for $\alpha(t) \in C(\mathbf{I}, \mathbf{R}_+)$

By inequality (4.151), we have

$$\alpha(t)|u(t)||u'(t)| \leq \epsilon \int_{t_0}^{t} \frac{|u'(s)|}{h(|u'(s)|)} ds + \int_{t_0}^{t} |\phi'(s)|A(|u(s)|)ds \tag{4.152}$$

By conditions (iv),(v) and setting $|u'(t)| \leq \lambda$, for $\lambda \geq 0$,

$$\frac{|(u(t))|}{L} \leq 1 + \frac{1}{\delta\lambda} \int_{t_0}^{t} |\phi'(s)|\omega(\frac{|u(s)|}{L})ds, \tag{4.153}$$

for $A \in \Psi$,$A(|u(t)) = \varpi(|u(t))|$ and

$$L = \frac{\epsilon}{\delta\lambda} Q \tag{4.154}$$

By Lemma 2.4 and condition(iii) we obtain

$$|u(t)| \leq L\Omega^{-1}\left(\Omega(1) + \frac{1}{\delta\lambda} l\right) \tag{4.155}$$

Substituting for L using equation (4.154), then inequality (4.155) becomes

$$|u(t)| \leq \frac{\epsilon}{\delta\lambda} Q\Omega^{-1}\left(\Omega(1) + \frac{1}{\delta\lambda} l\right) \tag{4.156}$$

Since

$$|u(t) - u(t_0)| \leq |u(t)|$$

we have

$$|u(t) - u_0(t)| \leq \frac{\epsilon}{\delta\lambda} Q\Omega^{-1}\left(\Omega(1) + \frac{1}{\delta\lambda} l\right) \tag{4.157}$$

Hence, equation (4.145) is Hyers-Ulam stable with Hyers-Ulam constant K given as

$$K = \frac{1}{\delta\lambda} Q\Omega^{-1}\left(\Omega(1) + \frac{1}{\delta\lambda} l\right)$$

The Hyers-Ulam stability of the Lienard equation is investigated as thus

$$u''(t) + c(t)f(u(t))u'(t) + a(t)g(u(t)) = 0 \tag{4.158}$$

with initial value $u(t_0) = u'(t_0) = 0$.

Theorem 4.12:

Let all the conditions of Theorem (4.6) remain valid with $P(t, u(t), u'(t)) = 0$. Equation (4.158) possess Hyers-Ulam stability and Hyers-Ulam constant
$K = (\frac{1}{\delta}L + \frac{1}{2}\lambda^2)\Omega^{-1}\left(\Omega(1) + \frac{\lambda^2}{\delta}m\right)$

Proof:

Since $P(t, u(t), u'(t)) = 0.$ and if there exists a solution $u(t) \in C^2(\mathbf{R}_+)$ which satisfies

$$|u''(t) + c(t)f(u(t))u'(t) + a(t)g(u(t))| \leq \epsilon \tag{4.159}$$

, then, from inequality (4.159) it follows that

$$-\epsilon \leq u''(t) + c(t)f(u(t))u'(t) + a(t)g(u(t)) \leq \epsilon \tag{4.160}$$

Multiplying inequality (4.160) by $u'(t)$, by applying the condition(i) of Theorem 4.10, integrating from t_0 and t

$$\int_{t_0}^{t} c(s)f(u(s))(u'(s))^2 ds + \int_{t_0}^{t} a(s)\frac{d}{ds}\mathbb{A}(u(s))ds \leq \epsilon \int_{t_0}^{t} u'(s)ds \tag{4.161}$$

Integrating by part, using $a'(t) \geq 0$ and $a(t) \geq \delta > 0$, taking the absolute value and let $\int_{t_0}^{t} |u'(s)|ds \leq L$, for $L > 0$, $|\mathbb{A}(u(t))| \geq |u(t)|$ we have

$$\frac{|u(t)|}{P} \leq 1 + \frac{(|u'(t)|)^2}{\delta}\int_{t_0}^{t} c(s)f(\frac{|u(s)|}{P})ds \tag{4.162}$$

where

$$P = \frac{\epsilon}{\delta}L$$

Let $|u'(t)| \leq \lambda$, $\frac{|u(t)|}{P} = z(t)$, and using Lemma 2.4, for $\varpi(z(t)) = f(z(t))$,

$$z(t) \leq \Omega^{-1}\left(\Omega(1) + \frac{\lambda^2}{\delta}\int_{t_0}^{t} c(s)ds\right) \tag{4.163}$$

Setting $\lim_{t\to\infty}\int_{t_0}^{t} c(s)ds \leq m < \infty$, for $m > 0$, substituting for $z(t)$, and P

$$|u(t)| \leq \frac{\epsilon}{\delta}L\Omega^{-1}\left(\Omega(1) + \frac{\lambda^2}{\delta}m\right) \tag{4.164}$$

where

$$K = \frac{1}{\delta}L\Omega^{-1}\left(\Omega(1) + \frac{\lambda^2}{\delta}m\right) \tag{4.165}$$

Therefore,

$|u(t) - u_0(t)| \leq |u(t)| \leq K\epsilon.$

4.2 Hyers-Ulam-Rassias Stability of Perturbed and Nonperturbed Nonlinear Second Order Differential Equation

4.2.1 Introduction

The goal of this segment is on Generalised Hyers-Ulam stability (also referred to as Hyers-Ulam-Rassias stability). This is an extension of Hyers-Ulam stability considered in the previous section. Here we shall obtain Hyers-Ulam-Rassias constant denoted by C_φ.

4.2.2 Hyers-Ulam-Rassias Stability of Differential Equation $u''(t) + f(t,u(t)) = P(t,u(t))$

The equation

$$u''(t) + B(t)Q(u(t)) = P(t,u(t)) \tag{4.166}$$

with initial conditions $u(t_0) = u'(t_0) = 0$. is considered in various ways.

Definition 4.7:

Equation (4.166) posses Hyers-Ulam-Rassias stability, if there exists $u(t) \in C^2(\mathbf{R}_+)$ be any solution satisfies inequality

$$|u''(t) + B(t)Q(u(t)) - P(t,u(t))| \leq \varphi(t) \tag{4.167}$$

where $\varphi(t)$ a nondecreasing and positive function defined as

$\varphi : \mathbf{I} \to \mathbf{R}_+$, and there exists solution $u_0(t) \in C^2(\mathbf{R}_+)$ of equation (4.166) such that

$$|u(t) - u_0(t)| \leq C_\varphi \varphi(t)$$

holds where C_φ is the Hyers-Ulam-Rassias constant.

Theorem 4.13:

In equation (4.166), $B(t)$ a nonnegative, nondecreasing continuous function on $\mathbf{R}_+$, $Q(u(t))$ a positive continuous function also defined for every real positive function $u(t)$ and given $\phi(t) \in (\mathbf{I}, \mathbf{R}_+)$ so that $|P(t,u(t))| \leq \phi(t)\varpi(|u(t)|)$, $\varpi(u(t)) \in \Psi$. Then, equation (4.166) is Hyers-Ulam-Rassias stable with Hyers-Ulam-Rassias constant is given as

$$C_\varphi = (\frac{\lambda}{2} + \frac{1}{\delta})\Omega^{-1}\left(\Omega(1) + \frac{\lambda}{\delta}y\right) \tag{4.168}$$

Proof:

From equation (4.167), we get

$$-\varphi(t) \leq u''(t) + B(t)Q(u(t)) - P(t,u(t)) \leq \varphi(t) \tag{4.169}$$

Multiplying inequality (4.169) by $u'(t)$,

$$-\varphi(t) \leq u''(t)u'(t) + B(t)Q(u(t))u'(t) - P(t,u(t))u'(t) \leq u(t)\varphi(t). \tag{4.170}$$

Integrating from t_0 to t and using the initial value, we have

$$\frac{1}{2}(u'(t))^2+\int_{t_0}^{t} B(s)Q(u(s))u'(s)ds-\int_{t_0}^{t} P(s,u(s))u'(s)ds \leq \int_{t_0}^{t} \varphi(s)u'(s)ds \quad (4.171)$$

Letting

$$\mathbb{A}(u(t)) = \int_{u(t_0)}^{u(t)} Q(s)ds, \quad (4.172)$$

then,

$$\int_{t_0}^{t} B(s)\frac{d}{ds}\mathbb{A}(u(s))ds + \frac{(u'(t))^2}{2} - \int_{t_0}^{t} P(s,u(s))u'(s)ds \leq \int_{t_0}^{t} \varphi(s)u'(s)ds. \quad (4.173)$$

Using integration by part with initial value and recall that $B(t)$ a nondecreasing function in t, then $\frac{d}{dt}B(t) \geq 0$, we obtain

$$B(t)|\mathbb{A}(u(t))| \leq \frac{|u'(t)|^2}{2} + |u'(t)|\int_{t_0}^{t} \varphi(s)ds + |u'(t)|\int_{t_0}^{t} |P(s,u(s))|ds \quad (4.174)$$

Let $B(t) \geq \delta$, $\delta > 0$, $|\mathbb{A}(u(t))| \geq |u(t)|$ and $|u'(t)| \leq \lambda$, $\lambda > 0$, and by the hypothesis of the Theorem 4.13, we arrive at

$$|u(t)| \leq (\frac{\lambda}{2} + \frac{1}{\delta})\int_{t_0}^{t} \lambda\varphi(s)ds + \frac{\lambda}{\delta}\int_{t_0}^{t} \phi(s)\varpi(|u(s)|)ds \quad (4.175)$$

Using Theorem 2.9 together with

$\lim_{t\to\infty}\int_{t_0}^{t} \phi(s)ds = y < \infty$, $\int_{t_0}^{t} \lambda\varphi(s)ds \leq \varphi(t)$,

where $\lambda \leq \frac{1}{t-t_0} \leq 1$ for $t \geq t_0 > 0$,

it is obvious that

$$|u(t)|) \leq (\frac{\lambda}{2} + \frac{1}{\delta})\varphi(t)\Omega^{-1}\left(\Omega(1) + \frac{\lambda}{\delta}y\right). \quad (4.176)$$

This leads to Hyers-Ulam-Rassis constant which is given as:

$$C_\varphi = (\frac{\lambda}{2} + \frac{1}{\delta})\left(\Omega(1) + \frac{\lambda}{\delta}y\right)$$

If $|P(t,u(t))| \leq K'\varpi(|u(t)|)$, where K' a positive constant, we have the following theorem

Theorem 4.14:

Let conditions of Theorem 4.13 remain valid expect that

$|P(t,u(t))| \leq K'\varpi(|u(t)|)$, K' is a positive constant. Then, the equation (4.166) is Hyers-Ulam-Rassias stable and Hyers-Ulam-Rassias constant is given as

$$C_\varphi = (\frac{\lambda}{2} + \frac{1}{\delta})\Omega^{-1}\left(\Omega(1) + \frac{K'}{\delta}L\right) \quad (4.177)$$

Proof:

Using the inequality (4.170), integrating by part, since $B(t)$ is a nondecreasing function in t, $\frac{d}{dt}B(t) \geq 0$, then equation (4.170) yields

$$B(t)|\mathbb{A}(u(t))| \leq \frac{|u'(t)|^2}{2} + \int_{t_0}^{t} |u'(s)|\varphi(s)ds + \int_{t_0}^{t} |u'(s)||P(s,u(s))|ds \quad (4.178)$$

Let $B(t) \geq \delta,\ \delta > 0,\ \mathbb{A}(u(t)) \geq |u(t)|,\ |u'(t)| \leq \lambda,\ \lambda > 0$, by the hypothesis of the Theorem 4.14, leads to

$$|u(t)| \leq (\frac{\lambda}{2} + \frac{1}{\delta}) \int_{t_0}^{t} \lambda\varphi(s)ds + \frac{K'}{\delta} \int_{t_0}^{t} |u'(s)|\varpi(|u(s)|)ds \tag{4.179}$$

By Theorem (2.2), letting $\int_{t_0}^{t} |u'(s)|ds \leq L,\ \int_{t_0}^{t} \lambda\varphi(s)ds \leq \varphi(t)$,

where $\lambda \leq \frac{1}{t - t_0} \leq 1$ for $t \geq t_0 > 0$,

we have

$$|u(t)|) \leq (\frac{\lambda}{2} + \frac{1}{\delta})\Omega^{-1} \left(\Omega(1) + \frac{K'}{\delta}L\right) \tag{4.180}$$

$$C_\varphi = (\frac{\lambda}{2} + \frac{1}{\delta})\Omega^{-1} \left(\Omega(1) + \frac{K'}{\delta}L\right)$$

Hyers-Ulam-Rassis stability of nonlinear Euler type equation is considered as our next result. The equation is given as

$$t^2u''(t) + f(u(t)) = P(t, u(t)), \quad t \geq 1 \tag{4.181}$$

where $t \in \mathbf{I},\ \ f \in C(\mathbf{R}_+, \mathbf{R}_+), P \in C(\mathbf{I} \times \mathbf{R}, \mathbf{R}),\ \ \mathbf{R}_+ = [0, \infty)$

Definition 4.8:

Equation (4.181)is Hyers-Ulam-Rassias stable, if $\exists$ a $C_\varphi > 0$, $\ni \varphi : \mathbf{I} \to \mathbf{R}_+$, a solution $u(t) \in C^2(\mathbf{I}, \mathbf{R}_+)$, of the inequality

$$|t^2u''(t) + f(u(t)) - P(t, u(t))| \leq \varphi(t) \tag{4.182}$$

for which the solution $u_0(t) \in C^2(\mathbf{I}, \mathbf{R}_+)$ of equation (4.181) satisfies

$$|u(t) - u_0(t)| \leq C_\varphi\varphi(t)$$

where C_φ is H-U-R constant.

Theorem 4.15:

Let the undermentioned conditions be given as:

(i) if $\phi(t) \in (\mathbf{I}, \mathbf{R}_+)$ then $|P(t, u(t))| \leq \phi(t)\varpi(|u(t)|)$ and $\int_{t_0}^{\infty} |u'(s)|ds \leq L$

(ii)$\lim_{t\to\infty} \int_{t_0}^{t} \phi(s)ds = y < \infty$

(iii) let $\lambda > 0$ and $\lambda \leq \frac{1}{t - t_0} \leq 1$ for $t \geq t_0 \geq 1$,

so that $\lambda \int_{t_0}^{t} \varphi(s)ds \leq \varphi(t)\ \ \forall t \in \mathbf{I}$,

equation (4.181) is Hyers-Ulam-Rassias stable and Hyers-Ulam-Rassias constant is given as:

$$C_\varphi = (|u''(\eta)|L + 1)\Omega^{-1} (\Omega(1) + \lambda y)$$

Proof:

From equation (4.182),

$$-\varphi(t) \leq t^2u''(t) + f(u(t)) - P(t, u(t)) \leq \varphi(t), \quad t \geq 1 \tag{4.183}$$

Multiply through by $u'(t)$ and take $\frac{1}{t^2} \leq 1$ for $t \geq 1$ integrating from t_0 to t, applying Lemma 1.1 that is there exists $\eta \in [t_0, t]$ such that

$$u''(\eta)\int_{t_0}^{t} u'(s)ds + \int_{t_0}^{t} f(u(s))u'(s)ds - \int_{t_0}^{t} P(s,u(s))ds \leq \int_{t_0}^{t} u'(s)\varphi(s)ds$$

Let equation

$$\mathbb{F}(u(t)) = \int_{u(t_0)}^{u(t)} f(s)ds, \tag{4.184}$$

be applied on the above equation, taking the absolute value and using $|u'(t)| \leq \lambda,$ for $\lambda > 0$

with condition(i)

$$|u(t)|) \leq (|u''(\eta)|L+1)\lambda\int_{t_0}^{t}\varphi(s)ds + \lambda\int_{t_0}^{t}\phi(s)\varpi(|u(s)|)ds \tag{4.185}$$

setting $|\mathbb{F}(u(t))| \geq |u(t)|$

By Theorem 2.9 and condition (ii)

$$|u(t)| \leq (|u''(\eta)|L+1)\lambda\int_{t_0}^{t}\varphi(s)ds\Omega^{-1}(\Omega(1)+\lambda y) \tag{4.186}$$

Using condition (iii),

$$|u(t) - u_0(t)| \leq |u(t)| \leq (|u''(\eta)|L+1)\Omega^{-1}\left(\Omega(1)+\lambda y\right)\varphi(t) \tag{4.187}$$

$C_\varphi = (|u''(\eta)|L+1)\Omega^{-1}\left(\Omega(1)+\lambda y\right)$

Lastly, if $|P(t,u(t))| \leq K'\varpi(|u(t)|)|$

Theorem 4.16:

The equation (4.181) is H-U-R stable if $|P(t,u(t))| \leq K'\varpi(|u(t)|)|$, where K' a positive constant with other conditions of Theorem 4.15 remain valid. Then, Hyers-Ulam-Rassias constant of equation (4.181) is given as

$C_\varphi = (|u''(\eta)|L+1)\Omega^{-1}\left(\Omega(1)+K'L\right)$

Proof:

Using the inequality (4.185) and taking $\frac{1}{t^2} \leq 1$ for $t \geq 1$

$$|u(t)|) \leq (|u''(\eta)|L+1)|u'(t)|\int_{t_0}^{t}\varphi(s)ds + K'\int_{t_0}^{t}|u'(s)|\varpi(|u(s)|)ds \tag{4.188}$$

setting $|\mathbb{F}(u(t))| \geq |u(t)|$, using the conditions of Theorem 4.15 and Theorem 2.9, we have

$$|u(t)| \leq (|u''(\eta)|L+1))\lambda\int_{t_0}^{t}\varphi(s)ds\Omega^{-1}(\Omega(1)+K'L). \tag{4.189}$$

This leads to

$$|u(t) - u_0(t)| \leq |u(t)| \leq (|u''(\eta)|L+1)\Omega^{-1}\left(\Omega(1)+K'L\right)\varphi(t), \tag{4.190}$$

The Hyers-Ulam-Rassis constant is given as

$C_\varphi = (|u''(\eta)|L+1)\Omega^{-1}\left(\Omega(1)+K'L\right)$

This concludes the proof.

4.2.3 Hyers-Ulam-Rassias Stability of Nonhomogeneous Second Order NonlinearOrdinary Differential Equation

This unit is concerned with examining Hyers-Ulam-Rassias stability of the general equation

$$u''(t) + f(t, u(t), u'(t)) = P(t, u(t), u'(t)) \tag{4.191}$$

in different forms. The first form is given as

$$[r(t)\phi(u(t))u'(t)]' + f(t, u(t), u'(t))u'(t) + \alpha(t)h(u(t)) = P(t, u(t), u'(t)), \tag{4.192}$$

with the initial conditions $u(t_0) = u'(t_0) = 0$, where $\alpha, r, : \mathbf{I} \to \mathbf{R}_+$,

$\phi : \mathbf{R}_+ \to \mathbf{R}_+$ $\quad f, P : \mathbf{R}_+ \times \mathbf{R^2} \to \mathbf{R}$ are continuous functions in their respective argument.

Definition 4.9:

Given (4.191), we define Hyers-Ulam-Rassias stability as thus, if there exists a positive constant C_φ with the following property: for every solution $u(t) \in C^2(\mathbf{R}_+)$, inequality

$$\begin{aligned}\big|[r(t)\phi(u(t))u'(t)]' + f(t, u(t), u'(t))u'(t) + \alpha(t)h(u(t)) - P(t, u(t), u'(t))\big| \\ \leq \varphi(t)\end{aligned} \tag{4.193}$$

holds for positive function u and $\varphi : \mathbf{I} \to \mathbf{R}_+$, if there exists $u_0(t) \in C^2(\mathbf{R}_+)$ solution of equation (4.191), then

$$|u(t) - u_0(t)| \leq C_\varphi \varphi(t)$$

where C_φ is neither independent on $\varphi(t)$ nor $u(t)$

If

$$f(t, u(t), u'(t)) = P(t, u(t), u'(t))$$

Theorem 4.17:

Let $r(t)$ be a positive polynomial function on $C(\mathbf{R}_+)$, if

i $|f(t, u(t), u'(t))| = |P(t, u(t), u'(t))| \leq \kappa(t)\varpi(|u(t)|)\,|u'(t)|$

where κ a nonnegative, continuous function on $\mathbf{R}_+$

and $\varpi \in \Psi$ and $u > 0$.

ii let $t \geq t_0 \geq 1$, so that $\dfrac{1}{t}\displaystyle\int_{t_0}^{t} \varphi(s)ds \leq \varphi(t), \forall t \in \mathbf{I}$,

with Hyers-Ulam-Rassias positive constant

$$C_\varphi = \frac{TE}{\delta}. \tag{4.194}$$

where

$$T = \Omega^{-1}\left(\Omega(1) + \frac{(\lambda + \lambda^2)}{\delta} m\varpi\,(E)\right) \tag{4.195}$$

and

$$E = F^{-1}\left(F(1) + \frac{l}{\delta}\right) \tag{4.196}$$

Proof:

From inequality (4.193), we get

$$\begin{aligned} -\varphi(t) \leq [r(t)\phi(u(t))u'(t)]' + f(t,u(t),u'(t))u'(t) + \alpha(t)h(u(t)) \\ -P(t,u(t),u'(t)) \leq \varphi(t). \end{aligned} \tag{4.197}$$

Integrating (4.197) twice from t_0 to t using Lemma 1.1, we get

$$\begin{aligned} \int_{t_0}^{t} r(s)\phi(u(s))u'(s)ds + t\int_{t_0}^{t} f(s,u(s),u'(s))u'(s)ds + t\int_{t_0}^{t} \alpha(s)h(u(s)) \\ -t\int_{t_0}^{t} P(s,u(s),u'(s))ds \leq t\int_{t_0}^{t} \varphi(s)ds \end{aligned} \tag{4.198}$$

Using equation (4.82), we have

$$\begin{aligned} \int_{t_0}^{t} r(s)\frac{d}{ds}\Phi(s)ds + t\int_{t_0}^{t} f(s,u(s),u'(s))u'(s)ds + t\int_{t_0}^{t} \alpha(s)h(u(s)) \\ -t\int_{t_0}^{t} P(s,u(s),u'(s))ds \leq t\int_{t_0}^{t} \varphi(s)ds \end{aligned}$$

Integrating by part, since $r(t)$ a nonnegative, nondecreasing, then, $\frac{d}{dt}r(t) \geq 0$ and there exists a positive constant δ such that $r(t) \geq \delta$ we obtain

$$\begin{aligned} \delta\Phi(u(t)) + t\int_{t_0}^{t} f(s,u(s),u'(s))u'(s)ds + t\int_{t_0}^{t} \alpha(s)h(u(s)) \\ -t\int_{t_0}^{t} P(s,u(s),u'(s))ds \leq t\int_{t_0}^{t} \varphi(s)ds \end{aligned}$$

Taking the absolute value and let $\frac{1}{t^2} \leq \frac{1}{t} \leq 1$ for $t \geq 1$ we get

$$\begin{aligned} \delta|\Phi(u(t))| \leq \frac{1}{t}\int_{t_0}^{t} \varphi(s)ds + \int_{t_0}^{t} |f(s,u(s),u'(s))|\,|u'(s)|\,ds \\ + \int_{t_0}^{t} \alpha(s)\,|h(u(s))| + \int_{t_0}^{t} |P(s,u(s),u'(s))|\,ds \end{aligned} \tag{4.199}$$

Using the condition (i) of the Theorem 4.17, we get

$$\begin{aligned} \delta\,|\Phi(u(t))| \leq \frac{1}{t}\int_{t_0}^{t} \varphi(s)ds + \int_{t_0}^{t} \kappa(t)\varpi(|u(t)|)\,|u'(t)|\,|u'(s)|\,ds \\ + \int_{t_0}^{t} \alpha(s)\,|h(u(s))| + \int_{t_0}^{t} \kappa(t)\varpi(|u(t)|)\,|u'(t)|\,ds. \end{aligned} \tag{4.200}$$

Factorised the inequality (4.200), we get

$$\begin{aligned} |\Phi(u(t))| \leq \frac{1}{t\delta}\int_{t_0}^{t} \varphi(s)ds + \frac{1}{\delta}\int_{t_0}^{t} \alpha(s)h(|u(s)|) + \frac{1}{\delta}(|u'(s)| \\ + \frac{1}{\delta}\,|u'(t)|^2)\int_{t_0}^{t} \kappa(s)\varpi(|u(s)|)ds. \end{aligned} \tag{4.201}$$

Setting $|\Phi(u(t)| \geq |u(t)|$ and $|u'(t)| \leq \lambda$ for $\lambda > 0$,

it follows that

$$|u(t)| \leq \frac{1}{t\delta}\int_{t_0}^{t} \varphi(s)ds + \frac{1}{\delta}\int_{t_0}^{t} \alpha(s)h(|u(s)|) + \frac{(\lambda+\lambda^2)}{\delta}\int_{t_0}^{t} \kappa(s)\omega(|u(s)|)ds \tag{4.202}$$

Applying Corollary 3.1, we get

$$|u(t)| \leq T(t)E(t)\frac{1}{t\delta}\int_{t_0}^{t}\varphi(s)ds, \tag{4.203}$$

where

$$T(t) = \Omega^{-1}\left(\Omega(1) + \frac{(\lambda+\lambda^2)}{\delta}\int_{t_0}^{t}\kappa(s)\varpi\left(E(s)\right)ds\right), \tag{4.204}$$

$$E(t) = F^{-1}\left(F(1) + \frac{1}{\delta}\int_{t_0}^{t}\alpha(s)ds\right) \tag{4.205}$$

and $B = \dfrac{\lambda+\lambda^2}{\delta}$, $A = \dfrac{1}{\delta}$

By conditions (ii) of Theorem 4.17, setting $\lim_{t\to\infty}\displaystyle\int_{t_0}^{t}\kappa(s)ds = m < \infty$ and $\lim_{t\to\infty}\displaystyle\int_{t_0}^{t}\alpha(s)ds = l < \infty$, where $l,\ m > 0$, we obtain

$$|u(t)| \leq \frac{TE}{\delta}\varphi(t), \tag{4.206}$$

where

$$T = \Omega^{-1}\left(\Omega(1) + \frac{(\lambda+\lambda^2)}{\delta}m\varpi\left(E\right)\right),$$

and

$$E = F^{-1}\left(F(1) + \frac{m}{\delta}\right).$$

Therefore,

$$|u(t) - u_0(t)| \leq |u(t)| \tag{4.207}$$

Hence,

$$|u(t) - u_0(t)| \leq \frac{TE}{\delta}\varphi(t) \tag{4.208}$$

This concludes the proof.

Theorem 4.18:

Let $P(t, u(t), u'(t))$ and $f(t, u(t), u'(t))$ be continuous functions on $(\mathbf{I}\times\mathbf{R^2})$. Furthermore, if $f(t, u(t), u'(t)) \neq P(t, u(t), u'(t))$ in equation (4.191), if $r(t)$ posses the same features as in the Theorem (4.17). Then equation (4.191) is stable in the sense of Hyers-Ulam-Rassias stability, if

are satisfied with Hyers-Ulam-Rassias constant.

(i)' $|f(t, u(t), u'(t))| \leq \kappa(t)\varpi(|u(t)|)\,|u'(t)|$,

(ii)'$|P(t, u(t), u'(t))| \leq g(t)\gamma(|u(t)|)\,|u'(t)|^n$ where $n \in \mathbf{N}$,

are satisfied with Hyers-Ulam-Rassias constant.

$$C_\varphi = \frac{1}{\delta}\Upsilon^{-1}\left[\Upsilon(K) + k\frac{\lambda^n}{\delta}\gamma\left[TE\right]\right]TE \tag{4.209}$$

Proof:

From inequality (4.197) and by applying conditions $(i)'$ and $(ii)\prime$ on inequality

(4.197) we get

$$|\Phi(u(t))| \leq \frac{1}{t\delta}\int_{t_0}^{t}\varphi(s)ds + \frac{1}{\delta}\int_{t_0}^{t}\alpha(s)h(|u(s)|)$$
$$+\frac{1}{\delta}\int_{t_0}^{t}\kappa(s)\varpi(|u(s)|)\left|u'(s)\right|^2 ds + \frac{1}{\delta}\int_{t_0}^{t}g(s)\gamma(|u(s)|)\left|u'(s)\right|^n ds \tag{4.210}$$

Setting $|u(t)| \leq |\Phi(u(t))|$, and $|u'(t)| \leq \lambda$, then we have

$$|u(t)| \leq \frac{1}{t\delta}\int_{t_0}^{t}\varphi(s)ds + \frac{1}{\delta}\int_{t_0}^{t}\alpha(s)h(|u(s)|) + \frac{1}{\delta}\lambda^2\int_{t_0}^{t}\kappa(s)\varpi(|u(s)|)ds$$
$$+\frac{1}{\delta}\lambda^n\int_{t_0}^{t}g(s)\gamma(|u(s)|)ds \tag{4.211}$$

Applying Theorem 3.17, we get

$$|u(t)| \leq \frac{1}{t\delta}\int_{t_0}^{t}\varphi(s)ds\Upsilon^{-1}\left[\Upsilon(K) + \frac{1}{\delta}\lambda^n\int_{t_0}^{t}g(s)\gamma\left[T(s)E(s)\right]ds\right]T(t)E(t), \tag{4.212}$$

where

$$T(t) = \Omega^{-1}\left(\Omega(1) + \frac{1}{\delta}\lambda^2\int_{t_0}^{t}\kappa(s)\varpi\left(E(s)\right)ds\right), \tag{4.213}$$

here B in Theorem 3.17 is given as $B = \dfrac{\lambda^2}{\delta}$.

Using the limits of integral in Theorem 4.17 and letting $\lim_{t_0\to\infty}\displaystyle\int_{t_0}^{t}g(s)ds \leq k$, we obtain

$$|u(t)| \leq \varphi(t)\frac{1}{\delta}\Upsilon^{-1}\left[\Upsilon(K) + k\frac{\lambda^n}{\delta}\gamma\left[TE\right]\right]TE, \tag{4.214}$$

we define

$$T = \Omega^{-1}\left(\Omega(1) + m\frac{\lambda^2}{\delta}\omega\left(E\right)\right), \tag{4.215}$$

and

$$E = F^{-1}\left(F(1) + \frac{l}{\delta}\right). \tag{4.216}$$

Hence,

$$|u(t) - u_0(t_0)| \leq |u(t)| \leq C_\varphi\varphi(t) \tag{4.217}$$

Therefore,

$$|u(t) - u_0(t)| \leq \varphi(t)\frac{1}{\delta}\Upsilon^{-1}\left[\Upsilon(K) + k\frac{\lambda^n}{\delta}\gamma\left[TE\right]\right]TE \tag{4.218}$$

where

$$C_\varphi = \frac{1}{\delta}\Upsilon^{-1}\left[\Upsilon(K) + k\frac{\lambda^n}{\delta}\gamma\left[TE\right]\right]TE$$

Let us consider equation (4.192) in the form

$$\left[r(t)\phi(u(t))u'(t)\right]' + \alpha(t)h(u(t)) = P(t, u(t), u'(t)) \quad \forall t > 0. \tag{4.219}$$

Theorem 4.19:

Equation (4.219) is Hyers-Ulam-Rassias stable if all conditions in Theorem (4.17) remain valid with Hyers-Ulam-Rassias constant given as

$$C_\varphi = \frac{1}{\delta}HE \tag{4.220}$$

where

$$H = \Upsilon^{-1}\left(\Upsilon(1) + m\frac{\lambda^n}{\delta}\gamma\left(E\right)\right) \tag{4.221}$$

and

$$E = F^{-1}\left(F(1) + \frac{l}{\delta}\right) \tag{4.222}$$

Proof:

Since $f(t, u(t), u'(t)) = 0$, let $|P(t, u(t), u'(t))| \leq g(t)\gamma(|u(t)|)\,|u'(t)|^n$ where $n \in \mathbf{N}$, from equation (4.192) we get

$$|u(t)| \leq \frac{1}{t\delta}\int_{t_0}^{t}\varphi(s)ds + \frac{1}{\delta}\int_{t_0}^{t}\alpha(s)h(|u(s)|)ds + \frac{1}{\delta}\lambda^n\int_{t_0}^{t}g(s)\gamma(|u(s)|)ds \tag{4.223}$$

Applying Corollary 3.1, we obtain

$$|u(t)| \leq \frac{1}{\delta}\int_{t_0}^{t}\varphi(s)dsH(t)E(t) \tag{4.224}$$

where

$$H(t) = \Upsilon^{-1}\left(\Upsilon(1) + \frac{\lambda^n}{\delta}\int_{t_0}^{t}g(s)\gamma\left(E(s)\right)ds\right) \tag{4.225}$$

and

$$E(t) = F^{-1}\left(F(1) + \frac{1}{\delta}\int_{t_0}^{t}\alpha(s)ds\right) \tag{4.226}$$

Using the limit of integral in the proof of Theorems 4.17 and 4.18 we obtain

$$|u(t)| \leq \varphi(t)\frac{1}{\delta}HE \tag{4.227}$$

Therefore,

$$|u(t) - u_0(t)| \leq |u(t)| \leq \varphi(t)\frac{1}{\delta}HE$$

Hence,

$$|u(t) - u_0(t)| \leq \varphi(t)\frac{1}{\delta}HE \tag{4.228}$$

Finally, we consider the case $f(t, u(t), u'(t)) = P(t, u(t), u'(t)) = 0$ in equation (4.192), then,

$$[r(t)\phi(u(t))u'(t)]' + \alpha(t)h(u(t)) = 0, \tag{4.229}$$

Theorem 4.20:

If $u(t) \in C^2(R)$ is a solution which satisfies the inequality

$$|\,[r(t)\phi(u(t))u'(t)]' + \alpha(t)h(u(t))| \leq \varphi(t), \tag{4.230}$$

then equation (4.230) is stable in the sense of Hyers-Ulam-Rassias stability, if there exists a $u_0(t) \in C^2(\mathbf{R}_+)$ such that

$$|u(t) - u_0(t)| \leq |u(t)| \leq C_\varphi\varphi(t)$$

note $u_0(t)$ is a solution of equation (4.230) and C_φ is Hyers-Ulam-Rassias constant given as

$$C_\varphi = \Omega^{-1}\left(\Omega(1) + \frac{l}{\delta}\right) \tag{4.231}$$

Proof:

From inequality (4.231), we get

$$-\varphi(t) \leq [r(t)\phi(u(t))u'(t)]' + \alpha(t)h(u(t)) \leq \varphi(t) \tag{4.232}$$

Integrating inequality (4.232) twice from t_0 to t using Lemma 1.1, taking $\frac{1}{t^2} \leq \frac{1}{t} \leq 1$ for $t \geq t_0 \geq 1$ and taking the absolute value, we get

$$|\Phi((u(t))| \leq \frac{1}{t\delta}\int_{t_0}^{t}|\varphi(s)|ds + \frac{1}{\delta}\int_{t_0}^{t}\alpha(s)|h(u(s))|ds, \forall t \geq 1 \tag{4.233}$$

Let $|\Phi(u(t))| \geq |u(t)|$, then

$$|u(t)| \leq \frac{1}{t\delta}\int_{t_0}^{t}|\varphi(s)|ds + \frac{1}{\delta}\int_{t_0}^{t}\alpha(s)h(|u(s)|)ds \tag{4.234}$$

By applying Theorem 2.9 we get

$$|u(t)| \leq \frac{1}{\delta}\Omega^{-1}\left(\Omega(1) + \frac{1}{\delta}\int_{t_0}^{t}\alpha(s)ds\right)\frac{1}{t}\int_{t_0}^{t}\varphi(s)ds \tag{4.235}$$

$$|u(t)| \leq \frac{1}{\delta}\Omega^{-1}\left(\Omega(1) + \frac{l}{\delta}\right)\varphi(t) \tag{4.236}$$

Provided $\lim_{t\to\infty}\int_{t_0}^{t}\alpha(s)ds = l < \infty$, and $\frac{1}{t}\int_{t_0}^{t}\varphi(s)ds \leq \varphi(t)$ Therefore,

$$|u(t) - u(t_0)| \leq |u(t)| \leq \frac{1}{\delta}\Omega^{-1}\left(\Omega(1) + \frac{l}{\delta}\right)\varphi(t)$$

Hence,

$$C_\varphi = \frac{1}{\delta}\Omega^{-1}\left(\Omega(1) + \frac{l}{\delta}\right)$$

Now, the extension of Euler type equation is given as

$$t^2u''(t) + f(u(t))tu'(t) + g(u(t)) = P(t, u(t), u'(t)) \tag{4.237}$$

together with initial conditions $u(t_0) = u'(t_0) = 0$ where $f, g, \mathbf{R} \to \mathbf{R}$ and $P : \mathbf{I} \times \mathbf{R}^2 \to \mathbf{R}$

Definition 4.10:

Given $C_\varphi > 0$, let $\varphi : \mathbf{I} \to \mathbf{R}$, then solution $u(t) \in C^2(\mathbf{I}, \mathbf{R})$, satisfies inequality

$$|t^2u''(t) + f(u(t))tu'(t) + g(u(t)) - P(t, u(t), u'(t))| \leq \varphi(t), \tag{4.238}$$

furthermore if there exists any solution $u_0(t) \in C^2(\mathbf{I}, \mathbf{R})$ of equation (4.179) with the initial conditions $u(t_0) = u'(t_0) = 0$ such that

$|u(t) - u_0(t)| \leq C_\varphi\varphi(t)$.

where C_φ a Hyers-Ulam-Rassias constant

Theorem 4.21:

The equation (4.238) together with its initial conditions is H-U-R stable provided:

(i) if there exists $\phi(t) \in C(\mathbf{I}, \mathbf{R}_+)$ such that $|P(t, u(t), u'(t))| \leq \phi(t)\varpi(|u(t)|)(|u'(t)|)^2$

(ii) $\lim_{t\to\infty}\int_{t_0}^{t}|u'(s)|ds = L < \infty$

and $\lim_{t\to\infty}\int_{t_0}^{t}\phi(s)ds = r < \infty$

(iii) let $\lambda > 0$ then $\lambda \int_{t_0}^{t} \varphi(s)ds \leq \varphi(t) \;\; \forall t \in \mathbf{I}$
where $\lambda \leq \frac{1}{t-t_0} \leq \gamma$ for $\gamma > 0$ and
$t \geq t_0 \geq 1$ and Hyers-Ulam-Rassias constant is given as

$$C_\varphi = (\frac{1}{2}\lambda + 1)\Omega^{-1}\left(\Omega(1) + r\lambda^3\varpi\left(F^{-1}\left(F(1)+\lambda L\right)\right)\right)F^{-1}\left(F(1)+\lambda L\right) \tag{4.239}$$

Proof:

From inequality (4.238), we get

$$-\varphi(t) \leq t^2u''(t) + f(u(t))tu'(t) + g(u(t)) - P(t,u(t),u'(t)) \leq \varphi(t) \tag{4.240}$$

Multiplying through by $u'(t)$ and $\frac{1}{t^2}$ for $t \geq 1$

$$\begin{aligned} u''(t)u'(t) + \frac{1}{t}f(u(t))(u'(t))^2 + \frac{1}{t^2}g(u(t))u'(t) - \frac{1}{t^2}P(t,u(t),u'(t))u'(t) \\ \leq \frac{1}{t^2}\varphi(t)u'(t) \end{aligned} \tag{4.241}$$

Integrating from t_0 to t and using equation (4.90)

$$\begin{aligned} \frac{1}{2}(u'(s))^2 + \frac{1}{t}\int_{t_0}^{t} f(u(s))(u'(s))^2ds + \frac{1}{t^2}\int_{t_0}^{t}\frac{G(u(s)}{ds}ds \\ -\frac{1}{t^2}\int_{t_0}^{t} P(s,u(s),u'(s))u'(s)ds \leq \frac{1}{t^2}\int_{t_0}^{t}\varphi(s)u'(s)ds \end{aligned} \tag{4.242}$$

Integrating and using initial conditions $u(t) = u'(t) = 0$, we obtain

$$\begin{aligned} \frac{1}{2}(u'(s))^2 + \frac{u'(t)}{t}\int_{t_0}^{t} u'(s)f(u(s))ds + \frac{1}{t^2}G(u(t)) - \\ \frac{1}{t^2}\int_{t_0}^{t} P(s,u(s),u'(s))u'(s)ds \leq \frac{1}{t^2}\int_{t_0}^{t}\varphi(s)u'(s)ds \end{aligned} \tag{4.243}$$

Let $\frac{1}{t^2} \leq \frac{1}{t} \leq 1$, where $t \geq t_0 \geq 1$ we obtain

$$\begin{aligned} G(u(t)) \leq \int_{t_0}^{t}\varphi(s)u'(s)ds - \frac{1}{2}(u'(s))^2 - u'(t)\int_{t_0}^{t} u'(s)f(u(s))ds \\ + \int_{t_0}^{t} P(s,u(s),u'(s))u'(s)ds \end{aligned} \tag{4.244}$$

Let $|G(u(t))| \geq |u(t)|$ and applying the condition (i)

$$\begin{aligned} |u(t)| \leq \int_{t_0}^{t}\varphi(s)|u'(s)|ds + \frac{1}{2}(|u'(s)|)^2 + |u'(t)|\int_{t_0}^{t}|u'(s)|f(|u(s)|)ds \\ +(|u'(t)|)^3\int_{t_0}^{t}\phi(s)\varpi(|u(t)|)ds \end{aligned} \tag{4.245}$$

From inequality (4.245) we have

$$\begin{aligned} |u(t)| \leq (\frac{1}{2}(|u'(s)|)+1)|u'(t)|\int_{t_0}^{t}\varphi(s)ds + |u'(t)|\int_{t_0}^{t}|u'(s)|f(|u(s)|)ds \\ +(|u'(t)|)^3\int_{t_0}^{t}\phi(s)\varpi(|u(t)|)ds \end{aligned} \tag{4.246}$$

Application of the Corollary 3.1, using conditions (ii), (iii) and let $|u(t)| \leq \lambda$.

$$\begin{aligned} |u(t) - u(t_0)| \leq |u(t)| \leq (\frac{1}{2}\lambda + 1) \\ \Omega^{-1}\left(\Omega(1) + r\lambda^3\varpi\left(F^{-1}\left(F(1)+\lambda L\right)\right)\right)F^{-1}\left(F(1)+\lambda L\right)\varphi(t) \end{aligned} \tag{4.247}$$

Furthermore, we investigate

$$t^2u''(t) + g(u(t)) = P(t, u(t), u'(t)) \tag{4.248}$$

with initial conditions $u(t_0) = u'(t_0) = 0$.

Theorem 4.22:

Let conditions of Theorem 4.21 remain valid, the equation (4.248) with the initial conditions $u(t_0) = u'(t_0) = 0$ is Hyers-Ulam-Rassias stable with Hyers-Ulam-Rassias constant given as.

$$C_\varphi = (\frac{1}{2}\lambda + 1)\Omega^{-1}(\Omega(1) + \lambda M) \tag{4.249}$$

Proof:

From inequality (4.238), if $f(u(t)tu'(t) = 0$, this leads to

$$-\varphi(t) \leq t^2u''(t) + g(u(t)) - P(t, u(t), u'(t)) \leq \varphi(t) \tag{4.250}$$

Multiply through by $u'(t)$ and $\frac{1}{t^2}$ for $t \geq 1$ yields

$$u''(t)u'(t) + \frac{1}{t^2}g(u(t))u'(t) - \frac{1}{t^2}P(t, u(t), u'(t))u'(t) \leq \frac{\varphi(t)u'(t)}{t^2} \tag{4.251}$$

Integrating from t_0 to t

$$\begin{aligned}\int_{t_0}^{t} u''(s)u'(s)ds + \frac{1}{t^2}\int_{t_0}^{t} g(u(s))u'(s)ds - \frac{1}{t^2}\int_{t_0}^{t} P(s, u(s), u'(s))u'st)ds \\ \leq \frac{1}{t^2}\int_{t_0}^{t}\varphi(s)u'(s)ds\end{aligned} \tag{4.252}$$

Simplifying inequality (4.252) we get

$$\begin{aligned}\frac{1}{2}(u'(s))^2 + \frac{1}{t^2}\int_{t_0}^{t} g(u(s))u'(s)ds - \frac{1}{t^2}\int_{t_0}^{t} P(s, u(s), u'(s))u'(s)ds \\ \leq \frac{1}{t^2}\int_{t_0}^{t}\varphi(s)u'(s)ds\end{aligned} \tag{4.253}$$

$$\begin{aligned}\frac{1}{t^2}\int_{t_0}^{t} g(u(s))u'(s)ds \leq \frac{1}{t^2}\int_{t_0}^{t}\varphi(s)u'(s)ds - \frac{1}{2}(u'(s))^2 \\ + \frac{1}{t^2}\int_{t_0}^{t} P(s, u(s), u'(s))u'(s)ds\end{aligned} \tag{4.254}$$

By equation (4.90) and letting $\frac{1}{t^2} \leq \frac{1}{t} \leq 1$, we get

$$\begin{aligned}\int_{t_0}^{t}\frac{d}{ds}G(u(s))ds \leq \int_{t_0}^{t}\varphi(s)u'(s)ds - \frac{1}{2}(u'(s))^2 \\ + \int_{t_0}^{t} P(s, u(s), u'(s))u'(s)ds\end{aligned} \tag{4.255}$$

Integrating and using initial conditions of the equation (4.237), we have

$$G(u(s)) \leq \int_{t_0}^{t}\varphi(s)u'(s)ds - \frac{1}{2}(u'(s))^2 + \int_{t_0}^{t} P(s, u(s), u'(t))u'(s)ds \tag{4.256}$$

Taking the absolute value, setting $|u'(t)| \leq \lambda$ for $\lambda > 0$ and by condition (i), inequality (4.256) becomes

$$|u(s)| \leq \lambda\int_{t_0}^{t}|u'(s)|\varphi(s)ds + \frac{1}{2}\lambda^2 + \lambda^3\int_{t_0}^{t}\phi(s)\varpi(|u(s)|)ds, \tag{4.257}$$

for $|G(u(t))| \geq |u(t)|$

$$|u(t)|) \leq R(t) + \lambda \int_{t_0}^{t} \phi(s)\omega(|u(s)|)ds \tag{4.258}$$

where

$$R(t) = (\frac{1}{2}\lambda + 1)\lambda \int_{t_0}^{t} \varphi(s)ds \tag{4.259}$$

$R(t)$ a nondecreasing monotonic, nonnegative function, by Theorem 2.9 with condition(ii), we obtain

$$|u(t)| \leq (\frac{1}{2}\lambda + 1)\Omega^{-1}(\Omega(1) + M)\lambda \int_{t_0}^{t} \varphi(s)ds$$

Using condition (iii), we have

$$|u(t)| \leq (\frac{1}{2}\lambda + 1)\Omega^{-1}(\Omega(1) + \lambda M)\varphi(t)$$

Hence,

$$|u(t) - u_0(t)| \leq |u(t)| \leq (\frac{1}{2}\lambda + 1)\Omega^{-1}(\Omega(1) + \lambda M)\varphi(t) \tag{4.260}$$

with

$$C_\varphi = (\frac{1}{2}\lambda + 1)\Omega^{-1}(\Omega(1) + \lambda M)$$

The next result is on Lienard type second order nonlinear perturbed DE

$$u'' + f(t, u(t), u'(t))u'(t) + a(t)\omega(u(t)) = P(t, u(t), u'(t)), \tag{4.261}$$

with initial condition $u(t_0) = u'(t_0) = 0$, for $a, \in C(\mathbf{I}, \mathbf{R}_+),\ \omega,\ \varpi \in C(\mathbf{R}_+, \mathbf{R}_+), f, P \in C(\mathbf{I} \times \mathbf{R} \times \mathbf{R}, \mathbf{R})$.

Definition 4.11:

The differential equation (4.261) is well defined and has Hyers-Ulam-Rassias stability if there exist $C_\varphi > 0$ constant such that for any solution $u_0(t) \in C^2(\mathbf{R}_+)$ of equation (4.261), the solution $u(t) \in C^2(\mathbf{I}, \mathbf{R}_+)$ of

$$|u'' + f(t, u(t), u'(t))u'(t) + a(t)\omega(u(t)) - P(t, u(t), u'(t))| \leq \varphi(t), \tag{4.262}$$

satisfies

$$|u(t) - u_0(t)| \leq C_\varphi \varphi(t) \quad \forall t \in \mathbf{I}.$$

where $\varphi : \mathbf{I} \to \mathbf{R}_+$.

Theorem 4.23:

Let $f(t, u(t).u'(t)) \neq P(t, u(t), u'(t))$, equation (4.261) is stable in the sense of H-U-R. If the following conditions are satisfied

(i)$|f(t, u(t), u'(t))| \leq \phi(t)g(|u(t)|)h(|u'(t)|)$,

(ii)$|P(t, u(t), u'(t))| \leq \alpha(t)\varpi(|u(t)|)|u'(t)|^n$,

where $\phi(t), \alpha(t) \in \mathbf{R}_+,\ g, h, \varpi$ are nonnegative, monotonic, nondecreasing, continuous. Suppose g, ϖ belong to class of Ψ and $\varphi : \mathbf{I} \to \mathbf{R}_+$, with Hyers-Ulam-Rassias constant

$$C_\varphi = \frac{1}{\delta}(1+\frac{\lambda}{2})\Omega^{-1}\left(\Omega(1)+\frac{m}{\delta}\lambda^{n+1}\omega\left(F^{-1}\left(F(1)+\frac{r}{\delta}h(\lambda)\lambda^2\right)\right)\right) \\ F^{-1}\left(F(1)+\frac{r}{\delta}h(\lambda)\lambda^2\right) \tag{4.263}$$

Proof:

From inequality (4.262) we obtain

$$-\varphi(t) \le u'' + f(t,u(t),u'(t))u'(t) + a(t)\omega(u(t)) - P(t,u(t),u'(t)) \le \varphi(t). \tag{4.264}$$

Multiplying equation (4.264) by $u'(t)$ and using equation

$$\mathbb{N}(u(t)) = \int_{u(t_0)}^{u(t)} \omega(s)ds, \tag{4.265}$$

we get

$$\frac{1}{2}(u'(t))^2 + \int_{t_0}^{t} f(s,u(s),u'(s))(u'(s))^2 ds + \int_{t_0}^{t} a(s)\frac{d}{ds}\mathbb{N}(u(s))ds \\ - \int_{t_0}^{t} P(s,u(s),u'(s))u'(s)ds \le \int_{t_0}^{t} u'(s)\varphi(s)ds. \tag{4.266}$$

if $a(t)$ a nonnegative, nondecreasing, then $a'(t) \ge 0$ and there exists $\delta > 0$ such that $a(t) \ge \delta$, by integration by part we have

$$\delta\mathbb{N}(u(t) \le \int_{t_0}^{t} u'(s)\varphi(s)ds - \frac{1}{2}(u'(t))^2 - \int_{t_0}^{t} f(s,u(s),u'(s))(u'(s))^2 ds \\ + \int_{t_0}^{t} P(s,u(s),u'(s))u'(s)ds. \tag{4.267}$$

Taking the absolute value and using the hypothesis of the Theorem 4.23 we obtain

$$|u(t)| \le \frac{\lambda}{\delta}\int_{t_0}^{t}\varphi(s)ds + \frac{1}{2\delta}\lambda^2 + h(\lambda)\lambda^2\int_{t_0}^{t}\phi(s)g(|u(s)|)ds \\ +\lambda^{n+1}\int_{t_0}^{t}\alpha(s)\varpi(|u(s)|ds \tag{4.268}$$

where $|u'(t)| \le \lambda$, $\lambda > 0$, and $\mathbb{N}(u(t))| \ge |u(t)|$,

Let $\beta(u(t)) = g(u(t)$, by application of Corollary 3.1, hence

$$|u(t)| \le \frac{1}{\delta}(1+\frac{\lambda}{2})\lambda\int_{t_0}^{t}\varphi(s)ds\Omega^{-1}(\Omega(1) \\ +\frac{1}{\delta}\lambda^{n+1}\int_{t_0}^{t}\alpha(s)\varpi\left(F^{-1}\left(F(1)+\frac{1}{\delta}h(\lambda)\lambda^2\int_{t_0}^{s}\phi(\tau)d\tau\right)\right)ds\right) \\ F^{-1}\left(F(1)+\frac{1}{\delta}h(\lambda)\lambda^2\int_{t_0}^{t}\phi(s)ds\right) \tag{4.269}$$

Setting $\lim_{t\to\infty}\int_{t_0}^{t}\alpha(s)ds = m < \infty$, $\lim_{t\to\infty}\int_{t_0}^{t}\phi(s)ds = r < \infty$,

by conditions (iii) of Theorem 4.21 let $\lambda\int_{t_0}^{t}\varphi(s)ds \le \varphi(t)$, we obtain

$$u(t) \le \varphi(t)\frac{1}{\delta}(1+\frac{\lambda}{2})\Omega^{-1}\left(\Omega(1)+m\frac{1}{\delta}\lambda^{n+1}\omega\left(F^{-1}\left(F(1)+\frac{1}{\delta}rh(\lambda)\lambda^2\right)\right)\right) \\ F^{-1}\left(F(1)+\frac{1}{\delta}rh(\lambda)\lambda^2\right), \tag{4.270}$$

Therefore,

$|u(t)-u(t_0)| \leq |u(t)| \leq C_\varphi \varphi(t).$

$$C_\varphi = \frac{1}{\delta}(1+\frac{\lambda}{2})\Omega^{-1}\left(\Omega(1)+\frac{m}{\delta}\lambda^{n+1}\omega\left(F^{-1}\left(F(1)+\frac{r}{\delta}h(\lambda)\lambda^2\right)\right)\right)$$
$$F^{-1}\left(F(1)+\frac{r}{\delta}h(\lambda)\lambda^2\right)$$

Further evaluation of equation (4.261) yields the subsequent result.

Theorem 4.24:

If $P(t,u(t),u'(t)) = f(t,u(t),u'(t))$ in equation (4.261). Then, the equation (4.261) has Hyers-Ulam-Rassias stability with Hyers-Ulam-Rassias constant

$$C_\varphi = \frac{1}{\delta}(1+\frac{\lambda}{2})\Omega^{-1}\left(\Omega(1)+\frac{r}{\delta}\lambda(\lambda+1)h(\lambda)\right) \tag{4.271}$$

Proof:

From inequality (4.262), multiplying through by $u'(t)$, we obtain

$$\begin{aligned}-u'(t)\varphi(t) \leq u''(t)u'(t) + f(t,u(t),u'(t))(u'(t)-1)u'(t)\\ +a(t)\omega(u(t))u'(t) \leq u'(t)\varphi(t).\end{aligned} \tag{4.272}$$

Integrating from t_0 to t with $a(t)$ a nonnegative,nondecreasing function, then $a'(t) \geq 0$ and there exists $\delta > 0$ such that $a(t) \geq \delta$ we obtain

$$\begin{aligned}|\mathbb{N}(u(t))| \leq \frac{1}{\delta}\int_{t_0}^{t}|u'(s)|\varphi(s)ds + \frac{1}{2\delta}(u'(t))^2\\ +\frac{1}{\delta}\int_{t_0}^{t}\phi(s)g(|u(s)|)h(|u'(t)|)(|u'(s)|+1)|u'(s)|ds.\end{aligned} \tag{4.273}$$

where $|f(t,u(t),u'(t))|$ is defined in the Theorem 4.23

Let $|u'(t)| \leq \lambda$, where $\lambda > 0$ and suppose $|u(t)| \leq |\mathbb{N}(u(t))|$, we obtain

$$|(u(t)| \leq \frac{\lambda}{\delta}(1+\frac{\lambda^2}{2})\int_{t_0}^{t}\varphi(s)ds + \frac{1}{\delta}\eta(\lambda+1)h(\lambda)\int_{t_0}^{t}\phi(s)g(|u(s)|)ds. \tag{4.274}$$

By application of Theorem 2.9 and the conditions defined in Theorem 4.21

$$|u(t)| \leq \varphi(t)\frac{1}{\delta}(1+\frac{\lambda^2}{2})\Omega^{-1}\left(\Omega(1)+\frac{r}{\delta}\eta(\eta+1)h(\lambda)\right) \quad t \in \mathbf{I}, \tag{4.275}$$

for

$$\frac{1}{t}\int_{t_0}^{t}\varphi(s)ds \leq \varphi(t)$$

Hence,

$|u(t)-u(t_0)| \leq |u(t)| \leq C_\varphi \varphi(t)$

where

$$C_\varphi = \frac{1}{\delta}(1+\frac{\lambda}{2})\Omega^{-1}\left(\Omega(1)+\frac{r}{\delta}\lambda(\lambda+1)h(\lambda)\right).$$

The stability of Lienard type equation is considered in two forms. This can be achieved by considering Hyers-Ulam-Rassis stability of the variants. Firstly, we consider the first variant as

$$u'' + c(t)f(u(t))u'(t) + a(t)g(u(t)) = P(t,u(t),u'(t)), \tag{4.276}$$

with initial conditions $u(t_0) = u'(t_0) = 0$ is considered.

Definition 4.12:

The inequality

$$|u'' + c(t)f(u(t))u'(t) + a(t)g(u(t)) - P(t, u(t), u'(t))| \leq \varphi(t) \tag{4.277}$$

holds for a solution $u(t) \in C^2(\mathbf{I}, \mathbf{R}_+)$, $\forall t \in \mathbf{I}$

for a positive function $\varphi(t)$ where $\varphi : \mathbf{I} \to [0, \infty)$, if there exists any solution $u_0(t) \in C^2(\mathbf{I}, \mathbf{R}_+)$ of (4.276), such that

$$|u(t) - u_0(t)| \leq C_\varphi \varphi(t) \quad \forall t \in \mathbf{I}$$

Where C_φ is called Hyers-Ulam-Rassias constant. Therefore, equation (4.276) is Hyers-Ulam-Rassias stable.

Theorem 4.25:

Suppose $a, c \in \mathbf{R}_+$ and f, g be functions which is monotonic, nonnegative, nondecreasing in u. Let $a(t)$ be nonnegative, nondecreasing function, then, $a'(t) \geq 0$, there exists $\delta > 0$ such that $a(t) \geq \delta$ with the following conditions remain valid:

$$\lim_{t\to\infty} \int_{t_0}^{t} c(s)ds = b < \infty \quad b > 0, \tag{4.278}$$

and

$$\lim_{t\to\infty} \int_{t_0}^{t} \alpha(s)ds = l < \infty, \quad l > 0, \tag{4.279}$$

then, equation (4.276) is Hyers-Ulam-Rassias stable with H-U-R constant

$$C_\varphi = (\frac{1}{\delta} + \frac{\lambda}{2})N^* M^* \tag{4.280}$$

where

$$N^* = \Omega^{-1}\left(\Omega(1) + \frac{\lambda^{(n+1)}}{\delta} l\varpi\left(M^*\right)\right)$$

and

$$M^* = F^{-1}\left(F(1) + \frac{\lambda^2}{\delta} b\right)$$

Proof:

From inequality (4.277), it is clear that

$$-\varphi(t) \leq u'' + c(t)f(u(t))u'(t) + a(t)g(u(t)) - P(t, u(t), u'(t)) \leq \varphi(t),$$

multiplying by $u'(t)$ to have

$$-u'(t)\varphi(t) \leq u''u'(t) + c(t)f(u(t))(u'(t))^2$$

$$+a(t)g(u(t))u'(t) - P(t, u(t), u'(t))u'(t) \leq u'(t)\varphi(t).$$

Using equation (4.90) and integrating from t_0 to t and using the hypothesis of Theorem 4.25 we obtain

$$\begin{aligned} \delta\,|G(u(t))| \leq \int_{t_0}^{t} |u'(s)|\,\varphi(s)ds + \frac{|u'(t)|^2}{2} + \int_{t_0}^{t} c(s)f(|u(s)|)(|u'(s)|)^2 ds \\ + \int_{t_0}^{t} |P(s, u(s), u'(s))|\,|u'(s)|\,ds. \end{aligned} \tag{4.281}$$

Suppose $|G(u(t)| \geq |u(t)|$, $\quad |P(t, u(t), u'(t))| \leq \alpha(t)\omega(|u(t)|)\,|u'(t)|^n$ for $n \in \mathbb{N}$(set of Natural numbers) and Setting $|u'(t)| \leq \lambda$ for $\lambda > 0$, we have

$$|u(t)| \leq (\frac{1}{\delta} + \frac{\lambda}{2})\lambda \int_{t_0}^{t} \varphi(s)ds + \lambda^2 \frac{1}{\delta} \int_{t_0}^{t} c(s)f(|u(s)|)ds + \frac{1}{\delta}\lambda^{(n+1)} \int_{t_0}^{t} \alpha(s)\varpi(|u(s)|)ds \tag{4.282}$$

Let $f(u(t)) = \beta(u(t))$, by Corollary 3.1, we have

$$|u(t)| \leq (\frac{1}{\delta} + \frac{\lambda}{2})\lambda \int_{t_0}^{t} \varphi(s)dsN(t)M(t)$$

where

$$N(t) = \Omega^{-1}\left(\Omega(1) + \frac{\lambda^{(n+1)}}{\delta} \int_{t_0}^{t} \alpha(s)\omega\left(M(s)\right)ds\right)$$

and

$$M(t) = F^{-1}\left(F(1) + \frac{\lambda^2}{\delta} \int_{t_0}^{t} c(s)ds\right)$$

Using the limit of integrals in the Theorem 4.25 and conditions (iii) of Theorem 4.21, we obtain

$$\lambda \int_{t_0}^{t} \varphi(s)ds \leq \varphi(t)$$

yields

$$|u(t)| \leq \varphi(t)N^*M^*$$

Hence,

$$|u(t) - u(t_0)| \leq |u(t)| \leq C_\varphi \varphi(t)$$

where

$$C_\varphi = N^*M^*$$

At this point, we consider Hyers-Ulam-Rassias stability of Lienard type equation in the form

$$u'' + c(t)f(u(t))u'(t) + a(t)g(u(t)) = P(t, u(t)), \tag{4.283}$$

with initial conditions $u(t_0) = u'(t_0) = 0$

Theorem 4.26:

Let the functions a, f, c, g, P be as defined in Theorem 4.25. Suppose the limits of integrals in Theorem 4.25 remain valid. Then, equation(4.277) is H-U-R stable with the H-U-R constant

$$C_\varphi = (\frac{1}{\delta} + \frac{\lambda}{2})J^* \tag{4.284}$$

and

$$J^* = \Omega^{-1}\left(\Omega(1) + \left(\frac{1}{\delta}LA\,|u(\xi)| + \frac{\lambda^2}{\delta}\right)b\right) \tag{4.285}$$

Proof:

Considering inequality (4.277), we get

$$-\varphi(t) \le u''(t) + c(t)f(u(t))u'(t) + a(t)g(u(t))u'(t) - P(t,u(t)) \le \varphi(t) \tag{4.286}$$

Multiplying inequality (4.286) by $u'(t)$, using equation(4.90), integrating from t_0 to t and by the hypothesis of the Theorem 4.25, we get

$$\begin{aligned}\delta\,|G(u(t))| \le \int_{t_0}^{t} |u'(s)|\,\varphi(s)ds + \frac{|u'(t)|^2}{2} + \int_{t_0}^{t} c(s)f(|u(s)|)(|u'(s)|)^2 ds \\ + \int_{t_0}^{t} |P(s,u(s))|\,|u'(s)|\,ds\end{aligned} \tag{4.287}$$

Suppose $|G(u(t)| \ge |u(t)|\,, |P(t,u(t))| \le A\,|u(t)|\,,$

$A > 0,\ \int_{t_0}^{\infty} |u'(s)|\,ds \le L \quad \text{for} \quad L > 0,\ |u'(t)| \le \lambda,\ \lambda > 0,$

using Theorem 1.1, $\exists\ \xi \in [t_0, t]\ \ni$

$$\begin{aligned}|u(t)| \le \frac{1}{\delta}\int_{t_0}^{t} |u'(s)|\,\varphi(s)ds + \frac{|u'^2(t)|}{2} \\ + \left(\frac{1}{\delta}LA\,|u(\xi)| + \frac{\lambda^2}{\delta}\right)\int_{t_0}^{t} c(s)f\left(|u(s)|\right)ds,\end{aligned} \tag{4.288}$$

By applying Theorem 2.9, we get

$|u(t)| \le (\frac{1}{\delta} + \frac{\lambda}{2})\Omega^{-1}\left(\Omega(1) + \left(\frac{1}{\delta}LA\,|u(\xi)| + \frac{\lambda^2}{\delta}\right) b\right)\varphi(t),$

By conditions (iii) of Theorem 4.21, we have

$\lambda \int_{t_0}^{t} \varphi(s)ds \le \varphi(t)$

Hence,

$|u(t) - u_0(t)| \le |u(t)| \le C_\varphi \varphi(t).$

Therefore,

$C_\varphi = (\frac{1}{\delta} + \frac{\lambda}{2})\Omega^{-1}\left(\Omega(1) + \left(\frac{1}{\delta}LA\,|u(\xi)| + \frac{\lambda^2}{\delta}\right) b\right)$

4.2.4 Hyers-Ulam-Rassias Stability Nonlinear Damped Ordinary Differential Equations

Damped nonlinear differential equation are examined in this unit. The area of our concern is second order differential equation. Different Hyers-Ulam-Rassias constants are going to be obtained by considering following equations :

$$(r(t)\psi(u(t))u'(t))' + p(t)u'(t) + q(t)f(u(t)) = P(t,u(t),u'(t)). \tag{4.289}$$

$$(r(t)u'(t))' + p(t)u'(t) + q(t)f(u(t)) = P(t,u(t),u'(t)). \tag{4.290}$$

with initial conditions $u(t_0) = u'(t_0) = 0$ where $t \in \mathbf{I} = [1,b)(b \le \infty)$, $r,p,q \in C(\mathbf{I},\mathbf{R})$, $f,\psi \in (\mathbf{R},\mathbf{R})$, $P \in C(\mathbf{I}\times\mathbf{R}^2,\mathbf{R})$, $\mathbf{R} = (-\infty,\infty)$ and $\mathbf{R}_+ = [0,\infty)$ The definitions of Hyers-Ulam-Rassias stability are given as thus:

Definition 4.13:

Hyers-Ulam-Rassias stability of (4.289) is defined if given a solution $u(t) \in C^2(\mathbf{R})$ which satisfies inequality

$$|(r(t)\psi(u(t))u'(t))' + p(t)u'(t) + q(t)f(u(t)) - P(t, u(t), u'(t))| \leq \varphi(t) \tag{4.291}$$

where $\varphi : \mathbf{I} \to \mathbf{R}_+$ and $u_0(t) \in C^2(\mathbf{R}_+)$ solution of equation(4.289) that makes

$|u(t) - u_0(t)| \leq C_\varphi \varphi(t)$

holds where C_φ is the Hyers-Ulam-Rassias constant.

Definition 4.14:

Equation (4.189) has Hyers-Ulam-Rassias stability, given a solution $u(t) \in C^2(\mathbf{I}, \mathbf{R})$ of inequality

$$|(r(t)u'(t))' + p(t)u'(t) + q(t)f(u(t)) - P(t, u(t), u'(t))| \leq \varphi(t) \tag{4.292}$$

there exists $u_0(t) \in C^2(\mathbf{I}, \mathbf{R})$ any solution of equation (4.289) for which

$|u(t) - u_0(t)| \leq C_\varphi \varphi(t)$

where C_φ is called H-U-R constant.

Firstly, we examine Hyers-Ulam-Rassias stability of equation (4.289) and obtain the Hyers-Ulam-Rassias constant.

Theorem 4.27:.

Suppose the following conditions are satisfied.

(i) Let $\displaystyle\int_{t_0}^{\infty} |u'(s)|ds \leq L$, for $L > 0$.

(ii) if $\phi(t) \in C(\mathbf{I}, \mathbf{R}_+)$ then $|P(t, u(t), u'(t))| \leq \phi(t)\varpi(|u(t)|)(|u'(t)|)^n$ where $n \in \mathbf{N}$

(iii) $\lim_{t\to\infty} \displaystyle\int_{t_0}^{t} q(s)ds = K_1 < \infty$ and

$\lim_{t\to\infty} \displaystyle\int_{t_0}^{t} \phi(s)ds = K_2 < \infty$, where $k_1, k_2 > 0$

(iv) let $|u'(t)| \leq \lambda$, where $\lambda > 0$ and $\displaystyle\frac{1}{t}\int_{t_0}^{t} \varphi(s)ds \leq \varphi(t)$ for $t \in \mathbf{I}$

Therefore, equation (4.289) is Hyers-Ulam-Rassias stable

and Hyers-Ulam-Rassias constant is given as

$$\begin{aligned} C_\varphi = \frac{(Lp(\xi) + 1)}{\delta}\Omega^{-1}\left(\Omega(1) + \frac{k_1}{\delta}\varpi\left(F^{-1}\left(F(1) + \frac{\lambda^n k_2}{\delta}\right)\right)\right) \\ F^{-1}\left(F(1) + \frac{\lambda^n k_2}{\delta}\right) \end{aligned} \tag{4.293}$$

Proof:

we begin the proof using equation (4.291), we obtain

$$\begin{aligned} -\varphi(t) \leq (r(t)\psi(u(t))u'(t))' \\ +p(t)u'(t) + q(t)f(u(t)) - P(t, u(t), u'(t)) \leq \varphi(t). \end{aligned} \tag{4.294}$$

The left hand side of inequality (4.294) is written as

$$(r(t)\psi(u(t))u'(t))' + p(t)u'(t) + q(t)f(u(t)) - P(t, u(t), u'(t)) \leq \varphi(t) \tag{4.295}$$

Integrating (4.295), we get

$$\begin{aligned} r(t)\psi(u(t))u'(t) + \int_{t_0}^{t} p(s)u'(s)ds + \int_{t_0}^{t} q(s)f(u(s))ds \\ - \int_{t_0}^{t} P(s, u(s), u'(s))ds \leq \int_{t_0}^{t} \varphi(s)ds \end{aligned} \tag{4.296}$$

By integrating (4.296) and applying Lemma 1.1, we arrives at

$$\begin{aligned} \int_{t_0}^{t} r(s)\psi(u(s))u'(s)ds + t\int_{t_0}^{t} p(s)u'(s)ds + t\int_{t_0}^{t} q(s)f(u(s))ds \\ -t\int_{t_0}^{t} P(s, u(s), u'(s))ds \leq t\int_{t_0}^{t} \varphi(s)ds \end{aligned} \tag{4.297}$$

Let

$$\Lambda(u(t)) = \int_{u(t_0)}^{u(t)} \psi(s)ds \tag{4.298}$$

Using equation (4.298) in inequality (4.297) yields

$$\begin{aligned} \int_{t_0}^{t} r(s)\frac{d}{ds}\Lambda(u(s))ds + t\int_{t_0}^{t} p(s)u'(s)ds + t\int_{t_0}^{t} q(s)f(u(s))ds \\ -t\int_{t_0}^{t} P(s, u(s), u'(s))ds \leq t\int_{t_0}^{t} \varphi(s)ds \end{aligned} \tag{4.299}$$

Integrating by part the equation (4.299), since $r(t)$ a nondecreasing, then $r'(t) \geq 0$ and there exists $\delta > 0$ such that $r(t) \geq \delta$, we have

$$\begin{aligned} \delta\lambda(u(t)) + t\int_{t_0}^{t} p(s)u'(s)ds + t\int_{t_0}^{t} q(s)f(u(s))ds \\ -t\int_{t_0}^{t} P(s, u(s), u'(s))ds \leq t\int_{t_0}^{t} \varphi(s)ds \end{aligned} \tag{4.300}$$

Multiplying equation (4.300) by $\frac{1}{t^2}$ and by applying Theorem 1.1 that is there exist $1 \leq \xi \leq t$ such that

$$\begin{aligned} \frac{\Lambda(u(t))}{t^2} + \frac{1}{t\delta}p(\xi)\int_{t_0}^{t} u'(s)ds + \frac{1}{t\delta}\int_{t_0}^{t} q(s)f(u(s))ds \\ -\frac{1}{t\delta}\int_{t_0}^{t} P(s, u(s), u'(s))ds \leq \frac{1}{t\delta}\int_{t_0}^{t} \varphi(s)ds \end{aligned} \tag{4.301}$$

Taking the absolute value of both sides and using conditions (i), (ii) and (iv). suppose $\frac{1}{t^2} \leq \frac{1}{t} \leq 1$ for $t \geq 1$ we have

$$\begin{aligned} \Lambda(|u(t)|) \leq \frac{1}{t\delta}\int_{t_0}^{t} \varphi(s)ds + \frac{Lp(\xi)}{\delta} + \frac{1}{\delta}\int_{t_0}^{t} q(s)f(|u(s)|)ds \\ +\frac{1}{\delta}\lambda^n \int_{t_0}^{t} \phi(s)\varpi(|u(s)|)ds \end{aligned} \tag{4.302}$$

Setting $\Lambda(|u(t)|) \geq |u(t)|$ it is clear that

$$|u(t)| \leq \frac{1}{\delta}(Lp(\xi)+1)\frac{1}{t}\int_{t_0}^{t}\varphi(s)ds + \frac{1}{\delta}\int_{t_0}^{t}q(s)f(|u(s)|)ds + \frac{1}{\delta}\lambda^n\int_{t_0}^{t}\phi(s)\varpi(|u(s)|)ds \tag{4.303}$$

By applying Corollary 3.1, we get

$$|u(t)| \leq \frac{1}{\delta}(Lp(\xi)+1)\frac{1}{t}\int_{t_0}^{t}\varphi(s)ds\Omega^{-1}\left(\Omega(1)+\frac{1}{\delta}\int_{t_0}^{t}\phi(s)\varpi\left(F^{-1}\left(F(1) + \frac{1}{\delta}\lambda^n\int_{t_0}^{t}q(\alpha)d\alpha\right)\right)ds\right)F^{-1}\left(F(1)+\frac{1}{\delta}\lambda^n\int_{t_0}^{t}q(s)ds\right) \quad t\in\mathbf{I} \tag{4.304}$$

Using the conditions (iii) and (iv), equation (4.304) becomes

$$|u(t)| \leq \frac{1}{\delta}(Lp(\xi)+1)\Omega^{-1}\left(\Omega(1)+\frac{k_1}{\delta}\varpi\left(F^{-1}\left(F(1) + \frac{\lambda^n k_2}{\delta}\right)\right)\right)F^{-1}\left(F(1)+\frac{\lambda^n k_2}{\delta}\right)\varphi(t) \quad t\in\mathbf{I} \tag{4.305}$$

Therefore, Hyers-Ulam-Rassias constant is

$$C_\varphi = \frac{1}{\delta}(Lp(\xi)+1)\Omega^{-1}\left(\Omega(1)+\frac{k_1}{\delta}\varpi\left(F^{-1}\left(F(1) + \frac{\lambda^n k_2}{\delta}\right)\right)\right)F^{-1}\left(F(1)+\frac{\lambda^n k_2}{\delta}\right) \quad t\in\mathbf{I}$$

Now, we consider the stability of equation (4.288).

Theorem 4.28:

Let all the conditions of Theorem 4.27 remain valid. Equation (4.290) is Hyers-Ulam-Rassias stable and Hyers-Ulam-Rassias constant is given as

$$C_\varphi = \frac{1}{\delta}L(\lambda r(\xi)+\lambda p(\eta)+1)\Omega^{-1}\left(\Omega(1)+\frac{1}{\delta}\lambda^{n+1}k_2\right)$$

Proof:

From equation (4.291), we get

$$-\varphi(t) \leq (r(t)u'(t))' + p(t)u'(t) + q(t)f(u(t)) - P(t,u(t),u'(t)) \leq \varphi(t) \tag{4.306}$$

Multiplying by inequality (4.306) by $u'(t)$ we obtain

$$(r(t)u'(t))'u'(t) + p(t)(u'(t))^2 + q(t)f(u(t))u'(t) - P(t,u(t),u'(t))u'(t) \leq u'(t)\varphi(t) \tag{4.307}$$

Integrating twice and applying Lemma 1.1, we get

$$u'(t)\int_{t_0}^{t}r(s)u'(s)ds + t\int_{t_0}^{t}p(s)(u'(s))^2ds + t\int_{t_0}^{t}q(s)f(u(s))u'(s)ds - t\int_{t_0}^{t}P(s,u(s),u'(s))u'(s)ds \leq t\int_{t_0}^{t}u'(s)\varphi(s)ds \tag{4.308}$$

Using equation (4.184) in inequality (4.308) and multiplying through by $\frac{1}{t^2}$

$$\frac{u'(t)}{t^2}\int_{t_0}^{t}r(s)u'(s)ds + \frac{1}{t}u'(t)\int_{t_0}^{t}p(s)u'(s)ds + \frac{1}{t}\int_{t_0}^{t}q(s)\frac{d}{ds}F(u(s))ds - \frac{1}{t}\int_{t_0}^{t}P(s,u(s),u'(s))u'(s)ds \leq \frac{1}{t}\int_{t_0}^{t}u'(s)\varphi(s)ds \tag{4.309}$$

Integrating by part, since $q(t)$ a nondecreasing, then $q'(t) \geq 0$ and there exists $\delta > 0$ such that $q(t) \geq \delta$, we get

$$\frac{u'(t)}{t^2}\int_{t_0}^{t} r(s)u'(s)ds + \frac{1}{t}u'(t)\int_{t_0}^{t} p(s)u'(s)ds + \frac{1}{t}\delta F(u(s) \\ -\frac{1}{t}\int_{t_0}^{t} P(s,u(s),u'(s))u'(s)ds \leq \frac{1}{t}\int_{t_0}^{t} u'(s)\varphi(s)ds \tag{4.310}$$

Let $\frac{1}{t^2} \leq \frac{1}{t} \leq 1$, applying Theorem 1.1 that is there exist $\xi, \eta \in [1,t]$ such that

$$\delta F(u(s)) \leq \frac{1}{t}\int_{t_0}^{t} u'(s)\varphi(s)ds - u'(t)r(\xi)\int_{t_0}^{t} u'(s)ds \\ -u'(t)p(\eta)\int_{t_0}^{t} u'(s)ds + \int_{t_0}^{t} P(s,u(s),u'(s))u'(s)ds \tag{4.311}$$

Taking the absolute value of both sides, hypothesis (i) and (ii) of Theorem 4.27, we obtain

$$\delta F(|u(s)|) \leq \frac{1}{t}\int_{t_0}^{t} |u'(s)|\varphi(s)ds + |u'(t)|r(\xi)L \\ +|u'(t)|p(\eta)L + \int_{t_0}^{t} \phi(t)\varpi(|u(s)|)(|u'(s)|)^{n+1}ds \tag{4.312}$$

Setting $|u'(t)| \leq \lambda$, and $F(|u(t)|) \geq |u(t)|$ it follows that

$$|u(t)| \leq \frac{1}{\delta}L(\lambda r(\xi) + \lambda p(\eta) + 1)\frac{1}{t}\int_{t_0}^{t} \varphi(s)ds + \frac{1}{\delta}\lambda^{n+1}\int_{t_0}^{t} \phi(s)\varpi(|u(s)|)ds \tag{4.313}$$

Applying Theorem 2.9 we get

$$|u(t)| \leq \frac{1}{\delta}L(\lambda r(\xi) + \lambda p(\eta) + 1)\frac{1}{t}\int_{t_0}^{t} \varphi(s)ds\Omega^{-1}\left(\Omega(1) + \frac{1}{\delta}\lambda^{n+1}\int_{0}^{t} \phi(s)ds\right) \tag{4.314}$$

By conditions (iii) and (iv) give

$$|u(t)| \leq \frac{1}{\delta}L(\lambda r(\xi) + \lambda p(\eta) + 1)\Omega^{-1}\left(\Omega(1) + \frac{1}{\delta}\lambda^{n+1}k_2\right)\varphi(t). \tag{4.315}$$

Therefore, Hyers-Ulam-Rassias constant is given as

$$C_\varphi = \frac{L(\lambda r(\xi) + \lambda p(\eta) + 1)}{\delta}\Omega^{-1}\left(\Omega(1) + \frac{\lambda^{n+1}k_2}{\delta}\right)$$

Lastly, we consider equation (4.290) when $P(t,u(t),u'(t)) = 0$

Theorem 4.29:

Let

$$(r(t)\psi(u(t))u'(t))' + p(t)u'(t) + q(t)f(u(t)) = 0 \tag{4.316}$$

together with initial conditions has H-U-R stability provided the conditions of Theorem 4.27 remain valid. The Hyers-Ulam-Rassias constant is given as

$$C_\varphi = \frac{1}{\delta}(Lp(\xi) + 1)\Omega^{-1}\left(\Omega(1) + \frac{k_1}{\delta}\right) \tag{4.317}$$

Proof:

From inequality (4.291), note that $P(t,u(t),u'(t)) = 0$ it then means

$$-\varphi(t) \leq (r(t)\psi(u(t))u'(t))' + p(t)u'(t) + q(t)f(u(t)) \leq \varphi(t). \tag{4.318}$$

Simplifying inequality (4.318) further, we get

$$(r(t)\psi(u(t))u'(t))' + p(t)u'(t) + q(t)f(u(t)) \leq \varphi(t) \tag{4.319}$$

By integrating twice, applying Lemma 1.1 we get

$$\int_{t_0}^{t} r(s)\psi(u(s))u'(s)ds + t\int_{t_0}^{t} p(s)u'(s)ds + t\int_{t_0}^{t} q(s)f(u(s))ds \leq t\int_{t_0}^{t}\varphi(s)ds \tag{4.320}$$

Using inequality (4.320) and multiplying through by $\frac{1}{t}$ for $t \geq t_0 \geq 1$

$$\frac{1}{t^2}\int_{t_0}^{t} r(s)\frac{d}{ds}\Lambda(u(s))ds + \frac{1}{t}\int_{t_0}^{t} p(s)u'(s)ds + \frac{1}{t}\int_{t_0}^{t} q(s)f(u(s))ds \leq \frac{1}{t}\int_{t_0}^{t}\varphi(s)ds \tag{4.321}$$

Integrating by part, since $r(t)$ a nondecreasing, then $r'(t) \geq 0$, there exists $\delta > 0$ such that $r(t) \geq \delta$ and by applying Theorem 1.1 there exist $1 \leq \xi \leq t$ such that

$$\frac{1}{t^2}\delta\Lambda(u(t)) \leq \frac{1}{t}\int_{t_0}^{t}\varphi(s)ds - p(\xi)\int_{t_0}^{t} u'(s)ds - \frac{1}{t}\int_{t_0}^{t} q(s)f(u(s))ds \tag{4.322}$$

Suppose $\frac{1}{t^2} \leq \frac{1}{t} \leq 1$, taking the absolute value of both sides and using conditions(i), setting $|u'(t)| \leq \lambda$, $|\Lambda(u(t))| \geq |u(t)|$ we arrive

$$|u(t)| \leq \frac{(Lp(\xi)+1)}{t\delta}\int_{t_0}^{t}\varphi(s)ds + \frac{1}{\delta}\int_{t_0}^{t} q(s)f(|u(s)|)ds \tag{4.323}$$

Using the Theorem 2.9 yields

$$|u(t)| \leq \frac{1}{\delta}(Lp(\xi)+1)\int_{t_0}^{t}\varphi(s)ds\Omega^{-1}\left(\Omega(1) + \frac{1}{\delta}\int_{0}^{t} q(s)ds\right) \tag{4.324}$$

Using the conditions (iii) and (iv) we conclude

$$|u(t)| \leq \frac{1}{\delta}(Lp(\xi)+1)\varphi(t)\Omega^{-1}\left(\Omega(1) + \frac{1}{\delta}k_1\right)\varphi(t) \tag{4.325}$$

Therefore, Hyers-Ulam-Rassias constant is

$$C_\varphi = \frac{1}{\delta}(Lp(\xi)+1)\Omega^{-1}\left(\Omega(1) + \frac{K_1}{\delta}\right)$$

At this point we consider the stability of damped nonlinear differential equation in the form,

$$u''(t) + nf(t)u'(t) + q(t)u(t) + Q(t,u(t)) = P(t,u(t),u'(t)) \tag{4.326}$$

with initial conditions $u(t_0) = u'(t_0) = 0$

where $n \in \mathbf{N}$, $f, q \in C(\mathbf{R})$, $Q \in (\mathbf{R},\mathbf{R})$ and $P \in (\mathbf{I} \times \mathbf{R^2}, \mathbf{R})$

Definition 4.15:

Equation (4.326) is stable in the sense of H-U-R, if $\exists\ u(t) \in C^2(\mathbf{I},\mathbf{R})$ satisfying

$$|u''(t) + nf(t)u'(t) + q(t)u(t) + Q(t,u(t)) - P(t,u(t),u'(t))| \leq \varphi(t), \tag{4.327}$$

let $u_0(t) \in C^2(\mathbf{I},\mathbf{R})$ be solution of (4.326) such that

$|u(t) - u_0(t)| \leq C_\varphi\varphi(t)$

and C_φ is called Hyers-Ulam-Rassias constant, while $\varphi(t)$ a nondecreasing, positive function defined as $\varphi : \mathbf{I} \to \mathbf{R}_+$

Theorem 4.30:

The equation (4.326) together with its the initial conditions is H-U-R stable provided the undermentioned conditions are established.

(i) Let $\int_{t_0}^{\infty} |u'(s)|ds \le L$, for $L > 0$. and $|Q(t, u(t))| \le r(t)\alpha(|u(t)|)$

(ii) If $\phi(t) \in C(\mathbf{I}, \mathbf{R}_+)$

then $|P(t, u(t), u'(t))| \le \phi(t)\varpi(|u(t)|)(|u'(t)|)^n$ where $n \in \mathbf{N}$

(iii)$\lim_{t\to\infty} \int_{t_0}^{t} r(s)ds = K_1 < \infty$ and

$\lim_{t\to\infty} \int_{t_0}^{t} q(s)ds = K_2 < \infty$, where $k_1, k_2 > 0$

(iv) let $\lambda > 0$, $|u'(t)| \le \lambda$, take $\lambda \int_{t_0}^{t} \varphi(s)ds \le \varphi(t)$ for $t \in \mathbf{I}$

therefore, Hyers-Ulam-Rassias constant

$$C_\varphi = \frac{1}{\delta}(\lambda + nf(\xi)L^2 + 1)\Omega^{-1}\left(\Omega(1) + \frac{1}{\delta}\lambda^{n+1}k_1\varpi\left(F^{-1}\left(F(1) + \frac{1}{\delta}\lambda k_2\right)\right)ds\right)F^{-1}\left(F(1) + \frac{1}{\delta}\lambda k_2\right) \quad t \in \mathbf{I} \tag{4.328}$$

Proof:

From inequality (4.276), simplified and multiplying by $u'(t)$, we obtain

$$u''(t)u'(t) + nf(t)(u'(t))^2 + q(t)u(t)u'(t) + Q(t, u(t))u'(t) - P(t, u(t), u'(t))u'(t) \le \varphi(t) \tag{4.329}$$

Integrating (4.329), we get

$$\int_{t_0}^{t} u''(s)u'(s)ds + n\int_{t_0}^{t} f(s)(u'(s))^2ds + \int_{t_0}^{t} q(s)u(s)u'(s)ds + \int_{t_0}^{t} Q(s, u(s))u'(s)ds - \int_{t_0}^{t} P(s, u(s), u'(s))u'(s)ds \le \int_{t_0}^{t} u'(s)\varphi(s)ds \tag{4.330}$$

Using equation (4.90), we have

$$\int_{t_0}^{t} u''(s)u'(s)ds + n\int_{t_0}^{t} f(s)(u'(s))^2ds + \int_{t_0}^{t} q(s)\frac{d}{ds}\mathbb{G}(u(s))ds + \int_{t_0}^{t} Q(s, u(s))u'(s)ds - \int_{t_0}^{t} P(s, u(s), u'(s))u'(s)ds \le \int_{t_0}^{t} u'(s)\varphi(s)ds \tag{4.331}$$

Integrating by part the equation (4.331), since the function $q(t) > 0$, nondecreasing, then $q'(t) \ge 0$, there exists $\delta > 0$ such that $q(t) \ge \delta$ and by applying Theorem 1.1, there exists $t_0 \le \xi \le t$ such that

$$(u'(t))^2 + nf(\xi)\int_{1}^{t}(u'(s))^2ds + \delta\mathbb{G}(u(t)) + \int_{t_0}^{t} Q(s, u(s))u'(s)ds - \int_{t_0}^{t} P(s, u(s), u'(s))u'(s)ds \le \int_{t_0}^{t} u'(s)\varphi(s)ds \tag{4.332}$$

Taking the absolute value of both sides, using conditions (i) and (ii) and setting $|\mathbb{G}(u(t))| \geq |u(t)|$ we arrive at

$$|u(t)| \leq \frac{1}{\delta}\int_{t_0}^{t} |u'(s)|\varphi(s)ds + \frac{|(u'(t))|^2}{\delta} + \frac{1}{\delta}nf(\xi)L^2$$
$$+\frac{1}{\delta}|u'(t)|\int_{t_0}^{t} r(s)\alpha(u(s))ds + \frac{1}{\delta}(|u'(t)|)^{n+1}\int_{t_0}^{t}\phi(t)\varpi(|u(t)|)ds \tag{4.333}$$

Setting $|u'(t)| \leq \lambda$,

$$|u(t)| \leq \frac{1}{\delta}(\lambda + nf(\xi)L^2 + 1)\lambda\int_{t_0}^{t}\varphi(s)ds$$
$$+\frac{1}{\delta}\lambda\int_{t_0}^{t} r(s)\alpha(u(s))ds + \frac{1}{\delta}\lambda^{n+1}\int_{t_0}^{t}\phi(t)\varpi(|u(t)|)ds \tag{4.334}$$

By applying Corollary 3.1 we obtain

$$|u(t)| \leq (\lambda + nf(\xi)L^2 + 1)$$
$$\lambda\int_{t_0}^{t}\varphi(s)ds\Omega^{-1}\left(\Omega(1) + \frac{1}{\delta}\lambda^{n+1}\int_{t_0}^{t}\phi(s)\varpi\left(F^{-1}\left(F(1)\right.\right.\right.$$
$$\left.\left.\left.+\frac{1}{\delta}\lambda\int_{t_0}^{t} r(\alpha)d\alpha\right)\right)ds\right)F^{-1}\left(F(1) + \frac{1}{\delta}\lambda\int_{t_0}^{t} r(s)ds\right) \tag{4.335}$$

Using the conditions (iii) and (iv) in inequality (4.335) to give

$$|u(t)| \leq \frac{1}{\delta}(\lambda + nf(\xi)L^2 + 1)\Omega^{-1}\left(\Omega(1) + \frac{1}{\delta}\lambda^{n+1}k_1\varpi\left(F^{-1}\left(F(1)\right.\right.\right.$$
$$\left.\left.\left.+\frac{1}{\delta}\lambda k_2\right)\right)ds\right)F^{-1}\left(F(1) + \frac{1}{\delta}\lambda k_2\right)\varphi(t) \quad t \in \mathbf{I} \tag{4.336}$$

we conclude that Hyers-Ulam-Rassias constant is given as

$$C_\varphi = \frac{1}{\delta}(\lambda + nf(\xi)L^2 + 1)\Omega^{-1}\left(\Omega(1) + \frac{1}{\delta}\lambda^{n+1}k_1\varpi\left(F^{-1}\left(F(1)\right.\right.\right.$$
$$\left.\left.\left.+\frac{1}{\delta}\lambda K_2\right)\right)ds\right)F^{-1}\left(F(1) + \frac{1}{\delta}\lambda k_2\right)$$

4.2.5 Hyers-Ulam-Rassias Stability of Homogeneous of Nonlinear Second Order Ordinary differential Equations

In this section, concentration shall be on investigation of H-U-R stability when $P(t, u(t)) = 0$ and $P(t, u(t), u'(t)) = 0$. First, we consider equation

$$u''(t) + f(t, u(t)) = 0 \tag{4.337}$$

with initial conditions $u(t_0) = u'(t_0) = 0$.

Definition 4.16:

Equation (4.337) is Hyers-Ulam-Rassias stable if given $C_\varphi > 0$ and solution $u(t) \in C^2(\mathbf{I}, \mathbf{R}_+)$ which satisfies

$$|u''(t) + f(t, u(t))| \leq \varphi(t), \tag{4.338}$$

then, there exists solution $u_0(t) \in C^2(\mathbf{R}_+)$ of equation (4.337) such that

$|u(t) - u_0(t)| \leq C_\varphi\varphi(t),$ for all $t \in \mathbf{I}$, where $\varphi \in (\mathbf{I}, \mathbf{R}_+)$ and C_φ is called Hyers-Ulam-Rassias constant.

Theorem 4.31:

Let

$$|f(t,u(t))| \le h_1(t)\left(|u(t)| + h_2(t)\left(\int_{t_0}^{t} g(s)\omega(u(s))ds\right)\right) \tag{4.339}$$

where $h_1, h_2, g \in C(\mathbf{R}_+)$. Furthermore, suppose the following conditions hold:

(i) for $s > 0$ the function $\varpi(s)$ is nondecreasing and $\varpi(\alpha u) \le \psi(\alpha)\varpi(u)$, for $\alpha \ge 1$, $u \ge 1$,

(ii) $\int_{t_0}^{\infty} h_1(s)ds = n < +\infty$, $\int_{1}^{t} h_1(s)h_2(s)g(s)(s-1)ds \le p$, $p > 0$

The equation (4.337) is H-U-R stable with H-U-R constant

$$C_\varphi = \Omega^{-1}\left(\Omega(1) + \exp(n)p\right) \tag{4.340}$$

Proof:

From inequality (4.338)

$$-\varphi(t) \le u''(t) + f(t,u(t)) \le \varphi(t) \tag{4.341}$$

Integrating twice and using Lemma 1.1, multiplying by $\frac{1}{t}$ for $t > 0$, since $t \ge 1$ taking the absolute value and substituting for $|f(t,u(t)|$ and applying Theorem 3.2, we have

$$\begin{aligned}|u(t)| \le \frac{1}{t}\int_{t_0}^{t} \varphi(s)ds\Omega^{-1}\Bigg(\Omega(1) + \int_{t_0}^{t} h_1(s)h_2(s)g(s)\Bigg(\exp\int_{t_0}^{s} h_1(\tau)d\tau \\ + \int_{t_0}^{s} \exp\left(\int_{\tau}^{s} h_1(\delta)d(\delta)\right)d\tau\Bigg)ds\Bigg)\end{aligned} \tag{4.342}$$

Given the following

$\lim_{t\to\infty}\int_{t_0}^{t} f(s)ds = n < \infty$ for $n > 0$,

$\int_{t_0}^{t} h_1(s)h_2(s)g(s)(s-1)ds \le p$

and

$\frac{1}{t}\int_{t_0}^{t} \varphi(s)ds \le \varphi(t)$

It shows that

$$|u(t) - u_0(t)| \le |u(t)| \le \varphi(t)\Omega^{-1}\left(\Omega(1) + \exp(n)p\right) \tag{4.343}$$

Hence,

$$|u(t) - u_0(t)| \le C_\varphi \varphi(t)$$

Therefore,

$$C_\varphi = \Omega^{-1}\left(\Omega(1) + \exp(n)p\right)$$

This concludes the proof.

The next result is given as

$$u''(t) + f(t,u(t),u'(t)) = 0, \tag{4.344}$$

with initial conditions $u(t_0) = u'(t_0) = 0$.

Definition 4.17:

The equation (4.344) is said to be H-U-R stable with respect to φ, if there exists $C_\varphi > 0$ such that for each solution $u \in C^2(\mathbf{I}, \mathbf{R}_+)$ satisfies

$$|u''(t) + f(t, u(t), u'(t))| \le \varphi(t) \tag{4.345}$$

also for any solution $u_0(t) \in C^2(\mathbf{I})$ satisfies (4.344) so that

$|u(t) - u_0(t)| \le C_\varphi \varphi(t)$ for all $t \in \mathbf{I}$

and $\varphi \in (\mathbf{I}, \mathbf{R}_+)$

Let

$$|f(t, u(t), u'(t))| \le h_1(t)(|u(t)|) \\ + h_1(t)h_2(t) \int_{t_0}^{t} g(s)\varpi(|u(s)|)ds + b(t)H(|u(t)|) \tag{4.346}$$

where $h_1(t), h_2(t), g(t), \varpi(u), H(u)$ are nondecreasing positive functions. The following theorem is given to establish our result.

Theorem 4.32:

Suppose conditions (i) and (ii) of the Theorem 4.31 remained valid with the additional one which states as $\lim_{t\to\infty} \displaystyle\int_{t_0}^{t} b(s)ds \le m < \infty$, for $m > 0$ then, the equation(4.344) is H-U-R stable with H-U-R Positive constant

$$C_\varphi = R^* F^{-1}\left(F(1) + mFH\left(R^*\right)\right) \tag{4.347}$$

Proof:

From inequality (4.345), it is clear that

$$-\varphi(t) \le u''(t) + f(t, u(t), u'(t)) \le \varphi(t) \tag{4.348}$$

Integrating twice, using Lemma 1.1 and multiplying by $\dfrac{1}{t}$ for $t > 0$ by absolute property and with aid of Theorem 3.), we obtain

$$|u(t)| \le \frac{1}{t}\int_{t_0}^{t} \varphi(s)dsR(t)F^{-1}\left(F(1) + \int_{t_0}^{t} b(s)H\left(R(s)\right)ds\right) \tag{4.349}$$

where

$$R(t) = \Omega^{-1}\left(\Omega(1) + \int_{t_0}^{t}\left(h_1(s)h_2(s)g(s)\exp\left(\int_{t_0}^{s} h_1(\tau)d\tau\right) + \int_{t_0}^{s}\exp\left(\int_{\tau}^{\alpha} h_1(\delta)d\delta\right)d\alpha\right)ds\right)$$

By applying the limit of integrals stated in Theorem 4.31 leads to

$$|u(t)| \le R^* F^{-1}\left(F(1) + nH\left(R^*\right)\right)\varphi(t) \tag{4.350}$$

where

$$R^* = \Omega^{-1}\left(\Omega(1) + \exp(n)p\right)$$

and
$\frac{1}{t}\int_{t_0}^{t}\varphi(s)ds \leq \varphi(t)$
Hence,

$|u(t) - u_0(t)| \leq |u(t)| \leq RF^{-1}\left(F(1) + mH\left(R\right)\right)\varphi(t)$

where

$C_{\varphi} = R^{*}F^{-1}\left(F(1) + mH\left(R^{*}\right)\right)$

To examine Hyers-Ulam-Rassias stability of

$$[r(t)\phi(u(t))u'(t)]' + g(t, u(t), u'(t))u'(t) + \alpha(t)h(u(t)) = 0 \tag{4.351}$$

with the initial conditions $u(t_0) = u'(t_0) = 0$ where $r, \alpha \in (\mathbf{I}, \mathbf{R}_{+}),\ \ \phi, : \mathbf{R}_{+} \to \mathbf{R}_{+},\quad g : \mathbf{R}_{+} \times \mathbf{R^2} \to \mathbf{R}$, some conditions may be stated in the subsequent theorem

Theorem 4.33:

If $u(t) \in C^2(R_{+})$ is any solution which satisfies the inequality

$$|\,[r(t)\phi(u(t))u'(t)]' + g(t, u(t), u'(t))u'(t) + \alpha(t)h(u(t))| \leq \varphi, \tag{4.352}$$

then, equation (4.351) is stable in the sense of Hyers-Ulam-Rassias, if it is happened that solution $u_0(t) \in C^2(\mathbf{R}_{+})$ of equation (4.351) satisfies

$|u(t) - u_0(t)| \leq |u(t)| \leq C_{\varphi}\varphi(t).$

Hyers-Ulam-Rassias constant is given as:
$C_{\varphi} = \frac{1}{\delta}T^{*}E^{*}$

Proof:

The proof here is similar to the proof of Theorem 4.27, detail will not be show. Using inequality (4.352), we arrive at

$$-\varphi(t) \leq [r(t)\phi(u(t))u'(t)]' + \alpha(t)h(u(t)) + g(t, u(t), u'(t))u'(t) \leq \varphi(t) \tag{4.353}$$

Integrating (4.353) twice from t_0 to t using Lemma 1.1, let $\frac{1}{t^2} \leq \frac{1}{t} \leq 1$
for $t \geq t_0 \geq 1$

and taking the absolute value

$$\begin{aligned}|R(u(t))| \leq \frac{1}{t\delta}\int_{t_0}^{t}|\varphi(s)|ds + \frac{1}{\delta}\int_{t_0}^{t}\alpha(s)|h(u(s))|ds \\ +\frac{1}{\delta}\int_{t_0}^{t}|g(s, u(s), u'(s))||u'(s)|ds.\end{aligned} \tag{4.354}$$

Let

$|g(t, u(t), u'(t))| \leq \kappa(t)\omega(|u(t)|)\left|u'(t)\right|^2$

and $|R(u(t))| \geq |u(t)|$, (4.354) equals

$$\begin{aligned}|u(t)| \leq \frac{1}{t\delta}\int_{t_0}^{t}|\varphi(s)|ds + \frac{1}{\delta}\int_{t_0}^{t}\alpha(s)h(|u(s)|)ds \\ +\frac{1}{\delta}(|u'(t)|^2\int_{t_0}^{t}\kappa(s)\omega(|u(s)|)ds.\end{aligned} \tag{4.355}$$

Applying Corollary 3.1

to have

$$|u(t)| \leq \frac{1}{\delta} T(t) E(t) \frac{1}{t} \int_{t_0}^{t} |\varphi(s)| ds \tag{4.356}$$

$$T(t) = \Omega^{-1} \left(\Omega(1) + \lambda^2 \int_{t_0}^{t} \kappa(s) \left(E(s) \right) ds \right) \tag{4.357}$$

and

$$E(t) = F^{-1} \left(F(1) + \frac{1}{\delta} \int_{t_0}^{t} \alpha(s) ds \right) \tag{4.358}$$

Let $\lim_{t\to\infty} \int_{t_0}^{t} \alpha(s)ds \leq l < \infty$, $\lim\limits_{t\to\infty} \int_{t_0}^{t} \kappa(s)ds \leq m < \infty$, where $m, l > 0$.

$$|u(t)| \leq \varphi(t) \frac{1}{\delta} T^* E^* \tag{4.359}$$

$$T^* = (\Omega^{-1} \left(\Omega(1) + \frac{m}{\delta} \lambda^2 \omega \left(E \right) \right) \tag{4.360}$$

and

$$E^* = F^{-1} \left(F(1) + \frac{l}{\delta} \right) \tag{4.361}$$

Therefore,

$$|u(t) - u_0(t)| \leq |u(t)| \leq \varphi(t) \frac{1}{\delta} T^* E^* \tag{4.362}$$

Hence,

$$|u(t) - u_0(t)| \leq \varphi(t) \frac{1}{\delta} T^* E^*. \tag{4.363}$$

This concludes the proof.

Euler Type equation is given as

$$t^2 u''(t) + g(u(t)) = 0 \tag{4.364}$$

with initial conditions $u(t_0) = u'(t_0) = 0$ is going to be considered in the next theorem.

Theorem 4.34:

Suppose $u(t) \in C^2(\mathbf{R}_+)$ is a solution that satisfies the inequality

$$|t^2 u''(t) + g(u(t))| \leq \varphi(t), \tag{4.365}$$

then, equation (4.364) is H-U-R stable if

$|u(t) - u_0(t)| \leq \varphi(t)$,

$u_0(t) \in C^2(\mathbf{R}_+)$ is any solution of equation (4.364) with H-U-R constant

$$C_\varphi = \frac{1}{|u''(\xi)|} \varphi(t) \Omega^{-1} \left(\Omega(1) + \frac{1}{|u''(\xi)|} L \right) \tag{4.366}$$

Proof:

From inequality (4.365), we obtain

$$-\varphi(t) \leq t^2 u''(t) + g(u(t)) \leq \varphi(t). \tag{4.367}$$

Multiplying inequality (4.367) by $u'(t)$ gives

$$-u'(t)\varphi(t) \leq t^2 u''(t) u'(t) + g(u(t)) u'(t) \leq u'(t) \varphi(t). \tag{4.368}$$

Integrating, using Lemma 1.1, there exists $\xi \in [t_0, t]$ such that

$$t^2u''(\xi)u(t) \leq \int_{t_0}^{t} \varphi(s)u'(s)ds - \int_{t_0}^{t} g(u(s))u'(s)ds \tag{4.369}$$

Taking the absolute value of both sides, dividing by $t^2|u''\xi|t^2$ we obtain

$$|u(t)| \leq \frac{1}{t^2|u''(\xi)|}\int_{t_0}^{t} \varphi(s)u'(s)ds + \frac{1}{|u''(\xi)|}\int_{t_0}^{t} |u'(s)|g(|u(s)|)ds \tag{4.370}$$

By Theorem 2.9, we get

$$|u(t)| \leq \frac{1}{t^2|u''(\xi)|}\int_{t_0}^{t} |u'(s)|\varphi(s)ds\Omega^{-1}\left(\Omega(1) + \frac{1}{|u''(\xi)|}\int_{t_0}^{t} |u'(s)|ds\right) \tag{4.371}$$

Setting $\frac{1}{t^2}\int_{t_0}^{t} |u'(s)|\varphi(s)ds \leq \lambda\varphi(t)$ and $\lim_{t\to\infty}\int_{t_0}^{t} |u'(s)|ds \leq L$ where $L > 0$ then inequality (4.371) becomes

$$|u(t)| \leq \frac{\lambda}{|u''(\xi)|}\varphi(t)\Omega^{-1}\left(\Omega(1) + \frac{1}{|u''(\xi)|}L\right) \tag{4.372}$$

Hence,

$$|u(t) - u_0(t)| \leq |u(t)| \leq C_\varphi\varphi(t)$$

Where

$$C_\varphi = \frac{\lambda}{|u''(\xi)|}\Omega^{-1}\left(\Omega(1) + \frac{1}{|u''(\xi)|}L\right).$$

In addition , we consider the stability of equation

$$t^2u''(t) + tf(u(t))u'(t) + g(u(t)) = 0 \tag{4.373}$$

with initial conditions $u(t_0) = u'(t_0) = 0$.

Theorem 4.35:

Equation (4.373) is Hyers-Ulam-Rassias stable if solution $u(t) \in C^2(\mathbf{I}, \mathbf{R}_+)$ satisfies the inequality

$$|t^2u''(t) + tf(u(t))u'(t) + g(u(t))| \leq \varphi(t) \tag{4.374}$$

such that

$$|u(t) - u_0(t)| \leq C_\varphi\varphi(t)$$

holds, $u_0(t) \in C^2(\mathbf{I}, \mathbf{R}_+)$ is any solution of equation(4.373), provided the under-mentioned are satisfied.

(i) Let $\phi(t) \in C(\mathbf{I}, \mathbf{R}_+)$, $|P(t, u(t), u'(t))| \leq \phi(t)\varpi(|u(t)|)(|u'(t)|)^2$

(ii) $\int_{t_0}^{\infty} |u'(s)ds \leq L < \infty$ and $\lim_{t\to\infty}\int_{t_0}^{t} \phi(s)ds = r < \infty$

(iii) Let $\lambda > 0$ and take $\lambda\int_{t_0}^{t} \varphi(s)ds \leq \varphi(t)$ for $t \in \mathbf{I}$

Hyers-Ulam-Rassias constant is given as

$$C_\varphi = (\lambda + 1)\Omega^{-1}(\Omega(1) + \lambda L) \tag{4.375}$$

Proof:

Simplifying further inequality (4.374) we arrive at

$$-\varphi(t) \leq t^2u''(t) + f(u(t))tu'(t) + g(u(t)) \leq \varphi(t) \tag{4.376}$$

Multiply through by $u'(t)$ and take $\frac{1}{t^2} \leq \frac{1}{t} \leq 1$ for $t \geq 1$ and integrating from t_0 to t by using equation (4.90) we obtain

$$\frac{1}{2}(u'(s))^2 + u'(t)\int_{t_0}^{t} u'(s)f(u(s))ds + \int_{t_0}^{t} \frac{G(u(s))}{ds}ds \quad \leq \int_{t_0}^{t} \varphi(s)u'(s)ds \tag{4.377}$$

Integrating, setting $|G(u(t))| \geq |u(t)|$, it is clear that

$$|u(t)| \leq \int_{t_0}^{t} \varphi(s)|u'(s)|ds + \frac{1}{2}(|u'(s)|)^2 + |u'(t)|\int_{t_0}^{t} |u'(s)|f(|u(s)|)ds \tag{4.378}$$

Applying Theorem (2.2), conditions (ii) and (iii), setting $|u'(t)| \leq \lambda$ for $\lambda > 0$,

$$|u(t)| \leq (\lambda+1)\Omega^{-1}\left(\Omega(1) + \lambda L\right)\varphi(t) \tag{4.379}$$

Hence,

$$|u(t) - u_0(t)| \leq |u(t)| \leq (\lambda+1)\Omega^{-1}\left(\Omega(1) + \lambda L\right)\varphi(t) \tag{4.380}$$

and

$$C_\varphi = (\lambda+1)\Omega^{-1}\left(\Omega(1) + \lambda L\right)$$

Our next consideration is Hyers-Ulam-Rassias stability of Lienard type equation

$$u''(t) + c(t)f(u(t))u'(t) + a(t)g(u(t))u'(t) = 0 \tag{4.381}$$

with initial conditions $u(t_0) = u'(t_0) = 0$.

Theorem 4.36:

Equation (4.381) is said to be H-U-R stable if the solution $u(t) \in C^2(\mathbf{I}, \mathbf{R}_+)$ satisfies the inequality

$$-\varphi(t) \leq u''(t) + c(t)f(u(t))u'(t) + a(t)g(u(t)) \leq \varphi(t) \tag{4.382}$$

such that

$$|u(t) - u_0(t)| \leq C_\varphi \varphi(t),$$

$u_0(t) \in C^2(\mathbf{I}, \mathbf{R}_+)$ is a solution of equation (4.381) and C_φ is Hyers-Ulam-Rassias positive constant given as

$$C_\varphi = \frac{1}{\delta}\Omega^{-1}\left(\Omega(1) + b\frac{\lambda^2}{\delta}\right), \tag{4.383}$$

Proof:

Using Inequality (4.383) and integrating from t_0 to t we obtain and applying equation (4.90), we get

$$\int_{t_0}^{t} c(s)f(u(s))(u'(s))^2 ds + \int_{t_0}^{t} a(s)\frac{d}{ds}G(u(s))ds \leq \int_{t_0}^{t} u'(s)\varphi(s)ds \tag{4.384}$$

Integration by part, $a(t)$ is an increasing function and $a'(t) \geq 0$, and there exists $\delta > 0$ taking the absolute value, let $|a(t)| \geq \delta$, and $|u'(t)| \leq \lambda$,

$$|u(t)| \leq \frac{1}{\delta}\int_{t_0}^{t} |u'(s)|\,\varphi(s)ds + \frac{\lambda^2}{\delta}\int_{t_0}^{t} c(s)f\left(|u(s)|\right)ds. \tag{4.385}$$

Applying Theorem 2.9, we obtain

$$|u(t)| \leq \frac{1}{\delta}\int_{t_0}^{t} |u'(s)|\,\varphi(s)ds\Omega^{-1}\left(\Omega(1) + \frac{\lambda^2}{\delta}\int_{t_0}^{t} c(s)ds\right)$$

Let $\lim_{t\to\infty}\int_{t_0}^{t} c(s)ds \le b < \infty$, where $b > 0$, where

$\int_{t_0}^{t} |u'(s)|\,\varphi(s)ds \le \varphi(t)$, by conditions (iii) of Theorem 4.21

Hence,

$|u(t)| \le \varphi(t)\frac{1}{\delta}\Omega^{-1}\left(\Omega(1) + b\frac{\lambda^2}{\delta}\right)$

Therefore,

$|u(t) - u_0(t)| \le |u(t)| \le C_\varphi \varphi(t)$

$C_\varphi = \frac{1}{\delta}\Omega^{-1}\left(\Omega(1) + b\frac{\eta^2}{\delta}\right)$

4.3 Hyers-Ulam Stability of Perturbed and Nonperturbed Nonlinear Third Order Differential Equation

4.3.0 Introduction

In this study, we consider Hyers-Ulam stability of third order nonlinear differential equation. The results here extend all the existence ones such as Qarawani (2012) and Algfiary and Jung (2014).

4.3.1 Hyers-Ulam Stability of a Perturbed Nonlinear Third Order Ordinary Differential Equation

The first equation to be considered under this section is:

$$u'''(t) + \beta(t)f(u(t))u''(t) + \alpha(t)g(u(t))u'(t) + \rho(t)\gamma(u(t)) = P(t, u(t), u'(t)). \tag{4.386}$$

with initial value $u(t_0) = u'(t_0) = u''(t) = 0$, where $u, \beta, \alpha, \rho \in C(\mathbf{I}, \mathbf{R}_+)$, $g, \gamma, f \in C(\mathbf{R}_+)$ and $P \in C(\mathbf{I} \times \mathbf{R}_+^2, \mathbf{R}_+)$,

Definition 4.18:

Hyers-Ulam stability of equation (4.386) is defined by giving $K > 0$, $\epsilon > 0$ and $u(t) \in C^3(\mathbf{I}, \mathbf{R}_+)$ be any solution whereby

$$|u'''(t) + \beta(t)f(u(t))u''(t) + \alpha(t)g(u(t))u'(t) + \rho(t)\gamma(u(t)) - P(t, u(t), u'(t))| \le \epsilon \tag{4.387}$$

holds, whenever the solution $u_0(t) \in C^3(\mathbf{I}, \mathbf{R}_+)$ of the equation (4.386) satisfies

$|u(t) - u_0(t)| \le K\epsilon$,

K called H-U constant.

Theorem 4.37:

Suppose $\beta(t)$ is an nondecreasing function, so $\beta'(t) \ge 0$, $|u''(t)| \le \delta$, $|u'''(\xi)| \le \mu$ where $\mu, \delta > 0$, and there exists $\delta > 0$ such that $\beta \ge \delta$, $\int_{t_0}^{\infty} |u'(s)|ds \le L$, $L > 0$, $|P(t, u(t), u'(t))| \le \gamma(t)\phi(|u(t)|)h(|u'(t)|)$ where $\varphi \in C(\mathbf{I}, \mathbf{R}_+)$, $h \in C^1(\mathbf{R}_+)$ and

$\phi \in C(R_+)$. Further more, let $\phi(u(t))$ belongs to class Ψ, then, equation (4.386) is H-U stable with H-U constant

$$K = \frac{(L+\psi L)}{\delta} F^{-1}\left[F(1) + dh(\lambda)\frac{\lambda}{\delta}\phi\left[D^*X^*\right]\right] D^*X^* \tag{4.388}$$

for

$$D^* = \Omega^{-1}\left(\Omega(1) + n\frac{\lambda}{\delta}\gamma\left(X^*\right)\right) X^* \tag{4.389}$$

and

$$X^* = F^{-1}\left(F(1) + m\frac{\lambda^2}{\delta}\right) \tag{4.390}$$

Proof:

Simplifying inequality (4.387) to have

$$-\epsilon \le u'''(t) + \beta(t)f(u(t))u''(t) + \alpha(t)g(u(t))u'(t) + \rho(t)\gamma(u(t)) - P(t, u(t), u'(t)) \le \epsilon.$$

Multiplying by $u'(t)$ to obtain

$$-u'(t)\epsilon \le u'(t)u'''(t) + u'(t)\beta(t)f(u(t))u''(t) + u'(t)\alpha(t)g(u(t))u'(t) + u'(t)\rho(t)\gamma(u(t)) - u'(t)P(t, u(t), u'(t)) \le u'(t)\epsilon.$$

Integrating from t_0 to t, using Lemma (1.1) with equation (4.184) and by Theorem 1.1, we get

$$\begin{aligned} u'''(\xi)\int_{t_0}^t u'(s)ds + \int_{t_0}^t \beta(s)u''(s)\frac{d}{ds}F(u(s))ds + \int_{t_0}^t \alpha(s)g(u(s))(u'(s))^2 ds \\ + \int_{t_0}^t \rho(s)\gamma(u(s))u'(s)ds - \int_{t_0}^t P(s, u(s), u'(t))u'(s)ds \le \epsilon \int_{t_0}^t u'(s)ds, \end{aligned} \tag{4.391}$$

for $\xi \in [t_0, t]$.

Integrating by part, given $\beta(t)$ a nondecreasing function, then $\beta'(t) \ge 0$,

$$\begin{aligned} u''(t)[\beta(t)\mathbb{F}(u(t))] \le \epsilon \int_{t_0}^t u'(s)ds - u'''(\xi)\int_{t_0}^t u'(s)ds \\ - \int_{t_0}^t u'(s)^2\alpha(s)g(u(s))ds - \int_{t_0}^t u'(s)\rho(s)\gamma(u(s))ds \\ + \int_{t_0}^t P(s, u(s), u'(s))u'(s)ds. \end{aligned} \tag{4.392}$$

Taking the absolute value, using the hypothesis in the Theorem 4.37 and $|u'(t)| \le \lambda$, where $\lambda > 0$, $\mathbb{F}(|u(t)|) \ge |u(t)|$, we get

$$\begin{aligned} \frac{|u(t)|}{N} \le 1 + \frac{\lambda^2}{\delta}\int_{t_0}^t \alpha(s)g\left(\frac{|u(s)|}{N}\right)ds + \frac{\lambda}{\delta}\int_{t_0}^t \rho(s)r\left(\frac{|u(s)|}{N}\right)ds + \\ h(\lambda)\frac{\lambda}{\delta}\int_{t_0}^t \gamma(s)\phi\left(\frac{|u(s)|}{N}\right)ds \end{aligned} \tag{4.393}$$

for

$$\frac{\epsilon(L+L\psi)}{\delta} = N. \tag{4.394}$$

Using equation (4.394), we get

$$A(t) \leq 1 + \frac{\lambda^2}{\delta}\int_{t_0}^{t} \alpha(s) g\left(A(s)\right) ds + \frac{\lambda}{\delta}\int_{t_0}^{t} \rho(s) r\left(A(s)\right) ds + h(\lambda)\frac{\lambda}{\delta}\int_{t_0}^{t} \gamma(s)\phi\left(A(s)\right) ds, \tag{4.395}$$

where $A(t) = \dfrac{|u(t)|}{N}$ and by applying Theorem 3.16, we get

$$A(t) \leq \Upsilon^{-1}\left[\Upsilon(1) + h(\lambda)\frac{\lambda}{\delta}\int_{t_0}^{t} \gamma(s)\phi\left[D(s)X(s)\right] ds\right] D(t)X(t)' \tag{4.396}$$

here, we define

$$D(t) = \Omega^{-1}\left(\Omega(1) + \frac{\lambda}{\delta}\int_{t_0}^{t} \rho(s)\gamma\left(X(s)\right)\right) ds, \tag{4.397}$$

and

$$X(t) = F^{-1}\left(F(1) + \frac{\lambda^2}{\delta}\int_{t_0}^{t} \alpha(s) ds\right). \tag{4.398}$$

Setting $\lim_{t\to\infty}\int_{t_0}^{t} \gamma(s)ds \leq d$, $\lim_{t\to\infty}\int_{t_0}^{t} \alpha(s)ds \leq m$,

$\lim_{t\to\infty}\int_{t_0}^{t} \rho(s)ds \leq n$, where $d, m, n > 0$. It is clear that

$$A(t) \leq \Upsilon^{-1}\left[\Upsilon(1) + dh(\lambda)\frac{\lambda}{\delta}\phi\left[DX\right]\right] DX \tag{4.399}$$

we define

$$D^* = \Omega^{-1}\left(\Omega(1) + n\frac{\lambda}{\delta}\gamma\left(X\right)\right) \tag{4.400}$$

and

$$X^* = F^{-1}\left(F(1) + m\frac{\lambda^2}{\delta}\right) \tag{4.401}$$

Replacing $A(t)$ in inequality (4.399), we obtain

$$\frac{|u(t)|}{N} \leq \Upsilon^{-1}\left[\Upsilon(1) + dh(\lambda)\frac{\lambda}{\delta}\phi\left[D^*X^*\right]\right] D^*X^* \tag{4.402}$$

It follows that

$$|u(t)| \leq N\Upsilon^{-1}\left[\Upsilon(1) + dh(\lambda)\frac{\lambda}{\delta}\phi\left[D^*X^*\right]\right] D^*X^* \tag{4.403}$$

Substituting the value of N, to obtain

$$|u(t)| \leq \epsilon\frac{(L + L\psi)}{\delta}G^{-1}\left[G(1) + dh(\lambda)\frac{\lambda}{\delta}\phi\left[D^*X^*\right]\right] D^*X^* \tag{4.404}$$

Hence,

$$|u(t) - u(t_0)| \leq |u(t)| \leq \epsilon K$$

K is given as

$$K = \frac{(L + L\psi)}{\delta}\Upsilon^{-1}\left[\Upsilon(1) + dh(\lambda)\frac{\lambda}{\delta}\phi\left[D^*X^*\right]\right] D^*X^*$$

We consider Hyers-Ulam stability of equation (4.386) by taking $\alpha(t)g(ut))u'(t) = 0$, the Lienard equation (4.386) reduces to

$$u'''(t) + \beta(t)f(u(t))u''(t) + \rho(t)\gamma(u(t)) = P(t, u(t), u'(t)). \tag{4.405}$$

with initial conditions $u(t_0) = u'(t_0) = u''(t_0) = 0$.

Theorem 4.38:

Equation (4.405) is Hyers-Ulam stable and H-U constant is given as

$$K = \frac{(L + L\psi)}{\delta}\Omega^{-1}\left(\Omega(1) + \frac{h(\lambda)\lambda}{\delta}\phi\left(F^{-1}\left(F(1) + \frac{\lambda}{\delta}n\right)\right)m\right) F^{-1}\left(F(1) + \frac{\lambda}{\delta}n\right) \tag{4.406}$$

with all the conditions prescribed in Theorem 4.37 remain valid.

Proof:

The proof begins from (4.387), multiplying by $u'(t)$, integrating from t_0 to t, using Lemma 1.1 and equation (4.184), by Theorem 1.1, there exist $\xi \in [t_0, t]$ such that

$$u'''(\xi)\int_{t_0}^{t} u'(s)ds + \int_{t_0}^{t} \beta(s)u''(s)\frac{d}{ds}F(u(s))ds + \int_{t_0}^{t} \rho(s)\gamma(u(s))u'(s)ds - \int_{t_0}^{t} P(s, u(s), u'(t))u'(s)ds \leq \epsilon \int_{t_0}^{t} u'(s)ds \tag{4.407}$$

Integrating by part and using hypothesis of Theorem 4.37 we obtain

$$u''(t)[\beta(t)\mathbb{F}(u(t))] \leq \epsilon \int_{t_0}^{t} u'(s)ds - u'''(\xi)\int_{t_0}^{t} u'(s)ds - \int_{t_0}^{t} u'(s)\rho(s)\gamma(u(s))ds + \int_{t_0}^{t} P(s, u(s), u'(s))u'(s)ds. \tag{4.408}$$

Taking the absolute value, using the hypothesis in the Theorem 4.37 and $|u'(t)| \leq \lambda$ for $\lambda > 0$, $\mathbb{F}(|u(t)|) \geq |u(t)|$

$$\frac{|u(t)|}{N} \leq 1 + \frac{\lambda}{\delta}\int_{t_0}^{t} \rho(s)r\left(\frac{|u(s)|}{N}\right)ds + h(\lambda)\frac{\lambda}{\delta}\int_{t_0}^{t} \gamma(s)\phi\left(\frac{|u(s)|}{N}\right)ds \tag{4.409}$$

let

$$\frac{\epsilon(L + L\psi)}{\delta} = N,$$

Equation(4.409) happens to be

$$\frac{|u(t)|}{N} \leq 1 + \frac{\lambda}{\delta}\int_{t_0}^{t} \rho(s)r\left(\frac{|u(t)|}{N}\right)ds + h(\lambda)\frac{\lambda}{\delta}\int_{t_0}^{t} \gamma(s)\phi\left(\frac{|u(t)|}{N}\right)ds \tag{4.410}$$

by applying Theorem 3.7, we get

$$\frac{|u(t)|}{N} \leq \Omega^{-1}\left(\Omega(1) + \frac{h(\lambda)\lambda}{\delta}\int_{t_0}^{t} \gamma(s)\phi\left(F^{-1}\left(F(1) + \frac{\lambda}{\delta}\int_{t_0}^{s} \rho(\alpha)d\alpha\right)\right)ds\right) F^{-1}\left(F(1) + \frac{\lambda}{\delta}\int_{t_0}^{t} \rho(s)ds\right) \tag{4.411}$$

Setting $\lim_{t\to\infty}\int_{t_0}^{t} \gamma(s)ds \leq m$, $\lim_{t\to\infty}\int_{t_0}^{t} \rho(s)ds \leq n$, where $m, n > 0$, we get

$$\frac{|u(t)|}{N} \leq \Omega^{-1}\left(\Omega(1) + \frac{h(\lambda)\lambda}{\delta}\phi\left(F^{-1}\left(F(1) + \frac{\lambda}{\delta}n\right)\right)m\right) F^{-1}\left(F(1) + \frac{\lambda}{\delta}n\right) \tag{4.412}$$

It further yields

$$|u(t)| \leq N\Omega^{-1}\left(\Omega(1)+\frac{h(\lambda)\lambda}{\delta}\phi\left(F^{-1}\left(F(1)+\frac{\lambda}{\delta}n\right)\right)m\right) \\ F^{-1}\left(F(1)+\frac{\lambda}{\delta}n\right) \tag{4.413}$$

Substituting the value of N, we have

$$|u(t)| \leq \epsilon\frac{(L+L\psi)}{\delta}\Omega^{-1}\left(\Omega(1)+\frac{h(\lambda)\lambda}{\delta}\phi\left(F^{-1}\left(F(1)+\frac{\lambda}{\delta}n\right)\right)m\right) \\ F^{-1}\left(F(1)+\frac{\lambda}{\delta}n\right). \tag{4.414}$$

Hence,

$$|u(t)-u(t_0)| \leq |u(t)| \leq \epsilon K$$

as

$$K = \frac{(L+L\psi)}{\delta}\Omega^{-1}\left(\Omega(1)+\frac{h(\lambda)\lambda}{\delta}\phi\left(F^{-1}\left(F(1)+\frac{\lambda}{\delta}n\right)\right)m\right) \\ F^{-1}\left(F(1)+\frac{\lambda}{\delta}n\right)$$

Furthermore, we investigate equation

$$(\alpha(t)p(u(t))u'(t))''+(\gamma(t)f(u(t))u'(t))'+\beta(t)g(u(t))u'(t) \\ +\rho(u(t)) = P(t,u(t),u'(t)) \tag{4.415}$$

with initial value $u(t_0)=u'(t_0)=u''(t_0)=0$, where $\beta,\gamma,\alpha,\in C(\mathbf{I},\mathbf{R}_+)$, $g,p,f,\rho\in C(\mathbf{R}_+)$. $P\in(\mathbf{I}\times{\mathbf{R}_+}^2,\mathbf{R}_+)$.

Definition 4.19:

Equation (4.415) is stable, if there exists $K>0$, $\epsilon>0$ and any solution $u(t)\in C^3(\mathbf{I},\mathbf{R}_+)$ satisfying

$$(\alpha(t)p(u(t))u'(t))''+(\gamma(t)f(u(t))u'(t))'+\beta(t)g(u(t))u'(t)+\rho(u(t)) \\ -P(t,u(t),u'(t))| \leq \epsilon \tag{4.416}$$

and if there exists any solution $u_0(t)\in C^3(\mathbf{I},\mathbf{R}_+)$ of (4.415) such that

$|u(t)-u_0(t)| \leq K\epsilon$,

therefore, equation (4.415) is Hyers-Ulam stable and

K is called Hyers-Ulam constant.

The result is given as thus.

Theorem 4.39;

Let the functions $\alpha,\ \gamma,\ \rho$ and β be continuous on $\mathbf{R}_+$ and P be continuous on $\mathbf{R}_+$, in addition, α is nondecreasing in t such that

$\alpha(t)\geq\phi$, $\alpha'(t)\geq 0$ on $\mathbf{R}_+$ and functions f,g belong to class Ψ. Suppose

(i) $\lim_{t\to\infty}\int_{t_0}^{t}\beta(s)ds \leq m<\infty,\ m>0.$

(ii) $\lim_{t\to\infty}\int_{t_0}^{t}\gamma(s)ds \le n < \infty,\ n > 0.$

(iii)$\lim_{t\to\infty}\int_{t_0}^{t}h(s)ds \le q < \infty,\ q > 0.$

then equation(4.415) is H-U stable with

$$K = \frac{(L+|r(u(\varphi))|L)}{2\phi|u'(\varepsilon)|}\Upsilon^{-1}\left[\Upsilon(1)+C_4qr\left[\Omega^{-1}\left(\Omega(1)+C_3mg\left(\mathbb{H}^*\right)\right)\right.\right.$$
$$\left.\left.\Omega^{-1}\left(\Omega(1)+C_2n\right)\right]\right]\Omega^{-1}\left(\Omega(1)+C_3mg\left(\mathbb{H}^*\right)\right)\mathbb{H}^*, \tag{4.417}$$

where

$\mathbb{H}^* = F^{-1}\left(F(1)+C_2n\right),$

$C_2 = \dfrac{2|u'(\eta)|\lambda}{2\phi|u'(\varepsilon)|},\quad C_3 = \dfrac{\lambda^2}{2\phi|u'(\varepsilon)|}$

and $C_4 = \dfrac{\lambda^{n+1}}{2\phi|u'(\varepsilon)|}$ are positive constants.

Proof:

From inequality (4.416), multiplying through by $u'(t)$, integrating from t_0 to t, using Theorem 1.1 there exists $t_0 \le \varepsilon \le t,\quad t_0 \le \eta \le t$, such that

$$\begin{aligned}u'(\varepsilon)\int_{t_0}^{t}(\alpha(s)p(u(s))u'(s))''ds + u'(\eta)\int_{t_0}^{t}(\gamma(s)f(u(s))u'(s))'ds\\ +\int_{t_0}^{t}\beta(s)g(u(s))(u'(s))^2ds+\int_{t_0}^{t}\rho(u(s))u'(s)ds\\ -\int_{t_0}^{t}P(s,u(s),u'(s))u'(s)ds \le \epsilon\int_{t_0}^{t}u'(s)ds\end{aligned} \tag{4.418}$$

By further simplification of equation (4.418) we obtain

$$\begin{aligned}&u'(\varepsilon)(\alpha(s)p(u(s))u'(s))' + u'(\eta)(\gamma(s)f(u(s))u'(s))\\ &+\int_{t_0}^{t}\beta(s)g(u(s))(u'(s))^2ds+\int_{t_0}^{t}\rho(u(s))u'(s)ds\\ &-\int_{t_0}^{t}P(s,u(s),u'(s))u'(s)ds \le \epsilon\int_{t_0}^{t}u'(s)ds.\end{aligned} \tag{4.419}$$

Integrating from t_0 to t, twice, using Lemma 1.1 and applying the Theorem 1.1, there exists $t_0 \le \varphi \le t$ such that

$$\begin{aligned}u'(\varepsilon)\int_{t_0}^{t}\alpha(s)p(u(s))u'(s)ds + tu'(\eta)\int_{t_0}^{t}\gamma(s)f(u(s))u'(s)ds\\ +\frac{t^2}{2}\int_{t_0}^{t}\beta(s)g(u(s))(u'(s))^2ds+\rho(u(\varphi))\frac{t^2}{2}\int_{t_0}^{t}u'(s)ds\\ -\frac{t^2}{2}\int_{1}^{t}P(s,u(s),u'(s))u'(s)ds \le \frac{t^2}{2}\epsilon\int_{t_0}^{t}u'(s)ds\end{aligned} \tag{4.420}$$

Multiplying both sides of inequality (4.420) by $\dfrac{2}{t^2}$ for $t \ge 1$ and

put

$$\mathbb{P}(u(t)) = \int_{u(t_0)}^{u(t)}p(u(s))ds, \tag{4.421}$$

then, inequality (4.420) becomes

$$\begin{aligned}&\frac{2}{t^2}u'(\varepsilon)\int_{t_0}^{t}\alpha(s)\frac{d}{ds}\mathbb{P}(u(s))ds+\frac{2}{t}u'(\eta)u'(t)\int_{t_0}^{t}\gamma(s)f(u(s))ds\\&+(u'(t))^2\int_{t_0}^{t}\beta(s)g(u(s))ds+\rho(u(\varphi))L-\int_{t_0}^{t}P(s,u(s),u'(s))u'(s)ds\leq\epsilon L\end{aligned}\tag{4.422}$$

Integrating (4.422) by part, take $\frac{1}{t^2}\leq 1$ for $t\geq 1$, using the hypothesis in the Theorem 4.39 and taking absolute value

$$\begin{aligned}2\phi|u'(\varepsilon)||\mathbb{P}(u(s))|&\leq\epsilon(L+|\rho(u(\varphi))|L)+2|u'(\eta)||u'(t)|\int_{t_0}^{t}\gamma(s)f(|u(s)|)ds\\&+|(u'(t))|^2\int_{t_0}^{t}\beta(s)g(|u(s)|)ds+\int_{t_0}^{t}|P(s,u(s),u'(s))||u'(s)|ds\end{aligned}\tag{4.423}$$

Setting $|P(t,u(t),u'(t))\leq h(t)r(|u(t)|)|u'(t)|^n$ where $h(t),\ r\in C(\mathbf{R}_+)$ and $n\in\mathbf{N}$, equation (4.423) becomes

$$\begin{aligned}2\phi|u'(\varepsilon)||\mathbb{P}(u(s))|&\leq\epsilon(L+|\rho(u(\varphi))|L)+2|u'(\eta)||u'(t)|\int_{t_0}^{t}\gamma(s)f(|u(s)|)ds\\&+|(u'(t))|^2\int_{t_0}^{t}\beta(s)g(|u(s)|)ds+|u'(t)|^{n+1}\int_{t_0}^{t}h(s)r(|u(s)|)ds\end{aligned}\tag{4.424}$$

Dividing both sides by $2\phi|u'(\varepsilon)|$, it leads to

$$\begin{aligned}|\mathbb{P}(u(s))|&\leq\frac{\epsilon(L+|\rho(u(\varphi))|L)}{2\phi|u'(\varepsilon)|}+\frac{2|u'(\eta)||u'(t)|}{2\phi|u'(\varepsilon)|}\int_{t_0}^{t}\gamma(s)f(|u(s)|)ds\\&+\frac{|(u'(t))|^2}{2\phi|u'(\varepsilon)|}\int_{t_0}^{t}\beta(s)g(|u(s)|)ds+\frac{|u'(t)|^{n+1}}{2\phi|u'(\varepsilon)|}\int_{t_0}^{t}h(s)r(|u(s)|)ds\end{aligned}\tag{4.425}$$

Let $|u'(t)|\leq\lambda$, $\quad C_1=\frac{\epsilon(L+|\rho(u(\varphi))|L)}{2\phi|u'(\varepsilon)|}\quad C_2=\frac{2|u'(\eta)|\lambda}{2\phi|u'(\varepsilon)|}\quad C_3=\frac{\lambda^2}{2\phi|u'(\varepsilon)|}$, $C_4=\frac{\lambda^{n+1}}{2\phi|u'(\varepsilon)|}$, using these in equation (4.425) to obtain

$$\begin{aligned}|\mathbb{P}(u(s))|\leq C_1+C_2\int_{t_0}^{t}\gamma(s)f(|u(s)|)ds+C_3\int_{t_0}^{t}\beta(s)g(|u(s)|)ds\\+C_4\int_{t_0}^{t}h(s)r(|u(s)|)ds\end{aligned}\tag{4.426}$$

Setting $\mathbb{P}(u(s))|\geq|u(t)|$, then, applying Theorem 3.16, we have

$$\begin{aligned}&\frac{|u(t)|}{C_1}\leq\Upsilon^{-1}\left[\Upsilon(1)+C_4\int_{t_0}^{t}h(s)r\left[\Omega^{-1}\left(\Omega(1)+C_3\int_{t_0}^{s}\beta(\alpha)g\left(\mathbb{H}(\alpha)d\alpha\right)\right)\right.\right.\\&\left.\left.\Omega^{-1}\left(\Omega(1)+C_2\int_{t_0}^{s}\gamma(\alpha)d\alpha\right)\right]ds\right]\Omega^{-1}\left(\Omega(1)+C_3\int_{t_0}^{t}\beta(s)g\left(\mathbb{H}(s)\right)ds\right)\mathbb{H}(t)\end{aligned}\tag{4.427}$$

where

$$\mathbb{H}(t))=F^{-1}\left(F(1)+C_2\int_{t_0}^{t}\gamma(\delta)d\delta\right)$$

By simplifying further using (i-iii) of the Theorem 4.39, we obtain

$$\begin{aligned}&\frac{|u(t)|}{C_1}\leq\Upsilon^{-1}\left[\Upsilon(1)+C_4qr\left[\Omega^{-1}\left(\Omega(1)+C_3mg\left(\mathbb{H}^*\right)\right)\right.\right.\\&\qquad\left.\left.\Omega^{-1}\left(\Omega(1)+C_2n\right)\right]\right]\Omega^{-1}\left(\Omega(1)+C_3mg\left(\mathbb{H}^*\right)\right)\mathbb{H}^*\end{aligned}\tag{4.428}$$

Here

$\mathbb{H}^* = F^{-1}\left(F(1) + C_2 n\right),$

and $\mathbb{H}^*$ a positive constant. we write

$$|u(t)| \le C_1 \Upsilon^{-1}\left[\Upsilon(1) + C_4 qr\left[\Omega^{-1}\left(\Omega(1) + C_3 mg\left(\mathbb{H}^*\right)\right)\right.\right.$$
$$\left.\left.\Omega^{-1}\left(\Omega(1) + C_2 n\right)\right]\right]\Omega^{-1}\left(\Omega(1) + C_3 mg\left(\mathbb{H}^*\right)\right)\mathbb{H}^* \tag{4.429}$$

Substituting $C_1,\ C_2,\ C_3$ and C_4

$$|u(t)| \le \epsilon\frac{(L + |\rho(u(\varphi))|L)}{2\phi|u'(\varepsilon)|}\Upsilon^{-1}\left[\Upsilon(1) + C_4 qr\left[\Omega^{-1}\left(\Omega(1) + C_3 mg\left(\mathbb{H}^*\right)\right)\right.\right.$$
$$\left.\left.\Omega^{-1}\left(\Omega(1) + C_2 n\right)\right]\right]\Omega^{-1}\left(\Omega(1) + C_3 mg\left(\mathbb{H}^*\right)\right)\mathbb{H}^* \tag{4.430}$$

Hence,

$|u(t) - u_0(t)| \le |u(t)| \le K\epsilon.$

By analysis

$$K = \frac{(L + |\rho(u(\varphi))|L)}{2\phi|u'(\varepsilon)|}\Upsilon^{-1}\left[\Upsilon(1) + C_4 qr\left[\Omega^{-1}\left(\Omega(1) + C_3 mg\left(\mathbb{H}^*\right)\right)\right.\right.$$
$$\left.\left.\Omega^{-1}\left(\Omega(1) + C_2 n\right)\right]\right]\Omega^{-1}\left(\Omega(1) + C_3 mg\left(\mathbb{H}^*\right)\right)\mathbb{H}^*$$

If $\beta(t)g(u(t))u'(t) = 0$ in equation (4.415) we consider its H-U stability in theorem below.

Theorem 4.40:

Let all undermentioned conditions of Theorem (4.39) remained valid. Suppose $\beta(t)g(u(t))u'(t) = 0$ in equation (4.415), then, the equation obtained is given as

$$(\alpha(t)p(u(t))u'(t))'' + (\gamma(t)f(u(t))u'(t))' + \rho(u(t)) = P(t, u(t), u'(t)) \tag{4.431}$$

with initial value $u(t_0) = u'(t_0) = u''(t_0) = 0$, is stable via H-U stability and H-U constant

$$K = \frac{(L + |\rho(u(\varphi))|L)}{2\phi|u'(\varepsilon)|}\Upsilon^{-1}\left(\Upsilon(1) + C_3 r\left[F^{-1}\left(F(1) + C_2 n\right)q\right)\right)$$
$$F^{-1}\left(F(1) + C_2 n\right) \tag{4.432}$$

$C_2 = \dfrac{\lambda^2}{2\phi|u'(\varepsilon)|}$ and $C_3 = \dfrac{\lambda^n + 1}{2\phi|u'(\varepsilon)|}$ are positive constants.

Proof:

From the inequality (4.416), if $\beta(t)g(u(t))u'(t) = 0$, multiplying through by $u'(t)$,

$$(\alpha(t)p(u(t))u'(t))''u'(t) + (\gamma(t)f(u(t))u'(t))'u'(t) + \rho(u(t))u'(t)$$
$$-P(t, u(t), u'(t))| \le \epsilon u'(t). \tag{4.433}$$

Integrating from t_0 to t, by Theorem 1.1, there exits $t_0 \le \varepsilon \le t, \quad t_0 \le \eta \le t.$ such that

$$u'(\varepsilon)\int_{t_0}^{t}(\alpha(s)p(u(s))u'(s))''ds + u'(\eta)\int_{t_0}^{t}(\gamma(s)f(u(s))u'(s))'ds$$
$$+\int_{t_0}^{t}\rho(u(s))u'(s)ds - \int_{t_0}^{t}P(s, u(s), u'(s))u'(s)ds \le \epsilon\int_{t_0}^{t}u'(s)ds, \tag{4.434}$$

Integrating from t_0 to t twice and applying Lemma 1.1, multiplying through by $\frac{2}{t^2}$ for $t \geq 0$ and applying the Theorem 1.1, there exists φ such that $t_0 \leq \varphi \leq t$, then

$$\begin{aligned}&\frac{2}{t^2}u'(\varepsilon)\int_{t_0}^{t}\alpha(s)p(u(s))u'(s)ds + \frac{2}{t}u'(\eta)\int_{t_0}^{t}\gamma(s)f(u(s))u'(s)ds\\&+\rho(u(\varphi))\int_{t_0}^{t}u'(s)ds - \int_{t_0}^{t}P(s,u(s),u'(s))u'(s)ds \leq \epsilon\int_{t_0}^{t}u'(s)ds\end{aligned} \tag{4.435}$$

Using equation (4.421) and further evaluation of (4.435), we obtain

$$\begin{aligned}&\frac{2}{t^2}u'(\varepsilon)\int_{t_0}^{t}\alpha(s)\frac{d}{ds}\mathbb{P}(u(s))ds + \frac{2}{t}u'(\eta)u'(t)\int_{t_0}^{t}\gamma(s)f(u(s))ds\\&\qquad+\rho(u(\varphi))L - \int_{t_0}^{t}P(s,u(s),u'(s))u'(s)ds \leq \epsilon L.\end{aligned} \tag{4.436}$$

Integrating (4.436) by part, $\frac{1}{t^2} \leq 1$, using the hypothesis in the Theorem 4.39 and taking absolute value

$$\begin{aligned}&2\phi|u'(\varepsilon)||\mathbb{P}(u(s))| \leq \epsilon(L + |\rho(u(\varphi))|L)\\&+2|u'(\eta)||u'(t)|\int_{t_0}^{t}\gamma(s)f(|u(s)|)ds + \int_{t_0}^{t}|P(s,u(s),u'(s))||u'(s)|ds.\end{aligned} \tag{4.437}$$

Setting $|P(t,u(t),u'(t)) \leq h(t)r(|u(t)|)|u'(t)|^n$ where $h(t),\ r \in C(\mathbf{R}_+)$ and $n \in \mathbf{N}$, then, equation (4.481) becomes

$$\begin{aligned}2\phi|u'(\varepsilon)||\mathbb{P}(u(s))| \leq \epsilon(L + |\rho(u(\varphi))|L) &+ 2|u'(\eta)||u'(t)|\int_{t_0}^{t}\gamma(s)f(|u(s)|)ds\\&+|u'(t)|^{n+1}\int_{t_0}^{t}h(s)r(|u(s)|)ds\end{aligned} \tag{4.438}$$

Divide both sides by $2\phi|u'(\varepsilon)|$, we obtain

$$\begin{aligned}|\mathbb{P}(u(s))| \leq \frac{\epsilon(L + |\rho(u(\varphi))|L)}{2\phi|u'(\varepsilon)|} &+ \frac{2|u'(\eta)||u'(t)|}{2\phi|u'(\varepsilon)|}\int_{t_0}^{t}\gamma(s)f(|u(s)|)ds\\&+\frac{|u'(t)|^{n+1}}{2\phi|u'(\varepsilon)|}\int_{t_0}^{t}h(s)r(|u(s)|)ds\end{aligned} \tag{4.439}$$

Let $|u'(t)| \leq \lambda, \quad C_1 = \frac{\epsilon(L + |\rho(u(\varphi))|L)}{2\phi|u'(\varepsilon)|} \quad C_2 = \frac{2|u'(\eta)|\lambda}{2\phi|u'(\varepsilon)|} \quad C_3 = \frac{\lambda^{n+1}}{2\phi|u'(\varepsilon)|}$, using these in equation (4.439) to get

$$|\mathbb{P}(u(s))| \leq C_1 + C_2\int_{t_0}^{t}\gamma(s)f(|u(s)|)ds + C_3\int_{t_0}^{t}h(s)r(|u(s)|)ds \tag{4.440}$$

Letting $|\mathbb{P}(u(s))| \geq |u(t)|$, by applying Theorem 3.7, the result is

$$\begin{aligned}&\frac{|u(t)|}{C_1} \leq \Upsilon^{-1}\left(\Upsilon(1) + C_3\int_{t_0}^{t}h(s)r\left[F^{-1}\left(F(1)\right.\right.\right.\\&\left.\left.\left.+C_2\int_{t_0}^{s}\gamma(\delta)d\delta\right)ds\right)\right)F^{-1}\left(F(1) + C_2\int_{t_0}^{t}\gamma(s)ds\right)\end{aligned} \tag{4.441}$$

By applying the conditions (ii-iii) of the Theorem 4.39, we obtain

$$|u(t)| \leq C_1\Upsilon^{-1}\left(\Upsilon(1) + C_3r\left[F^{-1}\left(F(1) + C_2n\right)q\right)\right)F^{-1}\left(F(1) + C_2n\right) \tag{4.442}$$

Substituting C_1, we have

$$|u(t)| \leq \frac{\epsilon(L+|\rho(u(\varphi))|L)}{2\phi|u'(\varepsilon)|}\Upsilon^{-1}\left(\Upsilon(1)+C_3 r\left[F^{-1}\left(F(1)+C_2 n\right)q\right)\right) \\ F^{-1}\left(F(1)+C_2 n\right) \tag{4.443}$$

Hence,

$|u(t)-u_0(t)| \leq |u(t)| \leq K\epsilon.$

where

$$K=\frac{(L+|\rho(u(\varphi))|L)}{2\phi|u'(\varepsilon)|}\Upsilon^{-1}\left(\Upsilon(1)+C_3 r\left[F^{-1}\left(F(1)+C_2 n\right)q\right)\right) \\ F^{-1}\left(F(1)+C_2 n\right) \tag{4.444}$$

Theorem 4.41

Let $(\gamma(t)f(u(t))u'(t))'=0$, equation (4.419) reduced to

$$(\alpha(t)p(u(t))u'(t))''+\beta(t)g(u(t))u'(t)+\rho(u(t))=P(t,u(t),u'(t)), \tag{4.445}$$

then equation (4.445) is H-U stable with

$$K=\frac{(L+|\rho(u(\varphi))|L)}{2\phi|u'(\varepsilon)|}\Upsilon^{-1}\left(\Upsilon(1)+C_3 r\left[\Omega^{-1}\left(\Omega(1)\quad +C_2 m\right)q\right)\right) \\ \Omega^{-1}\left(\Omega(1)+m\right) \tag{4.446}$$

where $C_2=\dfrac{\lambda^2}{2\phi|u'(\varepsilon)|}$, $\quad C3=\dfrac{\lambda^{n+1}}{2\phi|u'(\varepsilon)|}$ are positive constants

Proof:

From the inequality (4.415), put $(\gamma(t)f(u(t))u'(t))'=0$, multiplying through by $u'(t)$, integrating from t_0 to t trice and using Lemma 1.1 we get

$$\int_{t_0}^{t}\alpha(s)p(u(s))u'(s)u'(s)ds+\frac{t^2}{2}\int_{t_0}^{t}\beta(s)g(u(s))(u'(s))^2ds \\ +\frac{t^2}{2}\int_{t_0}^{t}\rho(u(s))u'(s)ds-\frac{t^2}{2}\int_{t_0}^{t}P(s,u(s),u'(s))u'(s)ds \leq \frac{t^2}{2}\epsilon\int_{t_0}^{t}u'(s)ds \tag{4.447}$$

Applying Theorem 1.1 that is there exist $\varepsilon,\phi\in[t_0,t]$ such that

$$u'(\varepsilon)\int_{t_0}^{t}\alpha(s)p(u(s))u'(s)ds+\frac{t^2}{2}\int_{t_0}^{t}\beta(s)g(u(s))(u'(s))^2ds \\ +\frac{t^2}{2}\rho(u(\phi))\int_{t_0}^{t}u'(s)ds-\frac{t^2}{2}\int_{t_0}^{t}P(s,u(s),u'(s))u'(s)ds \leq \frac{t^2}{2}\epsilon\int_{t_0}^{t}u'(s)ds \tag{4.448}$$

Multiplying through by $\dfrac{2}{t^2}$ for $t\geq 1$, we get

$$\frac{2}{t^2}u'(\varepsilon)\int_{t_0}^{t}\alpha(s)p(u(s))u'(s)ds+\int_{t_0}^{t}\beta(s)g(u(s))(u'(s))^2ds \\ +\rho(u(\phi))\int_{t_0}^{t}u'(s)ds-\int_{t_0}^{t}P(s,u(s),u'(s))u'(s)ds \leq \epsilon\int_{t_0}^{t}u'(s)ds \tag{4.449}$$

Use equation (4.421) on inequality (4.449), we have

$$\frac{2}{t^2}u'(\varepsilon)\int_{t_0}^{t}\alpha(s)\frac{d}{ds}\mathbb{P}(u(s))ds+(u'(t))^2\int_{t_0}^{t}\beta(s)g(u(s))ds \\ +\rho(u(\varphi))L-\int_{t_0}^{t}P(s,u(s),u'(s))u'(s)ds \leq \epsilon L \tag{4.450}$$

Integrating (4.450) by part, by using the hypothesis in the Theorem 4.39 and taking the absolute value, we have

$$2\phi|u'(\varepsilon)||\mathbb{P}(u(s))| \le \epsilon(L+|\rho(u(\varphi))|L) + |(u'(t))|^2\int_{t_0}^{t}\beta(s)g(|u(s)|)ds + \int_{t_0}^{t}|P(s,u(s),u'(s))||u'(s)|ds \tag{4.451}$$

Setting $|P(t,u(t),u'(t)) \le h(t)r(|u(t)|)|u'(t)|^n$ where $h(t),\ r \in C(\mathbf{R}_+)$ and $n \in \mathbf{N}$, then, equation(4.451) becomes

$$2\phi|u'(\varepsilon)||\mathbb{P}(u(s))| \le \epsilon(L+|\rho(u(\varphi))|L) + |(u'(t))|^2\int_{t_0}^{t}\beta(s)g(|u(s)|)ds + |u'(t)|^{n+1}\int_{t_0}^{t}h(s)r(|u(s)|)ds \tag{4.452}$$

Divide both sides by $2\phi|u'(\varepsilon)|$, we obtain

$$|\mathbb{P}(u(s))| \le \frac{\epsilon(L+|\rho(u(\varphi))|L)}{2\phi|u'(\varepsilon)|} + \frac{|(u'(t))|^2}{2\phi|u'(\varepsilon)|}\int_{t_0}^{t}\beta(s)g(|u(s)|)ds + \frac{|u'(t)|^{n+1}}{2\phi|u'(\varepsilon)|}\int_{t_0}^{t}h(s)r(|u(s)|)ds \tag{4.453}$$

Let $|u'(t)| \le \lambda$, $C_1 = \dfrac{\epsilon(L+|\rho(u(\varphi))|L)}{2\phi|u'(\varepsilon)|}$ $C_2 = \dfrac{\lambda^2}{2\phi|u'(\varepsilon)|}$, $C_3 = \dfrac{\lambda^{n+1}}{2\phi|u'(\varepsilon)|}$, using these in equation (4.453), we obtain

$$|\mathbb{P}(u(s))| \le C_1 + C_2\int_{1}^{t}\beta(s)g(|u(s)|)ds + C_3\int_{t_0}^{t}h(s)r(|u(s)|)ds \tag{4.454}$$

Letting $|\mathbb{P}(u(s))| \ge |u(t)|$ Applying Theorem 3.7, we arrive at

$$\frac{|u(t)|}{C_1} \le \Upsilon^{-1}\left(\Upsilon(1) + C_3\int_{t_0}^{t}h(s)r\left[\Omega^{-1}\left(\Omega(1) + C_2\int_{t_0}^{s}\beta(\delta)d\delta\right)ds\right)\right)\Omega^{-1}\left(\Omega(1) + C_2\int_{t_0}^{t}\beta(s)ds\right) \tag{4.455}$$

Taking advantage of conditions (i) and iii), to have

$$|u(t)| \le C_1\Upsilon^{-1}\left(\Upsilon(1) + C_3 r\left[\Omega^{-1}\left(\Omega(1) \quad + C_2 m\right)q\right)\right)\Omega^{-1}\left(\Omega(1) + m\right) \tag{4.456}$$

Substituting C_1, we obtain

$$|u(t)| \le \frac{\epsilon(L+|\rho(u(\varphi))|L)}{2\phi|u'(\varepsilon)|}\Upsilon^{-1}\left(\Upsilon(1) + C_3 r\left[\Omega^{-1}\left(\Omega(1) \quad + C_2 m\right)q\right)\right)\Omega^{-1}\left(\Omega(1) + m\right) \tag{4.457}$$

Hence,

$|u(t) - u_0(t)| \le |u(t)| \le K\epsilon,$

where

$$K = \frac{(L+|\rho(u(\varphi))|L)}{2\phi|u'(\varepsilon)|}\Upsilon^{-1}\left(\Upsilon(1) + C_3 r\left[\Omega^{-1}\left(\Omega(1) \quad + C_2 m\right)q\right)\right)\Omega^{-1}\left(\Omega(1) + m\right)$$

Equation (4.411) is considered when the function $\alpha(t)$ is absent.

Theorem 4.42:

Let the function $\alpha(t)$ be absent and $\rho(u(t)) = 0$ in equation (4.411), then, we have

$$(p(u(t))u'(t))'' + (\gamma(t)f(u(t))u'(t))' + \beta(t)g(u(t))u'(t) = P(t,u(t),u'(t)) \tag{4.458}$$

is Hyers-Ulam stable with

$$K = \frac{L}{2|u'(\varepsilon)|}\Upsilon^{-1}\left[\Upsilon(1) + C_4 qr\left[\Omega^{-1}\left(\Omega(1) + C_3 mg\left(\mathbb{H}^*\right)\right)\right.\right.$$
$$\left.\left.\Omega^{-1}\left(\Omega(1) + C_2 n\right)\right]\right]\Omega^{-1}\left(\Omega(1) + C_3 mg\left(\mathbb{H}^*\right)\right)\mathbb{H}^* \tag{4.459}$$

Proof:

From inequality (4.416), let the function $\alpha(t)$ be absent and $\rho(u(t)) = 0$, multiplying through by $u'(t)$, integrating from t_0 to t, we get

$$\int_{t_0}^{t}(p(u(s))u'(s))''u'(s)ds + \int_{t_0}^{t}(\gamma(s)f(u(s))u'(s))'u'(s)ds$$
$$+\int_{t_0}^{t}\beta(s)g(u(s))(u'(s))^2 ds - \int_{t_0}^{t}P(s,u(s),u'(s))u'(s)ds \le \epsilon\int_{t_0}^{t}u'(s)ds \tag{4.460}$$

Using Theorem 1.1 that is, there exist $t_0 \le \varepsilon \le t$, $\quad t_0 \le \eta \le t$, such that

$$u'(\varepsilon)\int_{t_0}^{t}(p(u(s))u'(s))''ds + u'(\eta)\int_{t_0}^{t}(\gamma(s)f(u(s))u'(s))'ds$$
$$+\int_{t_0}^{t}\beta(s)g(u(s))(u'(s))^2 ds - \int_{t_0}^{t}P(s,u(s),u'(s))u'(s)ds \le \epsilon\int_{t_0}^{t}u'(s)ds \tag{4.461}$$

By further simplification of equation (4.461) we obtain

$$u'(\varepsilon)(p(u(s))u'(s))' + u'(\eta)(\gamma(s)f(u(s))u'(s)) + \int_{t_0}^{t}\beta(s)g(u(s))(u'(s))^2 ds$$
$$-\int_{t_0}^{t}P(s,u(s),u'(s))u'(s)ds \le \epsilon\int_{t_0}^{t}u'(s)ds \tag{4.462}$$

Integrating from t_0 to t, twice , using Lemma 1.1.

$$u'(\varepsilon)\int_{t_0}^{t}p(u(s))u'(s)ds + tu'(\eta)\int_{t_0}^{t}\gamma(s)f(u(s))u'(s)ds$$
$$+\frac{t^2}{2}\int_{t_0}^{t}\beta(s)g(u(s))(u'(s))^2 ds - \frac{t^2}{2}\int_{1}^{t}P(s,u(s),u'(s))u'(s)ds \tag{4.463}$$
$$\le \frac{t^2}{2}\epsilon\int_{t_0}^{t}u'(s)ds$$

Multiplying through by $\frac{2}{t^2}$ for $t \ge 1$ and by using equation (4.421) in inequality (4.463) we have we obtain

$$\frac{2}{t^2}u'(\varepsilon)\int_{t_0}^{t}\frac{d}{ds}\mathbb{P}(u(s))ds + \frac{2}{t}u'(\eta)u'(t)\int_{t_0}^{t}\gamma(s)f(u(s))ds$$
$$+(u'(t))^2\int_{t_0}^{t}\beta(s)g(u(s))ds - \int_{t_0}^{t}P(s,u(s),u'(s))u'(s)ds \le \epsilon L \tag{4.464}$$

Using the hypothesis in the Theorem 4.39 and taking absolute value

$$2|u'(\varepsilon)||\mathbb{P}(u(s))| \le L\epsilon + 2|u'(\eta)||u'(t)|\int_{t_0}^{t}\gamma(s)f(|u(s)|)ds$$
$$+|(u'(t))|^2\int_{t_0}^{t}\beta(s)g(|u(s)|)ds + \int_{t_0}^{t}|P(s,u(s),u'(s))||u'(s)|ds \tag{4.465}$$

Setting $|P(t,u(t),u'(t)) \leq h(t)r(|u(t)|)|u'(t)|^n$ where $h(t),\ r \in C(\mathbf{R}_+)$ and $n \in \mathbf{N}$, inequality (4.465) becomes

$$2|u'(\varepsilon)||\mathbb{P}(u(s))| \leq L\epsilon + 2|u'(\eta)||u'(t)|\int_{t_0}^{t}\gamma(s)f(|u(s)|)ds$$
$$+|(u'(t))|^2\int_{t_0}^{t}\beta(s)g(|u(s)|)ds + |u'(t)|^{n+1}\int_{t_0}^{t}h(s)r(|u(s)|)ds \tag{4.466}$$

Dividing both sides by $2|u'(\varepsilon)|$, and letting $|u'(t)| \leq \lambda$ we obtain

$$|\mathbb{P}(u(s))| \leq \frac{L\epsilon}{2|u'(\varepsilon)|} + \frac{|u'(\eta)||u'(t)|}{|u'(\varepsilon)|}\int_{t_0}^{t}\gamma(s)f(|u(s)|)ds$$
$$+\frac{|(u'(t))|^2}{2|u'(\varepsilon)|}\int_{t_0}^{t}\beta(s)g(|u(s)|)ds + \frac{|u'(t)|^{n+1}}{2|u'(\varepsilon)|}\int_{t_0}^{t}h(s)r(|u(s)|)ds \tag{4.467}$$

Let $|u'(t)| \leq \lambda$, $C_1 = \dfrac{L\epsilon}{2|u'(\varepsilon)|}$ $C_2 = \dfrac{|u'(\eta)|\lambda}{|u'(\varepsilon)|}$ $C_3 = \dfrac{\lambda^2}{2|u'(\varepsilon)|}$, $C_4 = \dfrac{\lambda^{n+1}}{2|u'(\varepsilon)|}$, using these in equation (4.468) to obtain

$$|\mathbb{P}(u(s))| \leq C_1 + C_2\int_{t_0}^{t}\gamma(s)f(|u(s)|)ds + C_3\int_{t_0}^{t}\beta(s)g(|u(s)|)ds$$
$$+C_4\int_{t_0}^{t}h(s)r(|u(s)|)ds \tag{4.468}$$

Setting $|\mathbb{P}(u(s))| \geq |u(t)|$, then, applying Theorem 3.16, we have

$$\frac{|u(t)|}{C_1} \leq \Upsilon^{-1}\left[\Upsilon(1) + C_4\int_{t_0}^{t}h(s)r\left[\Omega^{-1}\left(\Omega(1) + C_3\int_{t_0}^{s}\beta(\alpha)g\left(\mathbb{H}(\alpha)d\alpha\right)\right)\right.\right.$$
$$\left.\left.\Omega^{-1}\left(\Omega(1) + C_2\int_{t_0}^{s}\gamma(\alpha)d\alpha\right)\right]ds\right]\Omega^{-1}\left(\Omega(1) + C_3\int_{t_0}^{t}\beta(s)g\left(\mathbb{H}(s)\right)ds\right)\mathbb{H}(t) \tag{4.469}$$

where

$$\mathbb{H}(t)) = F^{-1}\left(F(1) + C_2\int_{t_0}^{t}\gamma(\delta)d\delta\right)$$

By simplifying further using (i-iii) of the Theorem 4.39, we obtain

$$\frac{|u(t)|}{C_1} \leq \Upsilon^{-1}\left[\Upsilon(1) + C_4qr\left[\Omega^{-1}\left(\Omega(1) + C_3mg\left(\mathbb{H}^*\right)\right)\right.\right.$$
$$\left.\left.\Omega^{-1}\left(\Omega(1) + C_2n\right)\right]\right]\Omega^{-1}\left(\Omega(1) + C_3mg\left(\mathbb{H}^*\right)\right)\mathbb{H}^* \tag{4.470}$$

Here

$$\mathbb{H}^* = F^{-1}\left(F(1) + C_2n\right),$$

and $\mathbb{H}^*$ a positive constant. we write

$$|u(t)| \leq C_1\Upsilon^{-1}\left[\Upsilon(1) + C_4qr\left[\Omega^{-1}\left(\Omega(1) + C_3mg\left(\mathbb{H}^*\right)\right)\right.\right.$$
$$\left.\left.\Omega^{-1}\left(\Omega(1) + C_2n\right)\right]\right]\Omega^{-1}\left(\Omega(1) + C_3mg\left(\mathbb{H}^*\right)\right)\mathbb{H}^* \tag{4.471}$$

Substituting for C_1

$$|u(t)| \leq \epsilon\frac{L}{2|u'(\varepsilon)|}\Upsilon^{-1}\left[\Upsilon(1) + C_4qr\left[\Omega^{-1}\left(\Omega(1) + C_3mg\left(\mathbb{H}^*\right)\right)\right.\right.$$
$$\left.\left.\Omega^{-1}\left(\Omega(1) + C_2n\right)\right]\right]\Omega^{-1}\left(\Omega(1) + C_3mg\left(\mathbb{H}^*\right)\right)\mathbb{H}^* \tag{4.472}$$

Hence,

$$|u(t) - u_0(t)| \leq |u(t)| \leq K\epsilon.$$

By analysis

$$K = \frac{L}{2|u'(\varepsilon)|}\Upsilon^{-1}\left[\Upsilon(1) + C_4 qr\left[\Omega^{-1}\left(\Omega(1) + C_3 mg\left(\mathbb{H}^*\right)\right)\right.\right.$$
$$\left.\left.\Omega^{-1}\left(\Omega(1) + C_2 n\right)\right]\right]\Omega^{-1}\left(\Omega(1) + C_3 mg\left(\mathbb{H}^*\right)\right)\mathbb{H}^*$$

4.3.2 Hyers-Ulam Stability of Nonlinear Third Order Differential Equation with Forcing Term

Here we examine third order nonlinear DE with forcing term. The first equation to be investigated is given as:

$$u'''(t) + f(t, u(t), u'(t))u'(t) + g(u(t)) = P(t, u(t), u'(t)), \tag{4.473}$$

with initial conditions $u(t_0) = u'(t_0) = u''(t_0) = 0$ where $f, P, g \in C(\mathbf{R}_+)$,

Definition 4.20:

Given equation (4.473), Hyers-Ulam stability property is undermentioned, if the constant $K > 0$. For every $\epsilon > 0$, and $u(t) \in C^3(\mathbf{I}, \mathbf{R}_+)$, is any solution of

$$|u'''(t) + f(t, u(t), u'(t))u'(t) + g(u(t)) - P(t, u(t), u'(t))| \leq \epsilon, \tag{4.474}$$

if there exit $u_0(t) \in C^3(\mathbf{R}_+, \mathbf{R}_+)$ which is any solution satisfying (4.473) such that that

$|u(t) - u_0(t)| \leq K\epsilon.$

Note that K is Hyers-Ulam constant.

Theorem 4.43:

The equation (4.478) is H-U stable with H-U constant

$$K = LM^*N^*, \tag{4.475}$$

where

$$M^* = \Omega^{-1}\left[\Omega(1) + \lambda^{n+1} m\varpi\left(F^{-1}\left(F(1) + h(\lambda)\lambda^2 n\right)\right)\right] \tag{4.476}$$

and

$$N^* = (F^{-1}\left(F(1) + \lambda^2 h(\lambda)\right) \tag{4.477}$$

Proof:

We begin from inequality (4.474), multiplied by $u'(t)$, to get

$$\begin{aligned} u'(t)\epsilon \leq u'''(t)u'(t) + f(t, u(t), u'(t))(u'(t))^2 + g(u(t))u'(t) \\ -P(t, u(t), u'(t))u'(t) \leq u'(t)\epsilon. \end{aligned} \tag{4.478}$$

If $u''(t)$ a nondecreasing in t then, $u'''(t) \geq 0$, integrating thrice and making use of Lemma 1.1, multiplying through by $\frac{2}{t^2}$, for $t > 0$, using equation (4.96) to obtain

$$G(u(t)) \leq \int_{t_0}^{t} f(s,u(s),u'(s))(u'(s))^2 ds - \int_{t_0}^{t} P(s,u(s),u'(s))u'(s)ds \leq \int_{t_0}^{t} u'(s)\epsilon ds. \tag{4.479}$$

Taking the absolute value of both sides,

setting $|f(t,u(t),u'(t))| \leq \phi(t)f(|u(t)|)h(|u'(t)|)$,

$P(t,u(t),u'(t)| \leq \alpha(t)\varpi(|u(t)|)(|u'(t)|)^n$; for $n \in \mathbf{N}$,

where $h(t) \in (\mathbf{R}_+$, $|G(u(t))| \geq |u(t)|$, $\int_{t_0}^{t} |u'(s)|ds \leq L$, $|u'(t)| \leq \lambda$, where $\lambda \geq 0$, we get

$$\frac{|u(t)|}{\epsilon L} \leq 1 + h(\lambda)\lambda^2 \int_{t_0}^{t} \phi(s)f(\frac{|u(s)|}{\epsilon L})ds + \lambda^{n+1}\int_{t_0}^{t} \alpha(s)\omega(\frac{|u(s)|}{\epsilon L}ds.$$

By applying Theorem 3.16 we get

$$|u(t)| \leq \epsilon L M(t)N(t). \tag{4.480}$$

Here we define

$$M(t) = \Omega^{-1}\left[\Omega(1) + \lambda^{n+1}\int_{t_0}^{t} \alpha(s)\varpi\left(N(s)\right)ds\right] \tag{4.481}$$

and

$$N(t) = F^{-1}\left(F(1) + h(\lambda)\lambda^2\int_{t_0}^{t} \phi(s)ds.\right) \tag{4.482}$$

Let $\lim_{t_0\to\infty} \int_{t_0}^{t} \phi(s)ds = n < \infty$, $\lim\limits_{t_0\to\infty} \int_{t_0}^{t} \alpha(s)ds = m < \infty$,

where $n, m > 0$, using the (4.481) and (4.482) we have

$$|u(t)| \leq \epsilon L M^* N^*, \tag{4.483}$$

where M^* and N^* are defined (4.476) and (4.477) respectively. We conclude that

$|u(t) - u_0(t)| \leq |u(t)| \leq K\epsilon$,

Hence,

$$K = LM^*N^* \tag{4.484}$$

The next equation to be examined is given as:

$$u'''(t) + Q(t,u(t))u'(t) + g(u(t)) = P(t,u(t),u'(t)), \tag{4.485}$$

with initial conditions $u(t_0) = u'(t_0) = u''(t_0) = 0$ is closely investigated.

Definition 4.21:

If $u(t) \in C^3(\mathbf{I}, \mathbf{R}_+)$, is any solution of inequality

$$u'''(t) + Q(t,u(t))u'(t) + g(u(t)) - P(t,u(t),u'(t))| \leq \epsilon, \tag{4.486}$$

then, if $\exists$ any solution $u_0(t) \in C^3(\mathbf{R}_+, \mathbf{R}_+)$ of equation (4.486) such that

$|u(t) - u_0(t)| \leq K\epsilon$.

for K a positive H-U constant.

Theorem 4.44:

Equation (4.485) is Hyers-Ulam stable with Hyers-Ulam constant is given as $K = LE^*H^*$,

Here,

$$E^* = \Omega^{-1}\left[\Omega(1) + m\lambda^{2(n+1)}\varpi\left(H^*\right)\right] \tag{4.487}$$

and

$$H^* = F^{-1}\left(F(1) + n\lambda^2\right) \tag{4.488}$$

Proof:

From inequality (4.486), it is clear that

$$-\epsilon \le u'''(t) + Q(t,u(t))u'(t) + g(u(t)) - P(t,u(t),u'(t))| \le \epsilon, \tag{4.489}$$

Then, multiplying through by $u'(t)$ to have

$$-\epsilon \le u'(t)u'''(t) + Q(t,u(t))u'(t)u'(t) + g(u(t))u'(t) - u'(t)P(t,u(t),u'(t))| \le u'(t)\epsilon, \tag{4.490}$$

By applying equation (4.90),let function $u''(t)$ be a nondecreasing in t, then $u'''(t) \ge 0$, we get

$$\int_{t_0}^{t} Q(s,u(s))(u'(s))^2 ds + G(u(t)) - \int_{t_0}^{t} P(s,u(s),u'(s))u'(s)ds \le \int_{t_0}^{t} u'(s)\epsilon ds. \tag{4.491}$$

Integrating from t_0 to t thrice and applying Lemma 1.1 and taking the absolute value, we obtain

$$|G(u(t)| \le \epsilon \int_{t_0}^{t} |u'(s)|ds + \int_{t_0}^{t} |Q(s,u(s))||(u'(s))^2|ds + \int_{t_0}^{t} |P(s,u(s),u'(s))||u'(s)|ds. \tag{4.492}$$

Suppose $|\mathbb{G}(u(t))| \ge |u(t)|,\ |P(t,u(t),u'(t))| \le \alpha(t)\varpi(|u(t)|)|u'(t)|^{(2n+1)}$ and

$; |Q(t,u(t))| \le \phi(t)f(|u(t)|)$

$$|u(t)| \le \epsilon \int_{t_0}^{t} |u'(s)|ds + \int_{t_0}^{t} \phi(s)f(|u(s)|)|(u'(s))|^2 ds + \int_{t_0}^{t} \alpha(t)\varpi(|u(s)|)|u'(s)|^{2(n+1)}ds, \tag{4.493}$$

Setting $\int_{t_0}^{t} |u(s)|ds \le L,\ |u(t)| \le \lambda$ where $L, \lambda > 0$ gives

$$|u(t)| \le \epsilon L + \lambda^2 \int_{t_0}^{t} \phi(s)f(|u(s)|)ds + \lambda^{2(n+1)} \int_{t_0}^{t} \alpha(t)\omega(|u(s)|)ds, \tag{4.494}$$

$$\frac{|u(t)|}{\epsilon L} \leq 1+\lambda^2\int_{t_0}^{t}\phi(s)f(\frac{|u(s)|}{\epsilon L})ds+\lambda^{2(n+1)}\int_{t_0}^{t}\alpha(t)\omega(\frac{|u(s)|}{\epsilon L})ds. \tag{4.495}$$

By making use of Theorem 3.16 gives

$$|u(t)| \leq \epsilon LE(t)H(t). \tag{4.496}$$

where

$$E(t)=\Omega^{-1}\left[\Omega(1)+\lambda^{2(n+1)}\int_{t_0}^{t}\alpha(s)\omega\left(H(s)\right)ds\right] \tag{4.497}$$

and

$$H(t)=(F^{-1}\left(F(1)+\lambda^2\int_{t_0}^{t}\phi(\delta)d\delta\right). \tag{4.498}$$

This gives

$$|u(t)| \leq \epsilon LE^*H^*. \tag{4.499}$$

Note that E^* and H^* are defined in (4.487) and (4.488) respectively. Hence,

$K = LE^*H^*$

Therefore,

$|u(t) - u_0(t)| \leq |u(t)| \leq K\epsilon$

The next equation to be considered is given as:

$$u'''(t)+\beta(t)f(u(t))u'(t)+g(u(t))=P(t,u(t)) \tag{4.500}$$

with initial conditions $u(t_0) = u'(t_0) = u''(t_0) = 0$

Definition 4.22:

Differential equation (4.440) is Hyers-Ulam stable if

$u(t) \in C^3(\mathbf{I}, \mathbf{R}_+)$, is any solution of

$$|u'''(t)+\beta(t)r(u(t))u'(t)+g(u(t))-P(t,u(t))| \leq \epsilon, \tag{4.501}$$

then there exists $u_0(t) \in C^3(\mathbf{R}_+, \mathbf{R}_+)$ which is any solution satisfying (4.501) such that

$|u(t) - u_0(t)| \leq K\epsilon.$

for $K > 0$, is called Hyers-Ulam constant.

Theorem 4.45:

Suppose $\beta(t) \in C(\mathbf{R}_+)$, $f,\ g \in \Psi$. The nonlinear differential equation (4.501) has Hyers-Ulam stability property with Hyers-Ulam constant

$K = LW^*Z^*.$

where we define

$$W^*=\Omega^{-1}\left(\Omega(1)+B\lambda\omega\left(Z^*\right)\right) \tag{4.502}$$

and

$$Z^*=F^{-1}\left(F(1)+\lambda^2A\right) \tag{4.503}$$

Proof:

Evaluating inequality (4.501) we get

$$-\epsilon u'(t) \le u'''(t)u'(t) + \beta(t)f(u(t))(u'(t))^2 + g(u(t))u'(t) \\ -P(t,u(t))u'(t) \le \epsilon u'(t). \tag{4.504}$$

Integrating from t_0 to t thrice using Lemma 1.1, if $u''(t)$ be an increasing continuous function on $\mathbf{R}_+$ then $u'''(t) \ge 0$

$$\int_{t_0}^{t} \beta(s)f(u(s))(u'(s))^2 ds + \int_{t_0}^{t} g(u(s))u'(s)ds \\ - \int_{t_0}^{t} P(s,u(s))u'(s)ds \le \int_{t_0}^{t} \epsilon u'(s)ds \tag{4.505}$$

Using the equation(4.90), letting $|P(t,u(t))| \le \phi(t)\varpi(|u(t)|$, we get

$$|\mathbb{G}(u(t))| \le \epsilon \int_{t_0}^{t} |u'(s)|ds + (|u'(t)|)^2 \int_{t_0}^{t} \beta(s)f(|u(s)|)ds \\ +|u'(t)| \int_{t_0}^{t} \phi(s)\varpi(u(s))ds. \tag{4.506}$$

Setting $|\mathbb{G}(u(t))| \ge |u(t)|$, $|u'(t)| \le \lambda$, where $\lambda > 0$ we have

$$|u(t)| \le L\epsilon + \lambda^2 \int_{t_0}^{t} \beta(s)f(|u(s)|)ds + \lambda \int_{t_0}^{t} \phi(s)\varpi(u(s))ds. \tag{4.507}$$

Let $L\epsilon = Y$ we obtain

$$\frac{|u(t)|}{Y} \le 1 + \lambda^2 \int_{t_0}^{t} \beta(s)f\left(\frac{|u(s)|}{Y}\right)ds + \lambda \int_{t_0}^{t} \phi(s)\varpi\left(\frac{u(s)}{Y}\right)ds. \tag{4.508}$$

Applying Theorem 3.16 we arrive

$|u(t)| \le YMN(t)$

$$W(t) = \Omega^{-1}\left(\Omega(1) + \lambda \int_{t_0}^{t} \phi(s)\omega(Z(s))ds\right) \tag{4.509}$$

and

$$Z(t) = (F^{-1}\left(F(1) + \lambda^2 \int_{t_0}^{t} \beta(\delta)d\delta\right) \tag{4.510}$$

Suppose $\lim_{t\to\infty} \int_{t_0}^{t} \phi(s)ds = B < \infty$, $\lim_{t\to\infty} \int_{t_0}^{t} \beta(s)ds = A < \infty$ where A, $B > 0$ when these are used in (4.508), (4.509)and (4.510), to have

$|u(t)| \le YW^*Z^*$

Replacing Y to get

$|u(t) - u(t_0)| \le |u(t)| \le \epsilon LW^*Z^*$

Hence,

$|u(t) - u(t_0)| \le K\epsilon$

where

$$K = LW^*Z^* \tag{4.511}$$

4.3.3 Hyers-Ulam Stability of Nonperturbed Nonlinear Third Order Differential Equation

In this segment, we consider H-U stability of nonlinear differential equation whose

$P(t, u(t)) = 0$ and $P(t, u(t), u'(t)) = 0$.

Firstly, we examine equation

$$u'''(t) + \beta(t)f(u(t))u'(t) + g(u(t)) = 0, \tag{4.512}$$

with initial conditions $u(t_0) = u'(t_0) = u''(t_0) = 0$

Our result is presented in the following theorem:

Theorem 4.46:

Equation (4.512) has Hyers-Ulam stability property with Hyers-Ulam constant

$$K = LZ^*, \tag{4.513}$$

where

$$\int_{t_0}^{\infty} |u'(s)|ds \leq L, \quad L > 0$$

and

$$Z^* = F^{-1}\left(F(1) + \lambda^2 a\right)$$

Proof:

If there exist a solution $u(t) \in C^3(\mathbf{R}_+)$ satisfied inequality

$$|u'''(t) + \beta(t)f(u(t))u'(t) + g(u(t))| \leq \epsilon, \tag{4.514}$$

$\ni$

$$-\epsilon \leq u'''(t) + \beta(t)f(u(t))u'(t) + g(u(t)) \leq \epsilon, \tag{4.515}$$

Multiplying inequality (4.515) by $u'(t)$, we obtain

$$-\epsilon u'(t) \leq u'''(t)u'(t) + \beta(t)f(u(t))(u'(t))^2 + g(u(t))u'(t) \leq \epsilon u'(t). \tag{4.516}$$

Integrating from t_0 to t thrice, using Lemma 1.1, let $u''(t)$ be nondecreasing continuous function, then $u'''(t) \geq 0$ and applying equation (4.36) yields

$$\mathbb{G}(u(t)) \leq \epsilon \int_{t_0}^{t} u'(s)ds - \int_{t_0}^{t} \beta(s)f(u(s))(u'(s))^2 ds. \tag{4.517}$$

Let $|\mathbb{G}(u(t))| \geq |u(t)|$,$|u'(t)| \leq \lambda$, $\lambda > 0$ and by hypothesis of the Theorem 4.46 we obtain

$$\frac{|u(t)|}{Y} \leq 1 + \lambda^2 \int_{t_0}^{t} \beta(s)f\left(\frac{|u(s)|}{Y}\right) ds \tag{4.518}$$

for $L\epsilon = Y$

Applying Lemma 2.1 to inequality (4.518), we get

$$\frac{|u(t)|}{Y} \leq F^{-1}\left(F(1) + \lambda^2 \int_{t_0}^{t} \beta(s)ds\right) \tag{4.519}$$

By further evaluation of (4.519)

$$|u(t)| \leq YZ(t), \tag{4.520}$$

where $Z(t)$ is defined as

$$Z(t) = F^{-1}\left(F(1) + \lambda^2 \int_{t_0}^{t} \beta(s)ds\right) \tag{4.521}$$

Let $\lim_{t\to\infty} \displaystyle\int_{t_0}^{t} \beta(s)ds = a < \infty$ where $a > 0$, use all in (4.521), we obtain

$$|u(t)| \leq YZ, \tag{4.522}$$

for

$$Z^* = F^{-1}\left(F(1) + \lambda^2 a\right)$$

Replacing Y, in (4.522) we obtain

$$|u(t) - u(t_0)| \leq |u(t)| \leq \epsilon L Z^* \tag{4.523}$$

Hence, we arrive at

$$|u(t) - u(t_0)| \leq K\epsilon$$

where

$$K = LZ^* \tag{4.524}$$

The next interested equation to be considered is

$$u'''(t) + \beta(t)f(u(t))u''(t) + \alpha(t)g(u(t))u'(t) + \rho(t)\gamma(u(t)) = 0. \tag{4.525}$$

with initial conditions $u(t_0) = u'(t_0) = u''(t_0) = 0$. **Definition 4.23:**.

Equation (4.525) is stable in the sense of Hyers-Ulam, if there exists $K > 0,\ \ \epsilon > 0$ and $u(t) \in C^3(\mathbf{I}, \mathbf{R}_+)$ satisfying

$$|u'''(t) + \beta(t)f(u(t))u''(t) + \alpha(t)g(u(t))u'(t) + \rho(t)\gamma(u(t))| \leq \epsilon \tag{4.526}$$

whenever the solution $u_0(t) \in C^3(\mathbf{I}, \mathbf{R}_+)$ of the equation (4.525) satisfies

$$|u(t) - u_0(t)| \leq K\epsilon$$

where K is called H-U constant.

Theorem 4.47:

The nonlinear differential equation (4.525) has Hyers-Ulam stability property with Hyers-Ulam constant is given as

$$K = \frac{(L + \psi L)}{\mu\delta}\Omega^{-1}\left(\Omega(1) + \frac{\lambda}{\mu\delta}m\gamma\left(F^{-1}\left(F(1) + \frac{\lambda^2}{\mu\delta}n\right)\right)\right) F^{-1}\left(F(1) + \frac{\lambda^2}{\mu\delta}n\right), \tag{4.527}$$

Proof:

Simplifying (4.516), integrating from t_0 to t thrice, using Lemma 1.1, we get

$$\begin{aligned} t^2\int_{t_0}^t u'''(s)u'(s)ds + t^2\int_{t_0}^t \beta(s)u''(s)f(u(s))u'(s)ds \\ +t^2\int_{t_0}^t \alpha(s)g(u(s))(u'(s))^2ds + t^2\int_{t_0}^t \rho(s)\gamma(u(s))u'(s)ds \le \epsilon t^2\int_{t_0}^t u'(s)ds. \end{aligned} \tag{4.528}$$

Multiplying through by $\frac{2}{t^2}$, applying the equation (4.183) and by Theorem 1.1 there exist $\xi, \rho \in [t_0, t]$ such that

$$\begin{aligned} u'''(\xi)\int_{t_0}^t u'(s)ds + u''(\rho)\int_{t_0}^t \beta(s)\frac{d}{ds}\mathbb{F}(u(s))ds \\ +\int_{t_0}^t \alpha(s)g(u(s))(u'(s))^2ds\int_{t_0}^t \rho(s)\gamma(u(s))u'(s)ds \le \epsilon\int_{t_0}^t u'(s)ds \end{aligned} \tag{4.529}$$

Integrating inequality (4.529) by part, since $\beta(t)$ is a nondecreasing function, then $\beta'(t) \le 0$, we get

$$\begin{aligned} u'''(\xi)\int_{t_0}^t u'(s)ds + u''(\rho)\beta(t)\mathbb{F}(u(t)) \\ +(u'(t))^2\int_{t_0}^t \alpha(s)g(u(s))ds + u'(t)\int_{t_0}^t \rho(s)\gamma(u(s))ds \le \epsilon\int_{t_0}^t u'(s)ds \end{aligned} \tag{4.530}$$

Taking the absolute value of both sides, let $|u'(t)| \le \lambda, \quad |u'''(\xi)| \le \psi, \beta(t) \ge \delta$ $u''(\rho) \le \mu$ where $\psi,\ \lambda,\ \delta,\ \mu > 0$, and $|\mathbb{F}(u(t))| \ge |u(t)|$

$$\frac{|u(t)|}{N} \le 1 + \frac{\lambda^2}{\mu\delta}\int_{t_0}^t \alpha(s)g\left(\frac{|u(s)|}{N}\right)ds + \frac{\lambda}{\mu\delta}\int_{t_0}^t \rho(s)\gamma\left(\frac{|u(s)|}{N}\right)ds,$$

where $N = \epsilon\frac{(L+\psi L)}{\mu\delta}$ By applying Theorem 3.16, we get

$$\begin{aligned} |u(t)| \le N\Omega^{-1}\left(\Omega(1) + \frac{\lambda}{\mu\delta}\int_{t_0}^t \rho(s)\gamma\left(F^{-1}\left(F(1) + \frac{\lambda^2}{\mu\delta}\int_{t_0}^t \alpha(\beta)d\beta\right)\right)ds\right) \\ F^{-1}\left(F(1) + \frac{\lambda^2}{\mu\delta}\int_{t_0}^t \alpha(s)\right) \end{aligned} \tag{4.531}$$

The inequality (4.531) becomes

$$\begin{aligned} |u(t)| \le N\Omega^{-1}\left(\Omega(1) + \frac{\lambda}{\mu\delta}m\gamma\left(F^{-1}\left(F(1) + \frac{\lambda^2}{\mu\delta}n\right)\right)\right) \\ F^{-1}\left(F(1) + \frac{\lambda^2}{\mu\delta}n\right), \end{aligned} \tag{4.532}$$

Provided $\lim_{t\to\infty}\int_{t_0}^t \alpha(s)ds = n \le \infty$, $\lim_{t\to\infty}\int_{t_0}^t \rho(s)ds = m \le \infty$, where $n, m > 0$. Hence, replacing N, we obtain

$$\begin{aligned} |u(t)| \le \epsilon\frac{(L+\psi L)}{\mu\delta}\Omega^{-1}\left(\Omega(1) + \frac{\lambda}{\mu\delta}m\gamma\left(F^{-1}\left(F(1) + \frac{\lambda^2}{\mu\delta}n\right)\right)\right) \\ F^{-1}\left(F(1) + \frac{\lambda^2}{\mu\delta}n\right), \end{aligned} \tag{4.533}$$

Therefore,

$|u(t) - u(t_0)| \le |u(t_) \le \epsilon K,$

where

$$K=\frac{(L+\psi L)}{\mu\delta}\Omega^{-1}\left(\Omega(1)+\frac{\lambda}{\mu\delta}m\gamma\left(F^{-1}\left(F(1)+\frac{\lambda^2}{\mu\delta}n\right)\right)\right)$$
$$F^{-1}\left(F(1)+\frac{\lambda^2}{\mu\delta}n\right),$$

We consider equation

$$(\alpha(t)p(u(t))u'(t))''+(\gamma(t)f(u(t))u'(t))'+\beta(t)g(u(t))u'(t)+\rho(u(t))=0 \qquad (4.534)$$

with initial conditions $u(t_0)=u'(t_0)=u''(t_0)$, where $\beta,\gamma,\alpha\in C(\mathbf{I},\mathbf{R}_+)$, $g,p,f,\rho\in C(\mathbf{R}_+)$, for $\mathbf{I}=(1,\infty)$ $\mathbf{R}_+=[0,\infty]$ $\mathbf{R}=(-\infty,\infty)$, to be our next result.

Definition 4.24:

Equation (4.534) is stable if $K\geq 0$, $\epsilon>0$ and solution $u(t)\in C^3(\mathbf{I},\mathbf{R}_+)$ satisfying

$$|(\alpha(t)p(u(t))u'(t))''+(\gamma(t)f(u(t))u'(t))'+\beta(t)g(u(t))u'(t)+\rho(u(t))|\leq\epsilon \qquad (4.535)$$

whenever the $u_0(t)\in C^3(\mathbf{I},\mathbf{R}_+)$ which is solution of (4.534) so that

$|u(t)-u_0(t)|\leq K\epsilon$

where K is called Hyers-Ulam constant. Then, equation (4.534) is Hyers-Ulam stable.

Theorem 4.48:

Let $\alpha(t),\gamma(t),\beta(t)\in C(\mathbf{R}_+$ and $\alpha(t)\geq\phi$, $\alpha'(t)\leq 0$ with $f,g\in\Psi$. Suppose that

(i) $\lim_{t\to\infty}\displaystyle\int_{t_0}^{t}\beta(s)ds=m<\infty,\ m>0.$

(ii) $\lim_{t\to\infty}\displaystyle\int_{t_0}^{t}\gamma(s)ds=n<\infty,\ n>0.$

then equation (4.534) is stable in the sense of H-U stability and H-U constant is given as

$$K=\frac{L+|\rho(u(\varphi))|L}{2\phi|u'(\varepsilon)|}\Omega^{-1}\left(\Omega(1)+C_3mg\left(F^{-1}\left(F(1)+C_2n\right)\right)\right)$$
$$F^{-1}\left(F(1)+C_2n\right),$$

define $C_2=\dfrac{2|u'(\eta)|\lambda}{2\phi|u'(\varepsilon)|}$, $\quad C_3=\dfrac{\lambda^2}{2\phi|u'(\varepsilon)|}$,

Proof:

Evaluating inequality (4.535) to have

$$-\epsilon|(\alpha(t)p(u(t))u'(t))''+(\gamma(t)f(u(t))u'(t))'+\beta(t)g(u(t))u'(t)+\rho(u(t))|\leq\epsilon \quad (4.536)$$

Multiplying through by $u'(t)$, to obtain

$$\begin{aligned}&(\alpha(t)p(u(t))u'(t))''u'(t)+(\gamma(t)f(u(t))u'(t))'u'(t)\\&\quad+\beta(t)g(u(t))(u'(t))^2+\rho(u(t))u'(t)\leq\epsilon u'(t).\end{aligned} \qquad (4.537)$$

Integrating Inequality (4.537) from t_0 to t and by Theorem 1.1, there exists $\varepsilon,\ \eta,\ \mu \in [t_0, t]$ such that

$$\begin{aligned} &u'(\varepsilon)\int_{t_0}^{t}(\alpha(s)p(u(s))u'(s))''ds + u'(\eta)\int_{t_0}^{t}(\gamma(s)f(u(s))u'(s))'ds \\ &+(u'(\mu))^2\int_{t_0}^{t}\beta(s)g(u(s))ds + \int_{t_0}^{t}\rho(u(s))u'(s)ds \le \epsilon\int_{t_0}^{t}u'(s)ds \end{aligned} \tag{4.538}$$

Integrating twice from t_0 to t and applying Lemma 1.1 and multiplying through by $\frac{2}{t^2}$, for $t \geq 1$ we get

$$\begin{aligned} &\frac{2}{t^2}u'(\varepsilon)\int_{1}^{t}\alpha(s)p(u(s))u'(s)ds + \frac{2}{t}u'(\eta)\int_{1}^{t}\gamma(s)f(u(s))u'(s)ds \\ &+(u'(\mu))^2\int_{1}^{t}\beta(s)g(u(s))ds + \int_{t_0}^{t}\rho(u(s))u'(s)ds \le \epsilon\int_{t_0}^{t}u'(s)ds \end{aligned} \tag{4.539}$$

Using Theorem 1.1, $\exists\ t_0 \le \varphi \le t$ such that

$$\begin{aligned} &\frac{2}{t^2}u'(\varepsilon)\int_{t_0}^{t}\alpha(s)p(u(s))u'(s)ds + \frac{2}{t}u'(\eta)\int_{t_0}^{t}\gamma(s)f(u(s))u'(s)ds \\ &+(u'(\mu))^2\int_{t_0}^{t}\beta(s)g(u(s))ds + \rho(u(\varphi))\int_{t_0}^{t}u'(s)ds \le \epsilon\int_{t_0}^{t}u'(s)ds \end{aligned} \tag{4.540}$$

Using equation (4.421) in inequality (4.540), integrating by part and recall $\alpha(t) \geq \phi,\ \alpha'(t) \geq 0$ and taking the absolute value, we obtain

$$\begin{aligned} 2\phi|u'(\varepsilon)||\mathbb{P}(u(s))| \le \epsilon(L + |\rho(u(\varphi))|L) + 2|u'(\eta)||u'(t)|\int_{1}^{t}\gamma(s)f(|u(s)|)ds \\ +|(u'(t))|^2\int_{1}^{t}\beta(s)g(|u(s)|)ds \end{aligned} \tag{4.541}$$

Dividing inequality (4.541) by $2\phi|u'(\varepsilon)| > 0$,,

$$\begin{aligned} \mathbb{P}(u(s))(u(s))| \le \frac{\epsilon(L + |\rho(u(\varphi))|L)}{2\phi|u'(\varepsilon)|} + \frac{2|u'(\eta)||u'(t)|}{2\phi|u'(\varepsilon)|}\int_{t_0}^{t}\gamma(s)f(|u(s)|)ds \\ +\frac{|(u'(t))|^2}{2\phi|u'(\varepsilon)|}\int_{t_0}^{t}\beta(s)g(|u(s)|)ds \end{aligned} \tag{4.542}$$

Let $|u'(t)| \le \lambda, \quad C_1 = \frac{\epsilon(L + |\rho(u(\varphi))|L)}{2\phi|u'(\varepsilon)|} \quad C_2 = \frac{2|u'(\eta)|\lambda}{2\phi|u'(\varepsilon)|} \quad C_3 = \frac{\lambda^2}{2\phi|u'(\varepsilon)|}$, and setting $\mathbb{P}(u(s))(u(t)| \geq |u(t)|$ and dividing through by C_1, since f, g belong to the class Ψ,

$$\frac{|u(s)|}{C_1} \le 1 + C_2\int_{t_0}^{t}\gamma(s)f(\frac{|u(s)|}{C_1})ds + C_3\int_{t_0}^{t}\beta(s)g(\frac{|u(t)|}{C_1})ds \tag{4.543}$$

By applying Theorem 3.16, one obtains

$$\begin{aligned} \frac{|u(t)|}{C_1} \le \Omega^{-1}\left(\Omega(1) + C_3\int_{t_0}^{t}\beta(s)g\left(F^{-1}\left(F(1) + C_2\int_{t_0}^{s}\gamma(\delta)d\delta\right)\right)ds\right) \\ F^{-1}\left(F(1) + C_2\int_{t_0}^{t}\gamma(s)ds\right) \end{aligned} \tag{4.544}$$

Using conditions (i) and (ii), we have

$$\frac{|u(t)|}{C_1} \le \Omega^{-1}\left(\Omega(1) + C_3mg\left(F^{-1}\left(F(1) + C_2n\right)\right)\right)F^{-1}\left(F(1) + C_2n\right) \tag{4.545}$$

we conclude that

$$|u(t)-u_0(t)| \leq |u(t)| \leq C_1\Omega^{-1}\left(\Omega(1)+C_3mg\left(F^{-1}\left(F(1)+C_2n\right)\right)\right. \\ \left. F^{-1}\left(F(1)+C_2n\right)\right) \tag{4.546}$$

Replacing C_1, we have H-U constant given as

$$K=\frac{L+|\rho(u(\varphi))|L}{2\phi|u'(\varepsilon)|}\Omega^{-1}\left(\Omega(1)+C_3mg\left(F^{-1}\left(F(1)\right.\right.\right. \\ \left.\left.\left. +C_2n\right)\right)\right)F^{-1}\left(F(1)+C_2n\right)$$

4.4 Hyers-Ulam-Rassias Stability of Third Order Nonlinear Differential Equation with Forcing Term

4.4.1 Introduction

In this unit, Hyers-Ulam-Rassias stability of nonlinear third order ordinary DE with and without forcing term of different type of nonlinear equations are examined using the previous tools. Furthermore, we obtain the Hyers-Ulam-Rassias constant for each of equations considered.

4.4.2 Hyers-Ulam-Rassias Stability of Third Order Ordinary Differential Equation with Forcing Term

In this unit, the first equation to be considered is given as:

$$u'''(t)+f(t,u(t),u'(t))u''(t)+\alpha(t,u(t))u'(t) \\ +\beta(t)a(u(t))=P(t,u(t),u'(t)), \tag{4.547}$$

with initial conditions $u(t_0)=u'(t_0)=u''(t_0)=0$, where

$P\in C(\mathbf{I}\times\mathbf{R}_+{}^2,\mathbf{R}_+)$, $\alpha\in C(\mathbf{I}\times\mathbf{R}_+,\mathbf{R}_+)$, $\beta\in C(\mathbf{I},\mathbf{R}_+)$.

Definition 4.24:

Let $u(t)\in C^3(\mathbf{I},\mathbf{R}_+)$ be any solution of

$$|u'''(t)+f(t,u(t)u'(t))u''(t)+\alpha(t,u(t))u'(t)+\beta(t)a(u(t)) \\ -P(t,u(t),u'(t))|\leq\varphi(t), \tag{4.548}$$

then, equation (4.547) is Hyers-Ulam-Rassias stable, if in addition function φ is a nondecreasing, nonnegative and continuous on $\mathbf{R}_+$, and there exists any solution $u_0(t)\in C^3(\mathbf{R}_+)$ of equation (4.547) such that

$|u(t)-u_0(t)|\leq C_\varphi\varphi(t)$

C_φ a Hyers-Ulam-Rassias constant.

Theorem 4.49:

Suppose

$|P(t,u(t),u'(t))|\leq\kappa(t)\rho(|u(t)|)\varrho(|u'(t)|)$,

$|\alpha(t,u(t))|\leq\phi(t)\varpi(|u(t)|)$, where $\kappa(t), \phi(t)$ are all nonnegative functions on $C(\mathbf{R}_+)$ and the functions ϖ, ρ, ϱ are nonnegative, monotonic, nondecreasing. Let ρ, ϖ belong to class of ψ and $\varphi : \mathbf{I} \to [0, \infty)$, be an increasing positive function, equation (4.547) is stable in the sense of Hyers-Ulam-Rassias stability with Hyers-Ulam-Rassias constant

$$C_\varphi = \frac{1}{\delta}\Omega^{-1}\left(\Omega(1) + n_1\frac{1}{\delta}\varrho(\lambda)\lambda\left(F^{-1}\left(F(1) + \frac{n_2}{\delta}|u'(\eta)|^2\right)\right)\right) \\ F^{-1}\left(F(1) + \frac{n_2}{\delta}|u'(\eta)|^2\right) \tag{4.549}$$

Proof:

Simplifying inequality (4.548) to obtain

$$-\epsilon \le u'''(t) + f(t, u(t)u'(t))u''(t) + \alpha(t, u(t))u'(t) + \beta(t)a(u(t)) \\ -P(t, u(t), u'(t)) \le \varphi(t),$$

Multiplying by $u'(t)$ to have

$$u'(t)\varphi(t) \le u'(t)u'''(t) + f(t, u(t)u'(t))u''(t)u'(t) + \alpha(t, u(t))u'(t)u'(t) \\ +\beta(t)a(u(t))u'(t) - P(t, u(t), u'(t))u'(t) \le u'(t)\varphi(t). \tag{4.550}$$

Integrating thrice, using Lemma 1.1 and Theorem 1.1, there exist $\xi, \in [t_0, t]$ such that

$$u'(\xi)u''(t) + \int_{t_0}^{t} f(s, u(s), u'(s))u''(s)u'(s)ds + (u'(\eta))^2\int_{t_0}^{t}\alpha(s, u(s))ds \\ + \int_{t_0}^{t}\beta(s)a(u(s))u'(s)ds - \int_{t_0}^{t}P(s, u(s), u'(s))u'(s)ds \le \int_{t_0}^{t}u'(s)\varphi(s)ds. \tag{4.551}$$

Put

$$A(u(t)) = \int_{u_0}^{u(t)} a(s)ds. \tag{4.552}$$

Applying equation (4.552) in inequality (4.551) to have

$$u'(\xi)u''(t) + \int_{t_0}^{t} f(s, u(s), u'(s))u''(s)u'(s)ds + (u'(\eta))^2\int_{t_0}^{t}\alpha(s, u(s))ds \\ + \int_{t_0}^{t}\beta(s)\frac{d}{ds}A(u(s))ds - \int_{t_0}^{t}P(s, u(s), u'(s))u'(s)ds \le \int_{t_0}^{t}u'(s)\varphi(s)ds. \tag{4.553}$$

By integration by part we get

$$u'(\xi)u''(t) + \int_{t_0}^{t} f(s, u(s), u'(s))u''(s)u'(s)ds + (u'(\eta))^2\int_{t_0}^{t}\alpha(s, u(s))ds \\ +\beta(t)A(u(t)) - \int_{t_0}^{t}\beta'(s)A(u(s))ds - \int_{t_0}^{t}P(s, u(s), u'(s))u'(s)ds \le \\ \int_{t_0}^{t}u'(s)\varphi(s)ds. \tag{4.554}$$

Suppose $u'(t),\ \beta(t)$ are nondecreasing functions on $\mathbf{R}_+$, then $u''(t),\ \beta(t) \ge 0$, there exist $\delta,\ \lambda > 0$ such that $|u'(t)| \ge \lambda,\ \beta(t) \ge \delta$, taking the absolute value, it is clear that

$$\delta|A(u(t))| \leq \lambda \int_{t_0}^{t} \varphi(s)ds + |u'(\eta)|^2 \int_{t_0}^{t} |\alpha(s,u(s)|ds + \int_{t_0}^{t} |P(s,u(s),u'(s))||u'(s)|ds \tag{4.555}$$

Setting $|A(u(t))| \geq |u(t)|$ to have

$$|u(t)| \leq \frac{1}{\delta}\lambda \int_{t_0}^{t} \varphi(s)ds + \frac{1}{\delta}|u'(\eta)|^2 \int_{t_0}^{t} \phi(s)\varpi(|u(t)|)ds + \frac{1}{\delta}\varrho(\lambda)\lambda \int_{t_0}^{t} \kappa(s)\rho(|u(s)|)ds \tag{4.556}$$

By Corollary 3.1, the result equals

$$|u(t)| \leq \frac{1}{\delta}\lambda \int_{t_0}^{t} \varphi(s)ds\Omega^{-1}\left(\Omega(1) + \frac{1}{\delta}\varrho(\lambda)\lambda \int_{t_0}^{t} \kappa(s)\rho\left(F^{-1}\left(F(1) + \frac{1}{\delta}|u'(\eta)|^2 \int_{t_0}^{s} \phi(\gamma)d\gamma\right)\right)ds\right) F^{-1}\left(F(1) + \frac{1}{\delta}|u'(\eta)|^2 \int_{t_0}^{t} \phi(s)ds\right) \tag{4.557}$$

Let the $\lim_{t\to\infty} \int_{t_0}^{t} \kappa(s)ds = n_1 < \infty$, where $n_1 > 0$

$\lim_{t\to\infty} \int_{t_0}^{t} \kappa(s)ds = n_2 < \infty$, where $n_2 > 0$, enables one to get

$$|u(t)| \leq \varphi(t)\frac{1}{\delta}\Omega^{-1}\left(\Omega(1) + n_1\frac{1}{\delta}\varrho(\lambda)\lambda\left(F^{-1}\left(F(1) + \frac{n_2}{\delta}|u'(\eta)|^2\right)\right)\right) F^{-1}\left(F(1) + \frac{n_2}{\delta}|u'(\eta)|^2\right) \tag{4.558}$$

where

$$\lambda \int_{t_0}^{t} \varphi(s)ds \leq \varphi(t)$$

and constant

$$C_\varphi = \frac{1}{\delta}\Omega^{-1}\left(\Omega(1) + n_1\frac{1}{\delta}\varrho(\lambda)\lambda\left(F^{-1}\left(F(1) + \frac{n_2}{\delta}|u'(\eta)|^2\right)\right)\right) F^{-1}\left(F(1) + \frac{n_2}{\delta}|u'(\eta)|^2\right)$$

Hence,

$|u(t) - u_0(t)| \leq |u(t)| \leq C_\varphi\varphi(t).$

Now, we consider next equation given as

$$u'''(t) + f(t,u(t),u'(t))u''(t) + B\varpi(u(t))u'(t) + \beta(t)a(u(t)) = P(t,u(t),u'(t)) \tag{4.559}$$

with initial conditions $u(t_0) = u'(t_0) = u''(t_0) = 0$, where B is a positive constant.

Theorem 4.50:

Suppose

$|P(t,u(t),u'(t))| \leq \kappa(t)\rho(|u(t)|)\varrho(|u'(t)|),$

where $\kappa(t) \in C(\mathbf{R}_+)$. If constant $B \geq 0$ and other functions remain valid in Theorem (4.49). Then equation

$$u'''(t) + f(t,u(t),u'(t))u''(t) + B\varpi(u(t))u'(t) + \beta(t)a(u(t)) = P(t,u(t),u'(t))$$

has Hyers-Ulam-Rassias stability and Hyers-Ulam-Rassias constant is given as

$$C_\varphi = \frac{1}{\delta}\left(|1 + B\frac{\varpi(|u(\delta)|)}{\delta}L\right)\Omega^{-1}\left(\Omega(1) + \frac{1}{\delta}\frac{\varrho(\lambda)\lambda}{\delta}n_1\right). \tag{4.560}$$

Proof:

From (4.548), multiplying through by $u'(t)$ and integrating thrice, using Lemma 1.1 and Theorem 1.1 there exists $\xi, \delta \in [t_0, t]$, we get

$$\begin{aligned} u(\xi)u''(s) + \int_{t_0}^{t} f(s, u(s), u'(s))u''(s)u'(s)ds + u'(t)\varpi(u(\delta))\int_{t_0}^{t} u'(s)ds \\ + \int_{t_0}^{t} \beta(s)a(u(s))u'(s)ds - \int_{t_0}^{t} P(s, u(s), u'(s))u'(s)ds \leq u'(t)\int_{t_0}^{t}\varphi(s)ds, \end{aligned} \tag{4.561}$$

Use the equation (4.172) and other conditions use to prove Theorem 4.49 we have

$$\begin{aligned} |A(u(t))| \leq \frac{1}{\delta}|u'(t)|\int_{t_0}^{t}\varphi(s)ds + \frac{1}{\delta}B|u'(t)|\varpi(|u(\delta)|)\int_{t_0}^{t}|u'(s)|ds \\ + \frac{1}{\delta}\int_{t_0}^{t}|P(s, u(s), u'(s))||u'(s)|ds \end{aligned} \tag{4.562}$$

Setting $|A(u(t))| \geq |u(t)|,\ |u'(t)| \leq \lambda,\ \int_{t_0}^{t}|u'(s)|ds \leq L,$ for $L, \lambda > 0$ and

$$\begin{aligned} |u(t)| \leq \frac{1}{\delta}\left(1 + \frac{1}{\delta}B\varpi(|u(\delta)|)L\right)\lambda\int_{t_0}^{t}\varphi(s)ds \\ + \frac{1}{\delta}\varrho(\lambda)\lambda\int_{t_0}^{t}\kappa(s)\rho(|u(s)|)ds \end{aligned} \tag{4.563}$$

Using Theorem 2.9 on inequality (4.563) to get

$$|u(t)| \leq \frac{1}{\delta}\left(1 + B\frac{\varpi(|u(\delta)|)}{\delta}L\right)\Omega^{-1}\left(\Omega(1) + \frac{1}{\delta}\frac{\varrho(\lambda)\lambda}{\delta}n_1\right)\varphi(t) \tag{4.564}$$

Provided $\lim_{t\to\infty}\int_{t_0}^{t}\kappa(s)ds = n_1 < \infty$, where $n_1 > 0$ and

$$\lambda\int_{t_0}^{t}\varphi(s)ds \leq \varphi(t)$$

and the Hyers-Ulam-Rasssias constant

$$C_\varphi = \frac{1}{\delta}\left(1 + B\frac{\varpi(|u(\delta)|)}{\delta}L\right)\Omega^{-1}\left(\Omega(1) + \frac{1}{\delta}\frac{\varrho(\lambda)\lambda}{\delta}n_1\right)$$

Hence,

$$|u(t) - u_0(t)| \leq |u(t)| \leq C_\varphi\varphi t$$

This ends the proof.

In equation (4.547) if $P(t, u(t), u'(t)) = 0$, then, the following theorem is stated as

Theorem 4.51:

Given

$$u'''(t) + f(t, u(t), u'(t))u''(t) + \alpha(t, u(t))u'(t) + \beta(t)a(u(t)) = 0, \tag{4.565}$$

equation (4.565) has Hyers-Ulam-Rassias stability and Hyers-Ulam-Rassias constant is given as

$$C_\varphi = \frac{1}{\delta}\Omega^{-1}\left(\Omega(1) + \frac{1}{\delta}\lambda^2 r_1\right). \tag{4.566}$$

Take

$|\alpha(t,u(t))| \leq \phi(t)\varpi(|u(t)|)$,

where $\phi(t) \in C(\mathbf{R}_+)$ together with function ϖ a nonnegative, monotonic, nondecreasing.

Proof:

From definition of Hyers-Ulam-Rassias stability given above, it is clear that

$$-u'(t)\varphi(t) \leq u'''(t)u'(t) + f(t,u(t),u'(t))u''(t)u'(t) + \alpha(t,u(t)(u'(t))^2 + \beta(t)a(u(t))u'(t) \leq u'(t)\varphi(t) \tag{4.567}$$

Integrating (4.567) thrice, by Lemma 1.1 and by Theorem 1.1,

if there exists $\xi,\eta \in [t_0,t]$ such that

$$u(\xi)u''(t) + \int_{t_0}^{t} f(s,u(s),u'(s))u''(s)u'(s)ds + \int_{t_0}^{t} \alpha(s,u(s)(u'(s))^2 ds + \int_{t_0}^{t} \beta(s)a(u(s))u'(s)ds \leq u'(\eta)\int_{t_0}^{t}\varphi(s)ds. \tag{4.568}$$

Using the conditions and the steps in the proof of Theorem 4.49, we get

$$|A(u(t))| \leq \frac{1}{\delta}|u'(t)|\int_{t_0}^{t}\varphi(s)ds + \frac{1}{\delta}|u'(t)|^2\int_{t_0}^{t}\phi(s)\varpi(|u(s)|)ds \tag{4.569}$$

Let $(Au(t))| \geq |u(t)|$, $|u'(t)| \leq \lambda$ we have

$$|u(t)| \leq \frac{1}{\delta}\lambda\int_{t_0}^{t}\varphi(s)ds + \frac{\lambda^2}{\delta}\int_{t_0}^{t}\phi(s)\varpi(|u(s)|)ds \tag{4.570}$$

By Theorem 2.9, we obtain

$$|u(t)| \leq \frac{1}{\delta}\lambda\int_{t_0}^{t}\varphi(s)ds\Omega^{-1}\left(\Omega(1) + \frac{\lambda^2}{\delta}\int_{t_0}^{t}\phi(s)ds\right) \tag{4.571}$$

Let the $\lim_{t\to\infty}\int_{t_0}^{t}\phi(s)ds = r_1 < \infty$, then

$$|u(t)| \leq \frac{1}{\delta}\varphi(t)\Omega^{-1}\left(\Omega(1) + \frac{\lambda^2}{\delta}r_1\right) \tag{4.572}$$

where

$\lambda\int_{t_0}^{t}\varphi(s)ds \leq \varphi(t)$,

and the H-U-R constant is given as

$$C_\varphi = \frac{1}{\delta}\Omega^{-1}\left(\Omega(1) + \frac{\lambda^2}{\delta}r_1\right) \tag{4.573}$$

Hence,

$|u(t) - u_0(t)| \leq |u(t)| \leq C_\varphi\varphi(t)$

4.4.3 Hyers-Ulam-Rassias Stability of Nonhomogeneous and Homogenuous Nonlinear Third Order Ordinary Differential Equation

Different equations of nonhomogeneous and homogeneous nonlinear third order ordinary differential equations are going to be studied in this section. One of the equations is written as

$$u'''(t) + f(t, u(t), u'(t))u'(t) + \gamma(t)D(u(t)) = P(t, u(t), u'(t)) \tag{4.574}$$

with initial conditions $u(t_0) = u'(t_0) = u''(t_0) = 0$.

Definition 4.25:

Nonlinear equation (4.574) is Hyers-Ulam-Rassias stable, if there exits a positive constant C_φ called Hyers-Ulam-Rassias constant. For every continuous function φ which is nonnegative, nondecreasing and $u(t) \in C^3(\mathbf{I}, \mathbf{R}_+)$ is any solution of

$$|u'''(t) + f(t, u(t), u'(t))u'(t) + \gamma(t)D(u(t)) - P(t, u(t), u'(t))| \leq \varphi(t), \tag{4.575}$$

and $u_0(t) \in C^3(\mathbf{R}_+)$ is any solution of equation (4.574) that makes

$|u(t) - u_0(t)| \leq C_\varphi \varphi(t)$.

Theorem 4.52:

Let

$|f(t, u(t), u'(t))| \leq \phi(t)g(|u(t)|)h(|u'(t)|)$,

$|P(t, u(t), u'(t))| \leq \alpha(t)\omega(|u(t)||u'(t)|^n$,

where $\phi(t), \alpha(t)$ are nonnegative functions on $C(\mathbf{R}_+)$ and the functions g, h, ω are nonnegative, monotonic, nondecreasing. Furthermore, let function $\varphi(t)$ be defined as in Theorem 4.50, equation (4.574) is Hyers-Ulam-Rassias stable and Hyers-Ulam-Rassias constant is given as

$$\begin{aligned} C_\varphi = \Omega^{-1}\left(\Omega(1) + d_1\lambda^{n+1}\omega\left(F^{-1}\left(F(1) + d_2h(\lambda)\lambda^2\right)\right)\right) \\ F^{-1}\left(F(1) + d_2h(\lambda)\lambda^2\right) \end{aligned} \tag{4.576}$$

Proof:

From equation (4.575) we deduce that

$$\begin{aligned} -\varphi(t) \leq u'''(t) + f(t, u(t), u'(t))u'(t) + \gamma(t)D(u(t)) - P(t, u(t), u'(t)) \\ \leq \varphi(t), \end{aligned} \tag{4.577}$$

Multiplying equation (4.577) by $u'(t)$ we get

$$\begin{aligned} -u'(t)\varphi(t) \leq u'(t)u'''(t) + f(t, u(t), u'(t))(u'(t))^2 + u'(t)\gamma(t)D(u(t)) \\ -u'(t)P(t, u(t), u'(t)) \leq u'(t)\varphi(t). \end{aligned}$$

Integrating thrice using Lemma 1.1

$$\begin{aligned} (u'(t))^2 \int_{t_0}^t f(s, u(s), u'(s))ds + u'(t)\int_{t_0}^t \gamma(s)D(u(s))u'(s)ds \\ -u'(t)\int_{t_0}^t P(s, u(s), u'(s))ds \leq u'(t)\int_{t_0}^t \varphi(s)ds, \end{aligned} \tag{4.578}$$

Note: if $u'(t)$ a nondecreasing function on $C(\mathbf{R}_+)$, it is certain for $u''(t) \geq 0$, with advantage of this, we have

$$\mathbb{D}(u(t)) = \int_{u(t_0)}^{u(t)} D(s)ds, \tag{4.579}$$

together with $\gamma(t)$ being a nondecreasing function, then $\gamma'(t) \geq 0$ and taking the absolute value we get

$$\begin{aligned} |u'(t)||\gamma(t)||B(u(t))| \leq |u'(t)| \int_{t_0}^{t} \varphi(s)ds \\ +|(u'(t))|^2 \int_{t_0}^{t} |f(s,u(s),u'(s))|ds + |u'(t)| \int_{t_0}^{t} |P(s,u(s),u'(s))|ds, \end{aligned} \tag{4.580}$$

Setting $|u'(t)| \leq \lambda$ where $\lambda > 0$, $\gamma(t) \geq \eta$, $\eta > 0$ and using the hypothesis in the theorem and let $|u'(t)||\gamma(t)||B(u(t))| \geq |u(t)|$, to give

$$\begin{aligned} |u(t)| \leq \lambda \int_{t_0}^{t} \varphi(s)ds + h(\lambda)\lambda^2 \int_{t_0}^{t} \phi(s)g(|u(s)|)ds \\ +\lambda^{n+1} \int_{t_0}^{t} \alpha(s)\omega(|u(s)|ds, \end{aligned} \tag{4.581}$$

Using Corollary 3.1, $\lim_{t\to\infty} \int_{t_0}^{t} \phi(s)ds = d_1 < \infty$, $d_1 > 0$

$\lim_{t\to\infty} \int_{t_0}^{t} \kappa(s)ds = d_2 < \infty$, $d_2 > 0$, then

$$\begin{aligned} |u(t)| \leq \varphi(t)\lambda^2\Omega^{-1}\left(\Omega(1) + d_1\lambda^{n+1}\omega\left(F^{-1}\left(F(1) + d_2h(\lambda)\lambda^2\right.\right.\right. \\ \left.\left.\left.\right)\right)\right) F^{-1}\left(F(1) + d_2h(\lambda)\lambda^2\right) \end{aligned} \tag{4.582}$$

Provided $\lambda \frac{1}{t} \int_{t_0}^{t} \varphi(s)ds \leq \varphi(t)$

Hence,

$|u(t) - u_0(t)| \leq |u(t)| \leq C_\varphi \varphi(t)$

$C_\varphi = \Omega^{-1}\left(\Omega(1) + d_1\lambda^{n+1}\omega\left(F^{-1}\left(F(1) + d_2h(\lambda)\lambda^2\right)\right)\right) F^{-1}\left(F(1) + d_2h(\lambda)\lambda^2\right)$

The consequence of Theorem 4.52 is given as:

Theorem 4.53:

Let all the conditions of Theorem 4.52 remain valid. If

$|f(t,u(t),u'(t))| = |P(t,u(t),u'(t))| \leq \alpha(t)\omega(|u(t)||u'(t)|^n$.

Then, equation(4.574) has Hyers-Ulam-Rassiaa stability with Hyers-Ulam-Rassias constant

$$C_\varphi = \frac{1}{\eta}\Omega^{-1}\left(\Omega(1) + d_3\frac{(\lambda^2+\lambda)\lambda^n}{\eta}\right) \tag{4.583}$$

Proof

Follow the steps of proof of Theorem 4.51 to inequality (4.579), to get

$$\begin{aligned} |u'(t)||\gamma(t)||B(u(t))| \leq |u'(t)| \int_{t_0}^{t} \varphi(s)ds \\ +(|(u'(t))|^2 + |u'(t)|) \int_{t_0}^{t} |f(s,u(s),u'(s))|ds \end{aligned} \tag{4.584}$$

Using hypothesis giving in the Theorem 4.53 to obtain

$$|u'(t)||\gamma(t)||B(u(t))| \le |u'(t)| \int_{t_0}^{t} \varphi(s)ds$$
$$+(|(u'(t))|^2 + |u'(t)|) \int_{t_0}^{t} \alpha(s)\omega(|u(s)||u'(s)|^n ds. \tag{4.585}$$

By simplifying inequality (4.585) further to get

$$|u'(t)||\gamma(t)||B(u(t))| \le |u'(t)| \int_{t_0}^{t} \varphi(s)ds$$
$$+(|(u'(t))|^2 + |u'(t)|)|u'(t)|^n \int_{t_0}^{t} \alpha(s)\omega(|u(s)|ds \tag{4.586}$$

Setting $|u'(t)| \le \lambda$ where $\lambda > 0$, $|\gamma(t)| \ge \eta$, $\eta > 0$, using the hypothesis of the Theorem 4.53 and taking $|u'(t)||B(u(t))| \ge |u(t)|$, equals to

$$|u(t)| \le \frac{1}{\eta}\lambda \int_{t_0}^{t} \varphi(s)ds + \frac{(\lambda^2+\lambda)\lambda^n}{\eta} \int_{t_0}^{t} \alpha(s)\omega(|u(s)|ds \tag{4.587}$$

Applying the Theorem 2.9, let the $\lim_{t\to\infty} \int_{t_0}^{t} \alpha(s)ds = d_3 < \infty$, where $d_3 > 0$, then

$$|u(t)| \le \frac{1}{\eta}\Omega^{-1}\left(\Omega(1) + d_3\frac{(\lambda^2+\lambda)\lambda^{n-1}}{\eta}\right) \quad 0 < t \le b, \tag{4.588}$$

Provided $\lambda \int_{t_0}^{t} \varphi(s)ds \le \varphi(t)$.

Hence,

$|u(t) - u_0(t)| \le |u(t)| \le C_\varphi \varphi(t)$

where

$$C_\varphi = \frac{1}{\eta}\Omega^{-1}\left(\Omega(1) + d_3\frac{(\lambda^2+\lambda)\lambda^n}{\eta}\right)$$

If $P(t, u(t), u'(t)) = 0$. then, equation (4.574) reduce to

$$u'''(t) + f(t, u(t), u'(t))u'(t) + \gamma(t)D(u(t)) = 0 \tag{4.589}$$

with initial conditions $u(t_0) = u'(t_0) = u''(t_0) = 0$.

Theorem 4.54:

Suppose

$|f(t, u(t), u'(t))| \le \phi(t)g(|u(t)|)h(|u'(t)|)$,

where $\phi(t)$ a nonnegative function on $C(\mathbf{R}_+)$ and the functions g, h are nonnegative, monotonic, nondecreasing. $\varphi(t)$ in the same way as in Theorem 4.53, equation (4.589) is Hyers-Ulam-Rassias stable and Hyers-Ulam-Rassias constant is given as

$$C_\varphi = \frac{1}{\eta}\Omega^{-1}\left(\Omega(1) + \frac{d_1\lambda^2 h(\lambda)}{\eta}\right) \tag{4.590}$$

Proof:

By simple evaluation of (4.575) with $P(t, u(t), u'(t)) = 0$ to have

$-\varphi \le u'''(t) + f(t, u(t), u'(t))u'(t) + \gamma(t)D(u(t)) \le \varphi(t)$.

Multiplying by $u'(t)$, if $u''(t)$ a nondecreasing then $u'''(t) \geq 0$, using equation (4.579), we obtain

$$\begin{aligned}(u'(t))^2 \int_{t_0}^{t} f(s,u(s),u'(s))ds + u'(t) \int_{t_0}^{t} \gamma(s)\frac{d}{ds}B(u(s))ds \\ \leq u'(t) \int_{t_0}^{t} u'(s)\varphi(s)ds\end{aligned}, \tag{4.591}$$

Integrating by part, since $\gamma(t)$ is a nondecreasing function,

then $\gamma'(t) \geq 0$, $\gamma'(t) \geq 0$, taking the absolute value and applying the hypothesis of Theorem 4.54, we obtain

$$|u(t)| \leq \frac{1}{\eta}\lambda \int_{t_0}^{t} \varphi(s)ds + \frac{\lambda^2 h(\lambda)}{\eta} \int_{t_0}^{t} \phi(s)g(|u(s)|)ds, \tag{4.592}$$

for $|B(u(t))| \geq |u(t)|$. Applying the Theorem 2.9

$$|u(t)| \leq \varphi(t)\frac{1}{\eta}\Omega^{-1}\left(\Omega(1) + \frac{d_1\lambda^2 h(\lambda)}{\eta}\right) \tag{4.593}$$

provided $\lambda \int_{t_0}^{t} \varphi(s)ds \leq \varphi(t)$

Hence,

$|u(t) - u_0(t)| \leq |u(t)| \leq C_\varphi \varphi(t)$

one defines

$$C_\varphi = \frac{1}{\eta}\Omega^{-1}\left(\Omega(1) + \frac{d_1\lambda^2 h(\lambda)}{\eta}\right). \tag{4.594}$$

4.4.4 A Perturbed Nonlinear Third Order Ordinary Differential Equation

In continuation of our discussion on Hyers-Uulam-Rassias stability. Equation

$$\begin{aligned}(r(t)\phi(u(t))u'(t))'' + f(t,u(t),u'(t))u''(t) + g(t,u(t),u'(t))u'(t) \\ +\beta(t)\delta(u(t)) = P(t,u(t),u'(t))\end{aligned} \tag{4.595}$$

with initial value

$u(t_0) = u'(t_0) = u''(t_0) = 0$

has been giving deep attention in examining its stability via Hyers-Ulam-Rassias.

Definition 4.26:

The nonlinear differential equation (4.595) has Hyers-Ulam-Rassias stability properties, if there exits $C_\varphi > 0$ called H-U-R constant. For every continuous function φ which is nonnegative, nondecreasing and $u(t) \in C^3(\mathbf{I}, \mathbf{R}_+)$ is any solution that is satisfied equation

$$\begin{aligned}|\,(r(t)\phi(u(t))u'(t))'' + f(t,u(t),u'(t))u''(t) + g(t,u(t),u'(t))u'(t) \\ +\beta(t)\delta(u(t)) - P(t,u(t),u'(t))| \leq \varphi(t)\end{aligned} \tag{4.596}$$

there exists a solution $u_0(t) \in C^3(\mathbf{R}_+)$ of equation (4.595) such that $|u(t) - u_0(t)| \le C_\varphi \varphi(t)$.

Theorem 4.55:

Suppose the following conditions

(i) $|f(t, u(t), u'(t))| \le \phi(t)\gamma(|u(t)|)h(|u'(t)|)$

(ii) $|P(t, u(t), u'(t))| \le \alpha(t)\omega(|u(t)||u'(t)|^n$

(iii)$|g(t, u(t), u'(t))| \le \kappa(t)\psi(|u(t)|)|u'(t)|$

(iv) $\int_{t_0}^{\infty} |u'(s)|ds \le L$, where $L > 0$

are satisfied, $\phi(t), \alpha(t), \kappa(t) \in C(\mathbf{R}_+)$ and the functions γ, h, ω, ψ are nonnegative, monotonic, nondecreasing. Also, define $\varphi : \mathbf{I} \to [0, \infty]$, then, equation (4.595) is Hyers-Ulam-Rassias stability with

$$C_\varphi = (\kappa(\rho)\psi(|u(\rho)|)\lambda L + 1)\Gamma^{-1}\left[\Gamma(1) + m_1\omega\left(\Omega^{-1}\left(\Omega(1) + m_3(T^*)\right)\right)T^*\right]\Omega^{-1}\left(\Omega(1) + m_3\delta(T^*)\right)T^* \tag{4.597}$$

Proof:

From equation (4.596), it is clear that

$$-\varphi(t) \le (r(t)\phi(u(t))u'(t))'' + f(t, u(t), u'(t))u''(t) + g(t, u(t), u'(t))u'(t) + \beta(t)\delta(u(t)) - P(t, u(t), u'(t)) \le \varphi(t) \tag{4.598}$$

Multiplying inequality (4.598) by $u'(t)$ to have

$$-u'(t)\varphi(t) \le (r(t)\phi(u(t))u'(t))''\, u'(t) + f(t, u(t), u'(t))u''(t)u'(t) + g(t, u(t), u'(t))(u'(t))^2 + \beta(t)\delta(u(t))u'(t) - P(t, u(t), u'(t))u'(t) \le u'(t)\varphi(t) \tag{4.599}$$

Integrating the equation(4.599) trice and using Lemma 1.1

$$u'(t)\int_{t_0}^{t}(r(s)\phi(u(s))u'(s))\,ds + u'(t)\frac{t^2}{2}\int_{t_0}^{t} f(s, u(s), u'(s))u''(s)ds + u'(t)\frac{t^2}{2}\int_{t_0}^{t} g(s, u(s), u'(s))u'(s)ds + u'(t)\frac{t^2}{2}\int_{t_0}^{t}\beta(s)\delta(u(s))ds - u'(t)\frac{t^2}{2}\int_{t_0}^{t} P(s, u(s), u'(s))ds \le u'(t)\frac{t^2}{2}\int_{t_0}^{t}\varphi(s)ds \tag{4.600}$$

Use equation defines as

$$J(u(t)) = \int_{u(t_0)}^{u(t)} \phi(u(s))ds, \tag{4.601}$$

in inequality (4.599) to have

$$u'(t)\int_{t_0}^{t}\left(r(s)\frac{d}{ds}J(u(s))\right)dsu'(t)\frac{t^2}{2}\int_{t_0}^{t} f(s, u(s), u'(s))u''(s)ds + u'(t)\frac{t^2}{2}\int_{t_0}^{t} g(s, u(s), u'(s))u'(s)ds + u'(t)\frac{t^2}{2}\int_{t_0}^{t}\beta(s)\delta(u(s))ds - u'(t)\frac{t^2}{2}\int_{t_0}^{t} P(s, u(s), u'(s))ds \le \frac{t^2}{2}\int_{t_0}^{t}\varphi(s)ds \tag{4.602}$$

Integrating by part, since $r(t)$ is a nondecreasing, nonnnegative function on $\mathbf{R}_+$, then $r'(t) \geq 0$ and there exists $\delta > 0$ such that $\dfrac{r(t)}{t^2} \geq \delta$

$$\begin{aligned}\delta u'(t)J(u(t)) \leq u'(t)\int_{t_0}^{t}\varphi(s)ds - |u'(t)|\int_{t_0}^{t} f(s,u(s),u'(s))u''(s)ds \\ -u'(t)\int_{t_0}^{t} g(s,u(s),u'(s))u'(s)ds - u'(t)\int_{t_0}^{t}\beta(s)\delta(u(s))ds \\ +u'(t)\int_{t_0}^{t} P(s,u(s),u'(s))ds\end{aligned} \tag{4.603}$$

Taking the absolute value of both sides of inequality (4.603) to have

$$\begin{aligned}\delta|u'(t)||J(u(t))| \leq |u'(t)|\int_{t_0}^{t}\varphi(s)ds - |u'(t)|\int_{t_0}^{t}|f(s,u(s),u'(s))||u''(s)|ds \\ -|u'(t)|\int_{t_0}^{t}|g(s,u(s),u'(s))||u'(s|)ds - |u'(t)|\int_{t_0}^{t}\beta(s)\delta(|u(s|))ds \\ +|u'(t)|\int_{t_0}^{t}|P(s,u(s),u'(s))|ds.\end{aligned} \tag{4.604}$$

Setting $\delta|u'(t)||J(u(t))| \geq |u(t)|$ and using Theorem 1.1,

there exist $\xi, \rho \in [t_0, t]$ such that

$$\begin{aligned}|u(t)| \leq |u'(t)|\int_{t_0}^{t}\varphi(s)ds \\ +|u'(t)|u''(\xi)|\int_{t_0}^{t}|f(s,u(s),u'(s))|ds + |u'(t)||g(\rho,u(\rho),u'(\rho))|\int_{t_0}^{t}u'(s)ds \\ +|u'(t)|\int_{t_0}^{t}\beta(s)\delta(|u(s)|)ds + |u'(t)|\int_{t_0}^{t}|P(s,u(s),u'(s))|ds.\end{aligned} \tag{4.605}$$

Using the hypothesis of the Theorem 4.55, to obtain

$$\begin{aligned}|u(t)| \leq |u'(t)|\int_{t_0}^{t}\varphi(s)ds + |u'(t)|\int_{t_0}^{t}\phi(s)\gamma(|u(s)|)h(|u'(s)|)ds \\ +|u'(t)|\kappa(\rho)\psi(|u(\rho)|)|u'(\rho)|L + |u'(t)|\int_{t_0}^{t}\beta(s)\delta(|u(s)|)ds \\ +|u'(t)|^{n+1}\int_{t_0}^{t}\alpha(s)\omega(|u(s)|)ds\end{aligned} \tag{4.606}$$

It follows that

$$\begin{aligned}|u(t)| \leq (\kappa(\rho)\psi(|u(\rho)|)|u'(\rho)|L+1)|u'(t)|\int_{t_0}^{t}\varphi(s)ds \\ +|u'(t)|h(|u'(t)|)\int_{t_0}^{t}\phi(s)\gamma(|u(s)|)ds \\ +|u'(t)|\int_{t_0}^{t}\beta(s)\delta(|u(s)|)ds + |u'(t)|^{n+1}\int_{t_0}^{t}\alpha(s)\omega(|u(s)|ds\end{aligned} \tag{4.607}$$

Setting $|u'(t)| \leq \lambda$ where $\lambda > 0$, we obtain

$$\begin{aligned}|u(t)| \leq (\kappa(\rho)\psi(|u(\rho)|)\lambda L+1)\lambda\int_{t_0}^{t}\varphi(s)ds + \lambda h(\lambda)\int_{t_0}^{t}\phi(s)\gamma(|u(s)|)ds \\ +\lambda\int_{t_0}^{t}\beta(s)\delta(|u(s)|)ds + \lambda^{n+1}\int_{t_0}^{t}\alpha(s)\omega(|u(s)|ds\end{aligned} \tag{4.608}$$

By application of the Theorem 3.17

$$|u(t)| \leq (\kappa(\rho)\psi(|u(\rho)|)\lambda L + 1)\lambda \int_{t_0}^{t} \varphi(s)ds$$
$$\Gamma^{-1}\left[\Gamma(1) + \lambda^{n+1}\int_{t_0}^{t} \alpha(s)\omega\left(\Omega^{-1}\left(\Omega(1) + \right.\right.\right. \tag{4.609}$$
$$\left.\left.\left.\lambda\int_{t_0}^{s}\beta(\eta)\delta(T(\eta))d\eta\right)\right)T(s)ds\right]\Omega^{-1}\left(\Omega(1) + \lambda\int_{t_0}^{t}\beta(s)\delta(T(s))ds\right)T(t)$$

Where $T(t)$ is defined as

$$T(t) = F^{-1}\left(F(1) + \lambda h(\lambda)\int_{t_0}^{t}\phi(s)ds\right) \tag{4.610}$$

Let (i) $\lim_{t\to\infty}\int_{t_0}^{t}\alpha(s)ds \leq m_1$ where $m_1 > 0$,

(ii) $\lim_{t\to\infty}\int_{t_0}^{t}\phi(s)ds \leq m2$ where $m_2 > 0$

(iii) $\lim_{t\to\infty}\int_{t_0}^{t}\beta(s)ds \leq m_3$ where $m_3 > 0$

(iv)$\lambda\int_{t_0}^{t}\varphi(s)ds \leq \varphi(t)$

The inequality (4.609) becomes

$$|u(t)| \leq \varphi(t)(\kappa(\rho)\psi(|u(\rho)|)\lambda L + 1)\Gamma^{-1}\left[\Gamma(1) + m_1\lambda^{n+1}\omega\left(\Omega^{-1}\left(\Omega(1) + m_3(T)\right)\right)T\right]\Omega^{-1}\left(\Omega(1) + m_3\delta(T)\right)T \tag{4.611}$$

Where T a positive constant defined as

$$T = F^{-1}\left(F(1) + m_2\lambda h(\lambda)\right) \tag{4.612}$$

Therefore,

$$C_\varphi = (\kappa(\rho)\psi(|u(\rho)|)\lambda L + 1)\Gamma^{-1}\left[\Gamma(1) + m_1\lambda^{n+1}\omega\left(\Omega^{-1}\left(\Omega(1) + m_3(T)\right)\right)T\right]\Omega^{-1}\left(\Omega(1) + m_3\delta(T)\right)T$$

Hence,

$$|u(t) - u(t_0)| \leq |u(t)| \leq C_\varphi\varphi(t)$$

Theorem 4.56:

Let all the conditions of Theorem 4.55 remained valid. Suppose $g(t, u(t), u'(t)) = 0$, so (4.598) reduce

$$(r(t)\phi(u(t))u'(t))'' + f(t, u(t), u'(t))u''(t) + \beta(t)\delta(u(t)) = P(t, u(t), u'(t)). \tag{4.613}$$

In addition, Equation (4.613) posses Hyers-Ulam-Rassias stability with Hyers-Ulam-Rassias constant

$$C_\varphi = \Gamma^{-1}\left[\Gamma(1) + m_1\omega\left(\Omega^{-1}\left(\Omega(1) + m_3(T^*)\right)\right)T^*\right]\Omega^{-1}\left(\Omega(1) + m_3\delta(T^*)\right)T^* \tag{4.614}$$

where T a positive constant defined as

$$T = F^{-1}\left(F(1) + m_2h(\lambda)\right) \tag{4.615}$$

Proof:

Evaluate inequality (4.596), integrating the result trice using Lemma 1.1,

$$u'(t)\int_{t_0}^{t}(r(s)\phi(u(s))u'(s))\,ds+u'(t)\frac{t^2}{2}\int_{t_0}^{t}f(s,u(s),u'(s))u''(s)ds$$
$$+u'(t)\frac{t^2}{2}\int_{t_0}^{t}\beta(s)\delta(u(s))ds-u'(t)\frac{t^2}{2}\int_{t_0}^{t}P(s,u(s),u'(s))ds \qquad (4.616)$$
$$\le u'(t)\frac{t^2}{2}\int_{t_0}^{t}\varphi(s)ds.$$

Integrating by part, recall $r(t)$ is a nondecreasing function, implies $r'(t)\ge 0$

$$\frac{2}{t^3}r(t)u'(t)J(u(t))\le u'(t)\int_{t_0}^{t}\varphi(s)ds-u'(t)\int_{t_0}^{t}f(s,u(s),u'(s))u''(s)ds$$
$$-u'(t)\int_{t_0}^{t}\beta(s)\delta(u(s))ds+u'(t)\int_{t_0}^{t}P(s,u(s),u'(s))ds \qquad (4.617)$$

Since $r(t)$ is nonnegative ,nondecreasing function on $\mathbf{R}_+$, there exists $\delta>0$ such that $\frac{r(t)}{t^2}$, then setting $\delta|u'(t)|J(|u(t)|)\ge|u(t)|$ and using Theorem 1.1, there exist $\xi\in[t_0't]$ such that

$$|u(t)|\le|u'(t)|\int_{t_0}^{t}\varphi(s)ds+|u'(t)||u"(\xi)|\int_{t_0}^{t}|f(s,u(s),u'(s))|ds$$
$$+|u'(t)|\int_{t_0}^{t}\beta(s)\delta(|u(s)|)ds+|u'(t)|\int_{t_0}^{t}|P(s,u(s),u'(s))|ds. \qquad (4.618)$$

Using the hypothesis of Theorem 4.56, letting $|u'(t)|\le\lambda$ where $\lambda>0$, simplified further to obtain

$$|u(t)|\le\lambda\int_{t_0}^{t}\varphi(s)ds+\lambda h(\lambda)\int_{t_0}^{t}\phi(s)\gamma(|u(s)|)ds$$
$$+\lambda\int_{t_0}^{t}\beta(s)\delta(|u(s)|)ds+\lambda^{n+1}\int_{t_0}^{t}\alpha(s)\omega(|u(s)|ds \qquad (4.619)$$

By Theorem 3.17 we have

$$|u(t)|\le\lambda\int_{t_0}^{t}\varphi(s)ds\Gamma^{-1}\Bigg[\Gamma(1)+\lambda^{n+1}\int_{t_0}^{t}\alpha(s)\omega\left(\Omega^{-1}\left(\Omega(1)+\right.\right.$$
$$\left.\left.\lambda\int_{t_0}^{s}\beta(\eta)\delta(T(\eta))d\eta\right)\right)T(s)ds\Bigg]\Omega^{-1}\left(\Omega(1)+\lambda\int_{t_0}^{t}\beta(s)\delta(T(s))ds\right)T(t) \qquad (4.620)$$

note that $T(t)$ is defined as

$$T(t)=F^{-1}\left(F(1)+\lambda h(\lambda)\int_{t_0}^{t}\phi(s)ds.\right) \qquad (4.621)$$

Let

(i) $\lim_{t\to\infty}\int_{t_0}^{t}\alpha(s)ds\le m_1,$ where $m_1>0,$

(ii) $\lim_{t\to\infty}\int_{t_0}^{t}\phi(s)ds\le m_2,$ where $m_2>0$

(iii) $\lim_{t\to\infty}\int_{t_0}^{t}\beta(s)ds\le m_3,$ where $m_3>0$

(iv) $\lambda\int_{t_0}^{t}\varphi(s)ds\le\varphi(t).$

Then, the inequality (4.619) becomes

$$|u(t)| \le \lambda\varphi(t)\Gamma^{-1}\left[\Gamma(1) + m_1\lambda^{n+1}\omega\left(\Omega^{-1}\left(\Omega(1) + m_3(T)\right)\right)T^*\right]\Omega^{-1}\left(\Omega(1) + m_3\delta(T^*)\right)T^* \tag{4.622}$$

Taking T^* to be a positive constant defined as

$$T^* = F^{-1}\left(F(1) + m_2\lambda h(\lambda)\right) \tag{4.623}$$

Therefore,

$$C_\varphi = \Gamma^{-1}\left[\Gamma(1) + m_1\lambda\omega\left(\Omega^{-1}\left(\Omega(1) + m_3\lambda(T^*)\right)\right)T^*\right]\Omega^{-1}\left(\Omega(1) + m_3\lambda\delta(T^*)\right)T^*$$

Hence,

$$|u(t) - u(t_0)| \le |u(t)| \le C_\varphi\varphi(t)$$

Theorem 4.57:

Let equation

$$(r(t)\phi(u(t))u'(t))'' + g(t, u(t), u'(t))u'(t) + \beta(t)a(u(t)) = P(t, u(t), u'(t)) \le \varphi(t) \tag{4.624}$$

be derived from (4.595) by letting $f(t, u(t), u'(t)) = 0$ is said to be Hyers-Ulam-Rassias stable and Hyers-Ulam-Rassias constant denoted by

$$C_\varphi = \Gamma^{-1}\left[\Gamma(1) + \lambda^n m_1\omega\left(\Omega^{-1}\left(\Omega(1) + m_3\delta(H^*)\right)\right)H^*\right]\Omega^{-1}\left(\Omega(1) + m_3\delta(H^*)\right)H^* \tag{4.625}$$

$S(t)$ defined as

$$H^* = F^{-1}\left(F(1) + \lambda^2 k\right) \tag{4.626}$$

Proof:

Evaluating the inequality (4.596) to get

$$-\varphi(t) \le (r(t)\phi(u(t))u'(t))'' + g(t, u(t), u'(t))u'(t) + \beta(t)\delta(u(t)) - P(t, u(t), u'(t)) \le \varphi(t) \tag{4.627}$$

Multiplying inequality (4.627) by $u'(t)$ to get

$$-u'(t)\varphi(t) \le u'(t)\left(r(t)\phi(u(t))u'(t)\right)'' + g(t, u(t), u'(t))(u'(t))^2 + \beta(t)\delta(u(t))u'(t) - P(t, u(t), u'(t))u'(t) \le u'(t)\varphi(t) \tag{4.628}$$

Integrating the inequality (4.628) trice and using Lemma 1.1

$$u'(t)\int_{t_0}^{t}\left(r(s)\frac{d}{ds}J(u(s))\right)ds + u'(t)\frac{t^2}{2}\int_{t_0}^{t} g(s, u(s), u'(s))u'(s)ds + u'(t)\frac{t^2}{2}\int_{t_0}^{t}\beta(s)\delta(u(s))ds - u'(t)\frac{t^2}{2}\int_{t_0}^{t}P(s, u(s), u'(s))ds \le u'(t)\frac{t^2}{2}\int_{t_0}^{t}\varphi(s)ds \tag{4.629}$$

Integrating by part, since $r(t)$ a nonnegative, nondecreasing function on $\mathbf{R}_+$, implies $r'(t) \geq 0$,

$$u'(t)\frac{2}{t^3}r(t)J(u(t)) \leq u'(t)\frac{1}{t}\int_{t_0}^{t}\varphi(s)ds - u'(t)\frac{1}{t}\int_{t_0}^{t}g(s,u(s),u'(s))u'(s)ds \\ -u'(t)\frac{1}{t}\int_{t_0}^{t}\beta(s)\delta(u(s))ds + u'(t)\frac{1}{t}\int_{t_0}^{t}P(s,u(s),u'(s))ds \tag{4.630}$$

Taking absolute value, there exists $\delta > 0$ such that $\frac{r(t)}{t^2} \geq \delta$, then, setting $\delta|u'(t)|J(|u(t)|) \geq |u(t)|$, and by Theorem 1.1, there exists $\rho \in [t_0, t]$ such that

$$|u(t)| \leq |u'(t)|\int_{t_0}^{t}\varphi(s)ds + |u'(t)||u'(\rho)|\int_{t_0}^{t}|g(s,u(s),u'(s))u'(s)|ds \\ +|u'(t)|\int_{t_0}^{t}\beta(s)\delta(|u(s|))ds + |u'(t)|\int_{t_0}^{t}|P(s,u(s),u'(s))|ds \tag{4.631}$$

By hypothesis in Theorem 4.56 , if $|u'(t)| \leq \lambda$ where $\lambda > 0$ we get

$$|u(t)| \leq \lambda\int_{t_0}^{t}\varphi(s)ds + \lambda^3\int_{t_0}^{t}\kappa(s)\psi(|u(s)|)ds \\ +\lambda\int_{t_0}^{t}\beta(s)\delta(|u(s)|)ds + \lambda^{n+1}\int_{t_0}^{t}\alpha(s)\omega(|u(s)|ds \tag{4.632}$$

Applying Theorem 3.17 to get

$$|u(t)| \leq \lambda\int_{t_0}^{t}\varphi(s)ds\Gamma^{-1}\left[\Gamma(1) + \lambda^{n+1}\int_{t_0}^{t}\alpha(s)\omega\left(\Omega^{-1}\left(\Omega(1)+\right.\right.\right. \\ \left.\left.\left.\lambda\int_{t_0}^{s}\beta(\eta)\delta(H(\eta))d\eta\right)\right)H(s)ds\right]\Omega^{-1}\left(\Omega(1) + \lambda\int_{t_0}^{t}\beta(s)\delta(H(s))ds\right)H(t) \tag{4.633}$$

where

$$H(t) = F^{-1}\left(F(1) + \lambda^3\int_{t_0}^{t}\kappa(s)ds\right) \tag{4.634}$$

Letting

(i) $\lim_{t\to\infty}\int_{t_0}^{t}\alpha(s)ds \leq m_1$ where $m_1 > 0$,

(ii) $\lim_{t\to\infty}\int_{t_0}^{t}\kappa ds \leq k$ where $k > 0$

(iii) $\lim_{t\to\infty}\int_{t_0}^{t}\beta(s)ds \leq m_3$ where $m_3 > 0$

(iv) $\lambda\int_{t_0}^{t}\varphi(s)ds \leq \varphi(t)$.

Applying the limits of integrals to equation (4.634) becomes

$$|u(t)| \leq \varphi(t)\Gamma^{-1}\left[\Gamma(1) + \lambda^n m_1\omega\left(\Omega^{-1}\left(\Omega(1) + m_3\delta(H^*)\right)\right)H^*\right] \\ \Omega^{-1}\left(\Omega(1) + m_3\delta(H^*)\right)H^* \tag{4.635}$$

H^* is taking as positive constant

$$H^* = F^{-1}\left(F(1) + \lambda^3 k\right) \tag{4.636}$$

Hence,

$|u(t) - u(t_0)| \leq |u(t)| \leq C_\varphi\varphi(t)$,

where
$$C_\varphi = \Gamma^{-1}\left[\Gamma(1) + \lambda^{n+1} m_1 \omega \left(\Omega^{-1}\left(\Omega(1) + m_3\lambda\delta(H^*)\right)\right) H^*\right]$$
$$\Omega^{-1}\left(\Omega(1) + m_3\lambda\delta(H^*)\right) H^*$$
Last equation to be derived from (4.595) is given in the form
$$(r(t)\phi(u(t))u'(t))'' + f(t,u(t),u'(t))u''(t) + g(t,u(t),u'(t))u'(t) + \beta(t)\delta(u(t)) = 0 \tag{4.637}$$
with initial conditions $u(t_0) = u'(t_0) = u''(t_0) = 0$, if $P(t,u(t),u'(t)) = 0$.

Theorem 4.58:

Equation(4.637) is Hyers-Ulam-Rassias stability and given a well defined Hyers-Ulam-Rassias constant as
$$C_\varphi = \Omega^{-1}\left[\Omega(1) + m_3\lambda\delta\left(\Gamma^{-1}\left(\Gamma(1) + \lambda^3 k\psi(S)\right)\right) S^*\right] \quad \Gamma^{-1}\left(\Gamma(1) + \lambda^3 m_3\psi(S^*)\right) S^* \tag{4.638}$$

Proof:

From inequality (4.596), it is clear that
$$-\varphi(t) \leq (r(t)\phi(u(t))u'(t))'' + f(t,u(t),u'(t))u''(t) + g(t,u(t),u'(t))u'(t) + \beta(t)\delta(u(t)) \leq \varphi(t) \tag{4.639}$$
Multiplying inequality (4.639) by $u'(t)$ to have
$$-u'(t)\varphi(t) \leq (r(t)\phi(u(t))u'(t))'' u'(t) + f(t,u(t),u'(t))u''(t)u'(t) + g(t,u(t),u'(t))(u'(t))^2 + \beta(t)\delta(u(t))u'(t) \leq u'(t)\varphi(t) \tag{4.640}$$
Integrating the equation(4.640) trice and using Lemma 1.1
$$u'(t)\int_{t_0}^{t}(r(s)\phi(u(s))u'(s))\,ds + u'(t)\frac{t^2}{2}\int_{t_0}^{t} f(s,u(s),u'(s))u''(s)ds + u'(t)\frac{t^2}{2}\int_{t_0}^{t} g(s,u(s),u'(s))u'(s)ds + u'(t)\frac{t^2}{2}\int_{t_0}^{t}\beta(s)\delta(u(s))ds \leq u'(t)\frac{t^2}{2}\int_{t_0}^{t}\varphi(s)ds \tag{4.641}$$
Use the equation defined in (4.601) to have
$$u'(t)\int_{t_0}^{t}\left(r(s)\frac{d}{ds}J(u(s))\right)dsu'(t)\frac{t^2}{2}\int_{t_0}^{t} f(s,u(s),u'(s))u''(s)ds + u'(t)\frac{t^2}{2}\int_{t_0}^{t} g(s,u(s),u'(s))u'(s)ds + u'(t)\frac{t^2}{2}\int_{t_0}^{t}\beta(s)\delta(u(s))dsu'(t) \leq \frac{t^2}{2}\int_{t_0}^{t}\varphi(s)ds \tag{4.642}$$
Integrating by part, since $r(t)$ is a nondecreasing, nonnnegative function on $\mathbf{R}_+$, then $r'(t) \geq 0$ and there exists $\delta > 0$ such that $\dfrac{r(t)}{t^2} \geq \delta$
$$\delta u'(t)J(u(t)) \leq u'(t)\int_{t_0}^{t}\varphi(s)ds - |u'(t)|\int_{t_0}^{t} f(s,u(s),u'(s))u''(s)ds - u'(t)\int_{t_0}^{t} g(s,u(s),u'(s))u'(s)ds - u'(t)\int_{t_0}^{t}\beta(s)\delta(u(s))ds \tag{4.643}$$

Taking absolute value of inequality (4.643)

$$\begin{aligned}\delta|u'(t)||J(u(t))| \leq |u'(t)|\int_{t_0}^{t}\varphi(s)ds - |u'(t)|\int_{t_0}^{t}|f(s,u(s),u'(s))||u''(s)|ds \\ -|u'(t)|\int_{t_0}^{t}|g(s,u(s),u'(s))||u'(s|)ds - |u'(t)|\int_{t_0}^{t}\beta(s)\delta(|u(s|))ds.\end{aligned} \tag{4.644}$$

Setting $\delta|u'(t)||J(u(t))| \geq |u(t)|$ and using Theorem 1.1,

there exist $\xi,\rho \in [t_0,t]$ such that

$$\begin{aligned}|u(t)| \leq |u'(t)|\int_{t_0}^{t}\varphi(s)ds \\ +|u'(t)|u''(\xi)|\int_{t_0}^{t}|f(s,u(s),u'(s))|ds + |u'(t)||g(\rho,u(\rho),u'(\rho))|\int_{t_0}^{t}u'(s)ds \\ +|u'(t)|\int_{t_0}^{t}\beta(s)\delta(|u(s)|)ds\end{aligned} \tag{4.645}$$

Using the hypothesis of the Theorem 4.55, to obtain

$$\begin{aligned}|u(t)| \leq |u'(t)|\int_{t_0}^{t}\varphi(s)ds + |u'(t)|\int_{t_0}^{t}\phi(s)\gamma(|u(s)|)h(|u'(s)|)ds \\ +|u'(t)|\kappa(\rho)\psi(|u(\rho)|)|u'(\rho)|L + |u'(t)|\int_{t_0}^{t}\beta(s)\delta(|u(s)|)ds.\end{aligned} \tag{4.646}$$

Setting $|u'(t)| \leq \lambda$ where $\lambda > 0$, to have

$$\begin{aligned}|u(t)| \leq \lambda\int_{t_0}^{t}\varphi(s)ds + \lambda^2 h(\lambda)\int_{t_0}^{t}\phi(s)\gamma(|u(s)|)ds \\ +\lambda^3\int_{t_0}^{t}(\kappa(s)\psi(|u(s)|)ds\end{aligned} \tag{4.647}$$

By application of the Theorem 3.17 we have

$$\begin{aligned}|u(t)| \leq \lambda\int_{t_0}^{t}\varphi(s)ds\Omega^{-1}\left[\Omega(1) + \lambda\int_{t_0}^{t}\beta(s)\delta\left(\Gamma^{-1}\left(\Gamma(1)\right.\right.\right. \\ \left.\left.\left.+\lambda^3\int_{t_0}^{s}\kappa(\eta)\psi(S(\eta))d\eta\right)\right)S(s)ds\right]\Omega^{-1}\left(\Omega(1) + \lambda^3\int_{t_0}^{t}\kappa(s)\psi(S(s))ds\right)S(t)\end{aligned} \tag{4.648}$$

Let

$$S(t) = F^{-1}\left(F(1) + \lambda h(\lambda)\int_{t_0}^{t}\phi(s)ds\right) \tag{4.649}$$

Assuming the following limits are ascertained

(i) $\lim_{t\to\infty}\int_{t_0}^{t}\kappa(s)ds \leq k$ where $k > 0$,

(ii) $\lim_{t\to\infty}\int_{t_0}^{t}\phi(s)ds \leq m2$ where $m_2 > 0$

(iii) $\lim_{t\to\infty}\int_{t_0}^{t}\beta(s)ds \leq m_3$ where $m_3 > 0$

(iv) $\lambda\int_{t_0}^{t}\varphi(s)ds \leq \varphi(t)$.

Inequality (4.648) reduce

$$|u(t)| \leq \varphi(t)\Omega^{-1}\left[\Omega(1) + m_3\lambda\delta\left(\Gamma^{-1}\left(\Gamma(1) + \lambda^3 k\psi(S^*)\right)\right)S\right] \\ \Gamma^{-1}\left(\Gamma(1) + \lambda^3 m_3\psi(S^*)\right)S^* \tag{4.650}$$

where S^* a positive constant defined as

$$S^* = F^{-1}\left(F(1) + m_2\lambda\lambda h(\lambda)\right) \tag{4.651}$$

Therefore,

$$\Omega^{-1}\left[\Omega(1) + m_3\lambda\delta\left(\Gamma^{-1}\left(\Gamma(1) + \lambda^3 k\psi(S^*)\right)\right)S^*\right]\Gamma^{-1}\left(\Gamma(1) + \lambda^2 m_3\lambda\psi(S^*)\right)S^*$$

Hence,

$$|u(t) - u(t_0)| \leq |u(t)| \leq C_\varphi\varphi(t)$$

4.4.5 Hyers-Ulam-Rassias Stability of Nonlinear Third Order Damped Ordinary Differential Equation with Forcing Term

The stability of nonlinear third order damped equations with forcing term are considered and their Hyers-Ulam-Rassias constants are obtained.

$$(a(t)\psi(u(t))u'(t))'' + r(t)u''(t) + p(t)u'(t) + q(t)f(u(t)) = P(t, u(t), u'(t)). \tag{4.652}$$

$$(a(t)\psi(u(t))u'(t))'' + p(t)u'(t) + q(t)f(u(t)) = P(t, u(t), u'(t)). \tag{4.653}$$

$$(a(t)u'(t))'' + r(t)u''(t) + p(t)u'(t) + q(t)f(x(t)) = P(t, u(t), u'(t)). \tag{4.654}$$

$$(a(t)u'(t))'' + p(t)u'(t) + q(t)f(u(t)) = P(t, u(t), u'(t)). \tag{4.655}$$

with initial conditions $u(t_0) = u'(t_0) = u''(t_0) = 0$, where $t \in \mathbf{I} = [t_0, b)(b \leq \infty)$, $a, r, p, q \in C(\mathbf{I}, \mathbf{R})$, $f, \psi \in (\mathbf{R}, \mathbf{R})$, $P \in C(\mathbf{I} \times \mathbf{R}^2, \mathbf{R})$, $\mathbf{R} = (-\infty, \infty)$ and $\mathbf{R}_+ = [0, \infty)$. The definitions of aforementioned equations are given below.

Definition 4.27:

Equation (4.652) is Hyers-Ulam-Rassias stable, if there exists $u(t) \in C^3(\mathbf{R})$ any solution satisfies inequality

$$|(a(t)\psi(u(t))u'(t))'' + r(t)u''(t) + p(t)u'(t) + q(t)f(u(t)) - P(t, u(t), u'(t))| \\ \leq \varphi(t) \tag{4.656}$$

where a positive function defined as $\varphi : \mathbf{I} \to \mathbf{R}_+$ and there exists solution $u_0(t) \in C^3(\mathbf{R}_+)$ of equation (4.652) such that

$$|u(t) - u_0(t)| \leq C_\varphi\varphi(t)$$

holds for C_φ is taking as Hyers-Ulam-Rassias constant.

Definition 4.28:

Let solution $u(t) \in C^3(\mathbf{I}, \mathbf{R})$ satisfying

$$|(a(t)u'(t))'' + r(t)u''(t) + p(t)u'(t) + q(t)f(u(t)) - P(t, u(t), u'(t))| \leq \varphi(t), \tag{4.657}$$

also taking $u_0(t) \in C^3(\mathbf{I}, \mathbf{R})$ to be any of solution of (4.654) which satisfies

$|u(t) - u_0(t)| \leq C_\varphi \varphi(t)$

defined $\varphi : \mathbf{I} \to \mathbf{R}_+$ and C_φ denotes H-U-R constant. Therefore equation (4.654) is Hyer-Ulam-Rassias stability.

Firstly, we consider equation (4.652) as follows:

Theorem 4.59:

The equation (4.652) together with its initial conditions is Hyers-Ulam-Rassias stable provided under-listed are obeyed.

(i) Let $\int_{t_0}^{\infty} |u'(s)|ds \leq L$, $\int_{t_0}^{\infty} u''(s)ds \leq N$ for $L, N > 0$.

(ii) If it happens that $\phi(t) \in C(\mathbf{I}, \mathbf{R}_+)$

so that $|P(t, u(t), u'(t))| \leq \phi(t)\varpi(|u(t)|)(|u'(t)|)^n$ where $n \in \mathbf{N}$

(iii)$\lim_{t\to\infty} \int_{t_0}^{t} q(s)ds = k_1 < \infty$ and $\lim_{t\to\infty} \int_{t_0}^{t} \phi(s)ds = k_2 < \infty$, where $k_1, k_2 > 0$.

(iv) Assume $\lambda > 0$ implies $|u'(t)| \leq \lambda$, and $\lambda \int_{t_0}^{t} \varphi(s)ds \leq \varphi(t)$ for $t \in \mathbf{I}$

Hence, Hyers-Ulam-Rassias constant

$$C_\varphi = (Lp(\xi) + r(\rho)N + 1)\Omega^{-1}\left(\Omega(1) + \lambda k_1 \varpi\left(F^{-1}\left(F(1) + \lambda^{n+1}k_2\right)\right)\right) F^{-1}\left(F(1) + \lambda^{n+1}k_2\right) \quad t \in \mathbf{I} \tag{4.658}$$

Proof:

From equation(4.656), we get

$$-\varphi(t) \leq (a(t)\psi(u(t))u'(t))'' + r(t)u''(t) + p(t)u'(t) + q(t)f(u(t)) - P(t, u(t), u'(t)) \leq \varphi(t) \tag{4.659}$$

Equation (4.659) reduce to

$$(a(t)\psi(u(t))u'(t))'' + r(t)u''(t) + p(t)u'(t) + q(t)f(u(t)) - P(t, u(t), u'(t)) \leq \varphi(t) \tag{4.660}$$

Integrating (4.660), equals

$$(a(t)\psi(u(t))u'(t))' + \int_{t_0}^{t} r(s)u''(s) + \int_{t_0}^{t} p(s)u'(s)ds + \int_{t_0}^{t} q(s)f(u(s))ds - \int_{t_0}^{t} P(s, u(s), u'(s))ds \leq \int_{t_0}^{t} \varphi(s)ds \tag{4.661}$$

By integrating (4.661)

$$a(t)\psi(u(t))u'(t) + \int_{t_0}^{t}\int_{t_0}^{t} r(s)u''(s)ds^2 + \int_{t_0}^{t}\int_{t_0}^{t} p(s)u'(s)ds^2 + \int_{t_0}^{t}\int_{t_0}^{t} q(s)f(u(s))ds^2 - \int_{t_0}^{t}\int_{t_0}^{t} P(s, u(s), u'(s))ds^2 \leq \int_{t_0}^{t}\int_{t_0}^{t} \varphi(s)ds^2 \tag{4.662}$$

Integrating and applying Lemma 1.1, to get

$$\begin{aligned}&\int_{t_0}^{t} a(s)\psi(u(s))u'(s)ds + \frac{t^2}{2}\int_{t_0}^{t} r(s)u''(s)ds + \frac{t^2}{2}\int_{t_0}^{t} p(s)u'(s)ds\\&+\frac{t^2}{2}\int_{t_0}^{t} q(s)f(u(s))ds - \frac{t^2}{2}\int_{t_0}^{t} P(s,u(s),u'(s))ds \le \frac{t^2}{2}\int_{t_0}^{t}\varphi(s)ds\end{aligned}\tag{4.663}$$

Multiplying by $u'(t)$, using equation (4.172), integrating by part, if $a(t)$ a nondecreasing, it gives $a'(t) \ge 0$, to have

$$\begin{aligned}&u'(t)a(t)\Lambda(u(t)) + u'(t)\frac{t^2}{2}\int_{t_0}^{t} r(s)u''(s)ds + u'(t)\frac{t^2}{2}\int_{t_0}^{t} p(s)u'(s)ds\\&\quad+u'(t)\frac{t^2}{2}\int_{t_0}^{t} q(s)f(u(s))ds - u'(t)\frac{t^2}{2}\int_{t_0}^{t} P(s,u(s),u'(s))ds\\&\qquad\le u'(t)\frac{t^2}{2}\int_{t_0}^{t}\varphi(s)ds\end{aligned}\tag{4.664}$$

Multiplying by $\frac{1}{t^2}$, applying Theorem 1.1 there exist $t_0 \le \xi, \rho \le t$ such that

$$\begin{aligned}&\frac{1}{t^2}u'(t)a(t)\Lambda(u(t)) + u'(t)r(\rho)\int_{t_0}^{t} u''(s)ds\\&+u'(t)p(\xi)\frac{1}{t}\int_{t_0}^{t} u'(s)ds + u'(t)\frac{1}{t}\int_{t_0}^{t} q(s)f(u(s))ds\\&-u'(t)\frac{1}{t}\int_{t_0}^{t} P(s,u(s),u'(s))ds \le u'(t)\frac{1}{t}\int_{t_0}^{t}\varphi(s)ds\end{aligned}\tag{4.665}$$

Taking the absolute value of both sides, by conditions (i) and (ii) of Theorem 4.59, we get

$$\begin{aligned}&\frac{1}{t^2}|u'(t)|a(t)\Lambda(|u(t)|) \le |u'(t)|\int_{t_0}^{t}\varphi(s)ds + |u'(t)|r(\rho)N + Lp(\xi)\\&\quad+|u'(t)|\int_{t_0}^{t} q(s)f(|u(s)|)ds + (|u'(t)|)^{n+1}\int_{t_0}^{t}\phi(s)\varpi(|u(s)|)ds\end{aligned}\tag{4.666}$$

Since $a(t)$ a nonnegative, nondecreasing function on $\mathbf{R}_+$ then there exists $\delta > 0$ such that $\frac{1}{t^2}a(t) \ge \delta$, hence, setting $|u'(t)| \le \lambda$, and $\delta\Lambda(|u(t)|) \ge |u(t)|$, it is clear that

$$\begin{aligned}|u(t)| \le (Lp(\xi) + r(\rho)N + 1)\lambda\int_{t_0}^{t}\varphi(s)ds + \lambda\int_{t_0}^{t} q(s)f(|u(s)|)ds&\\+\lambda^{n+1}\int_{t_0}^{t}\phi(s)\varpi(|u(s)|)ds&\end{aligned}\tag{4.667}$$

Applying the Corollary 3.1, we get

$$\begin{aligned}&|u(t)| \le (Lp(\xi) + r(\rho)N + 1)\\&\lambda\int_{t_0}^{t}\varphi(s)ds\Omega^{-1}\left(\Omega(1) + \lambda\int_{t_0}^{t}\phi(s)\varpi\left(F^{-1}\left(F(1)\right.\right.\right.\\&\left.\left.\left.+\lambda^{n+1}\int_{t_0}^{t} q(\alpha)d\alpha\right)\right)ds\right)F^{-1}\left(F(1) + \lambda^{n+1}\int_{t_0}^{t} q(s)ds\right)\end{aligned}\tag{4.668}$$

Employing conditions (iii) and (iv) of Theorem 4.59 to obtain

$$|u(t)| \leq (Lp(\xi) + r(\rho)N + 1)\Omega^{-1}\left(\Omega(1) + \lambda k_1 \varpi\left(F^{-1}\left(F(1) + \lambda^{n=1}k_2\right)\right)\right) \\ F^{-1}\left(F(1) + \lambda^{n+1}k_2\right)\varphi(t) \quad t \in \mathbf{I} \tag{4.669}$$

Therefore, Hyer-Ulam-Rassias constant is

$$C_\varphi = (Lp(\xi) + r(\rho)N + 1)\Omega^{-1}\left(\Omega(1) + \lambda k_1 \varpi\left(F^{-1}\left(F(1) + \lambda^{n+1}k_2\right)\right)\right) \\ F^{-1}\left(F(1) + \lambda^{n+1}k_2\right) \quad t \in \mathbf{I}.$$

If $r(t)u"(t) = 0$, the equation (4.652) reduce to

$$(a(t)\psi(u(t))u'(t))'' + p(t)u'(t) + q(t)f(u(t)) = P(t, u(t), u'(t)), \tag{4.670}$$

with initial conditions $u(t_0) = u'(t_0) = u"(t) = 0$,

Theorem 4.60:

The equation (4.670) together with its initial conditions is Hyers-Ulam-Rassias stable provided the conditions enumerated in the Theorem 4.59 are satisfied. Then, Hyers-Ulam-Rassia constant

$$C_\varphi = (Lp(\xi) + 1)\Omega^{-1}\left(\Omega(1) + \lambda k_1 \varpi\left(F^{-1}\left(F(1) + \lambda^{n+1}k_2\right)\right)\right) \\ F^{-1}\left(F(1) + \lambda^{n+1}k_2\right) \quad t \in \mathbf{I}. \tag{4.671}$$

Proof:

From equation(4.656), let $r(t)u"(t) = 0$, to get

$$-\varphi(t) \leq (a(t)\psi(u(t))u'(t))'' + p(t)u'(t) + q(t)f(u(t)) - P(t, u(t), u'(t)) \\ \leq \varphi(t) \tag{4.672}$$

Consider the left hand side of Inequality (4.672) to have

$$(a(t)\psi(u(t))u'(t))'' + p(t)u'(t) + q(t)f(u(t)) \\ -P(t, u(t), u'(t)) \leq \varphi(t) \tag{4.673}$$

Integrating (4.673) to get

$$(a(t)\psi(u(t))u'(t))' + \int_{t_0}^{t} p(s)u'(s)ds + \int_{t_0}^{t} q(s)f(u(s))ds \\ - \int_{t_0}^{t} P(s, u(s), u'(s))ds \leq \int_{t_0}^{t} \varphi(s)ds \tag{4.674}$$

By integrating (4.674) we have

$$a(t)\psi(u(t))u'(t) + \int_{t_0}^{t}\int_{t_0}^{t} p(s)u'(s)dsds + \int_{t_0}^{t}\int_{t_0}^{t} q(s)f(u(s))dsds \\ - \int_{t_0}^{t}\int_{t_0}^{t} P(s, u(s), u'(s))dsds \leq \int_{t_0}^{t}\int_{t_0}^{t} \varphi(s)dsds \tag{4.675}$$

Integrating and applying Lemma 1.1, to get

$$\int_{t_0}^{t} a(s)\psi(u(s))u'(s)ds + \frac{t^2}{2}\int_{t_0}^{t} p(s)u'(s)ds$$
$$+\frac{t^2}{2}\int_{t_0}^{t} q(s)f(u(s))ds - \frac{t^2}{2}\int_{t_0}^{t} P(s,u(s),u'(s))ds \tag{4.676}$$
$$\leq \frac{t^2}{2}\int_{t_0}^{t} \varphi(s)ds$$

Multiplying by $u'(t)$, using equation (4.298) integrating by part, if $a(t)$ a nondecreasing, it gives $a'(t) \geq 0$, to have

$$\begin{aligned} u'(t)a(t)\Lambda(u(t)) + u'(t)\frac{t^2}{2}\int_{t_0}^{t} p(s)u'(s)ds + u'(t)\frac{t^2}{2}\int_{t_0}^{t} q(s)f(u(s))ds \\ -u'(t)\frac{t^2}{2}\int_{t_0}^{t} P(s,u(s),u'(s))ds \leq u'(t)\frac{t^2}{2}\int_{t_0}^{t}\varphi(s)ds \end{aligned} \tag{4.677}$$

Multiplying by $\frac{1}{t^2}$, applying Theorem 1.1 there exist $t_0 \leq \xi, \leq t$ such that

$$\begin{aligned} \frac{1}{t^2}u'(t)a(t)\Lambda(u(t)) + u'(t)p(\xi)\frac{1}{t}\int_{t_0}^{t} u'(s)ds + u'(t)\frac{1}{t}\int_{t_0}^{t} q(s)f(u(s))ds \\ -u'(t)\frac{1}{t}\int_{t_0}^{t} P(s,u(s),u'(s))ds \leq u'(t)\frac{1}{t}\int_{t_0}^{t}\varphi(s)ds \end{aligned} \tag{4.678}$$

Taking the absolute value of both sides, by conditions (i) and (ii) of Theorem 4.59, we get

$$\begin{aligned} \frac{1}{t^2}|u'(t)|a(t)\Lambda(|u(t)|) \leq |u'(t)|\int_{t_0}^{t}\varphi(s)ds + Lp(\xi) \\ +|u'(t)|\int_{t_0}^{t} q(s)f(|u(s)|)ds + (|u'(t)|)^{n+1}\int_{t_0}^{t}\phi(s)\varpi(|u(s)|)ds \end{aligned} \tag{4.679}$$

Since $a(t)$ a nonnegative, nondecreasing function on $\mathbf{R}_+$ then there exists $\delta > 0$ such that $\frac{1}{t^2}a(t) \geq \delta$, hence, setting $|u'(t)| \leq \lambda$, and $\delta\Lambda(|u(t)|) \geq |u(t)|$, it is clear that

$$\begin{aligned} |u(t)| \leq (Lp(\xi)+1)\lambda\int_{t_0}^{t}\varphi(s)ds + \lambda\int_{t_0}^{t} q(s)f(|u(s)|)ds \\ +\lambda^{n+1}\int_{t_0}^{t}\phi(s)\varpi(|u(s)|)ds \end{aligned} \tag{4.680}$$

Applying the Corollary 3.1, we get

$$\begin{aligned} |u(t)| \leq (Lp(\xi)+1) \\ \lambda\int_{t_0}^{t}\varphi(s)ds\Omega^{-1}\left(\Omega(1) + \lambda\int_{t_0}^{t}\phi(s)\varpi\left(F^{-1}\left(F(1)\right.\right.\right. \\ \left.\left.\left.+\lambda^{n+1}\int_{t_0}^{t} q(\alpha)d\alpha\right)\right)ds\right)F^{-1}\left(F(1) + \lambda^{n+1}\int_{t_0}^{t} q(s)ds\right) \end{aligned} \tag{4.681}$$

Employing conditions (iii) and (iv) of Theorem 4.59 to obtain

$$\begin{aligned} |u(t)| \leq (Lp(\xi)+1)\Omega^{-1}\left(\Omega(1) + \lambda k_1\varpi\left(F^{-1}\left(F(1) + \lambda^{n=1}k_2\right)\right)\right) \\ F^{-1}\left(F(1) + \lambda^{n+1}k_2\right)\varphi(t) \quad t \in \mathbf{I} \end{aligned} \tag{4.682}$$

Therefore, Hyer-Ulam-Rassias constant is
$C_\varphi = (Lp(\xi)+1)\Omega^{-1}\left(\Omega(1)+\lambda k_1 \varpi\left(F^{-1}\left(F(1)+\lambda^{n+1}k_2\right)\right)\right)$

$$F^{-1}\left(F(1)+\lambda^{n+1}k_2\right) \quad t \in \mathbf{I}.$$

In our next consideration, equation (4.652) is considered in the form

$$(a(t)\psi(u(t))u'(t))'' + r(t)u''(t) + p(t)u'(t) + q(t)f(u(t)) = 0. \tag{4.683}$$

with initial conditions $u(t_0) = u'(t_0) = u"(t_0) = 0$.

Theorem 4.61:

The equation (4.683) together with initial conditions posses Hyers-Ulam-Rassias stability provided conditions in Theorem 4.59 remain valid. The Hyers-Ulam-Rassias constant is given as:

$$C_\varphi = (Lp(\xi) + r(\rho)N + 1)\varphi(t)\Omega^{-1}\left(\Omega(1) + k_1\lambda\right) \tag{4.684}$$

Proof:

From equation(4.656), we get

$$\begin{aligned}-\varphi(t) &\le (a(t)\psi(u(t))u'(t))'' + r(t)u''(t)\\ &\quad +p(t)u'(t) + q(t)f(u(t)) \le \varphi(t)\end{aligned} \tag{4.685}$$

From inequality (4.685) we have

$$(a(t)\psi(u(t))u'(t))'' + r(t)u''(t) + p(t)u'(t) + q(t)f(u(t)) \le \varphi(t) \tag{4.686}$$

Integrating (4.686) to have

$$\begin{aligned}(a(t)\psi(u(t))u'(t))' + \int_{t_0}^{t} r(s)u''(s) + \int_{t_0}^{t} p(s)u'(s)ds + \int_{t_0}^{t} q(s)f(u(s))ds\\ \le \int_{t_0}^{t}\varphi(s)ds\end{aligned} \tag{4.687}$$

By integrating (4.687)

$$\begin{aligned}a(t)\psi(u(t))u'(t) + \int_{t_0}^{t}\int_{t_0}^{t} r(s)u''(s)dsds + \int_{t_0}^{t}\int_{t_0}^{t} p(s)u'(s)dsds\\ + \int_{t_0}^{t}\int_{t_0}^{t} q(s)f(u(s))dsds \le \int_{t_0}^{t}\int_{t_0}^{t}\varphi(s)ds^2\end{aligned} \tag{4.688}$$

Integrating and applying Lemma 1.1, to get

$$\begin{aligned}\int_{t_0}^{t} a(s)\psi(u(s))u'(s)ds + \frac{t^2}{2}\int_{t_0}^{t} r(s)u''(s)ds + \frac{t^2}{2}\int_{t_0}^{t} p(s)u'(s)ds\\ +\frac{t^2}{2}\int_{t_0}^{t} q(s)f(u(s))ds \le \frac{t^2}{2}\int_{t_0}^{t}\varphi(s)ds\end{aligned} \tag{4.689}$$

Multiplying by $u'(t)$, using equation (4.298), integrating by part, if $a(t)$ a nondecreasing, it gives $a'(t) \ge 0$, to have

$$\begin{aligned}u'(t)a(t)\Lambda(u(t)) + u'(t)\frac{t^2}{2}\int_{t_0}^{t} r(s)u''(s)ds + u'(t)\frac{t^2}{2}\int_{t_0}^{t} p(s)u'(s)ds\\ +u'(t)\frac{t^2}{2}\int_{t_0}^{t} q(s)f(u(s))ds \le u'(t)\frac{t^2}{2}\int_{t_0}^{t}\varphi(s)ds\end{aligned} \tag{4.690}$$

Multiplying by $\frac{1}{t^2}$, applying Theorem 1.1 there exist $t_0 \leq \xi, \rho \leq t$ such that

$$\begin{aligned}
&\frac{1}{t^2}u'(t)a(t)\Lambda(u(t)) + u'(t)r(\rho)\int_{t_0}^{t} u''(s)ds \\
&+u'(t)p(\xi)\frac{1}{t}\int_{t_0}^{t} u'(s)ds + u'(t)\frac{1}{t}\int_{t_0}^{t} q(s)f(u(s))ds \\
&\leq u'(t)\frac{1}{t}\int_{t_0}^{t} \varphi(s)ds
\end{aligned} \tag{4.691}$$

Taking the absolute value of both sides, by condition (i) of Theorem 4.59, we get

$$\begin{aligned}
\frac{1}{t^2}|u'(t)|a(t)\Lambda(|u(t)|) \leq |u'(t)|\int_{t_0}^{t} \varphi(s)ds + |u'(t)|r(\rho)N + Lp(\xi) \\
+|u'(t)|\int_{t_0}^{t} q(s)f(|u(s)|)ds + \varpi(|u(s)|)ds
\end{aligned} \tag{4.692}$$

Since $a(t)$ a nonnegative, nondecreasing function on $\mathbf{R}_+$ then there exists $\delta > 0$ such that $\frac{1}{t^2}a(t) \geq \delta$, hence, setting $|u'(t)| \leq \lambda$, and $\delta\Lambda(|u(t)|) \geq |u(t)|$, it is clear that

$$|u(t)| \leq (Lp(\xi) + r(\rho)N + 1)\lambda\int_{t_0}^{t} \varphi(s)ds + \lambda\int_{t_0}^{t} q(s)f(|u(s)|)ds \tag{4.693}$$

Applying Theorem 2.9, we get

$$|u(t)| \leq (Lp(\xi) + r(\rho)N + 1)\lambda\int_{t_0}^{t} \varphi(s)ds\Omega^{-1}\left(\Omega(1) + \lambda\int_{t_0}^{t} q(s)ds\right) \tag{4.694}$$

Employing conditions (iii) and (iv) of Theorem 4.59 to obtain

$$|u(t)| \leq (Lp(\xi) + r(\rho)N + 1)\Omega^{-1}\left(\Omega(1) + \lambda k_1\right) \tag{4.695}$$

Therefore, Hyers-Ulam-Rassias constant is

$$C_\varphi = (Lp(\xi) + r(\rho)N + 1)\Omega^{-1}\left(\Omega(1) + \lambda k_1\right)$$

If $r(t)u''(t) = 0$, and $P(t, u(t), u'(t)) = 0$ equation (4.652) reduce to

$$(a(t)\psi(u(t))u'(t))'' + p(t)u'(t) + q(t)f(u(t)) = 0. \tag{4.696}$$

Theorem 4.62:

The equation (4.696) together with initial conditions has Hyers-Ulam-Rassias stability provided conditions of Theorem 4.59 remain valid. The Hyers-Ulam-Rassias constant is given as.

$$C_\varphi = (Lp(\xi) + 1)\varphi(t)\Omega^{-1}\left(\Omega(1) + k_1\lambda\right) \tag{4.697}$$

Proof:

The equation (4.683) together with initial conditions posses Hyers-Ulam-Rassias stability provided conditions in Theorem 4.59 remain valid. The Hyers-Ulam-Rassias constant is given as:

$$C_\varphi = (Lp(\xi) + 1)\varphi(t)\Omega^{-1}\left(\Omega(1) + k_1\lambda\right) \tag{4.698}$$

Proof:

From equation(4.656), we get

$$-\varphi(t) \leq (a(t)\psi(u(t))u'(t))'' + p(t)u'(t) + q(t)f(u(t)) \leq \varphi(t) \tag{4.699}$$

From inequality (4.699) we have

$$(a(t)\psi(u(t))u'(t))'' + p(t)u'(t) + q(t)f(u(t)) \leq \varphi(t) \tag{4.700}$$

Integrating (4.700) to have

$$\begin{aligned}(a(t)\psi(u(t))u'(t))' + \int_{t_0}^{t} p(s)u'(s)ds + \int_{t_0}^{t} q(s)f(u(s))ds \\ \leq \int_{t_0}^{t} \varphi(s)ds\end{aligned} \tag{4.701}$$

By integrating (4.701)

$$\begin{aligned}a(t)\psi(u(t))u'(t) + \int_{t_0}^{t}\int_{t_0}^{t} p(s)u'(s)dsds \\ + \int_{t_0}^{t}\int_{t_0}^{t} q(s)f(u(s))dsds \leq \int_{t_0}^{t}\int_{t_0}^{t} \varphi(s)ds^2\end{aligned} \tag{4.702}$$

Integrating and applying Lemma 1.1, to get

$$\begin{aligned}\int_{t_0}^{t} a(s)\psi(u(s))u'(s)ds + \frac{t^2}{2}\int_{t_0}^{t} p(s)u'(s)ds \\ + \frac{t^2}{2}\int_{t_0}^{t} q(s)f(u(s))ds \leq \frac{t^2}{2}\int_{t_0}^{t} \varphi(s)ds\end{aligned} \tag{4.703}$$

Multiplying by $u'(t)$, using equation (4.298), integrating by part, if $a(t)$ a nondecreasing, it gives $a'(t) \geq 0$, to have

$$\begin{aligned}u'(t)a(t)\Lambda(u(t)) + u'(t)\frac{t^2}{2}\int_{t_0}^{t} p(s)u'(s)ds \\ + u'(t)\frac{t^2}{2}\int_{t_0}^{t} q(s)f(u(s))ds \leq u'(t)\frac{t^2}{2}\int_{t_0}^{t} \varphi(s)ds\end{aligned} \tag{4.704}$$

Multiplying by $\frac{1}{t^2}$, applying Theorem 1.1 there exist $t_0 \leq \xi \leq t$ such that

$$\begin{aligned}\frac{1}{t^2}u'(t)a(t)\Lambda(u(t)) + u'(t)p(\xi)\frac{1}{t}\int_{t_0}^{t} u'(s)ds \\ + u'(t)\frac{1}{t}\int_{t_0}^{t} q(s)f(u(s))ds \leq u'(t)\frac{1}{t}\int_{t_0}^{t} \varphi(s)ds\end{aligned} \tag{4.705}$$

Taking the absolute value of both sides, by condition (i) of Theorem 4.59, we get

$$\begin{aligned}\frac{1}{t^2}|u'(t)|a(t)\Lambda(|u(t)|) \leq |u'(t)|\int_{t_0}^{t} \varphi(s)ds + Lp(\xi) \\ + |u'(t)|\int_{t_0}^{t} q(s)f(|u(s)|)ds + \varpi(|u(s)|)ds\end{aligned} \tag{4.706}$$

Since $a(t)$ a nonnegative, nondecreasing function on $\mathbf{R}_+$ then there exists $\delta > 0$ such that $\frac{1}{t^2}a(t) \geq \delta$, hence, setting $|u'(t)| \leq \lambda$, and $\delta\Lambda(|u(t)|) \geq |u(t)|$, it is clear that

$$|u(t)| \leq (Lp(\xi) + +1)\lambda \int_{t_0}^{t} \varphi(s)ds + \lambda \int_{t_0}^{t} q(s)f(|u(s)|)ds \tag{4.707}$$

Applying Theorem 2.9, we get

$$|u(t)| \leq (Lp(\xi) + 1)\lambda \int_{t_0}^{t} \varphi(s)ds\Omega^{-1} \left(\Omega(1) + \lambda \int_{t_0}^{t} q(s)ds\right) \tag{4.708}$$

Employing conditions (iii) and (iv) of Theorem 4.59 to obtain

$$|u(t)| \leq (Lp(\xi) + 1)\Omega^{-1} (\Omega(1) + \lambda k_1) \tag{4.709}$$

Therefore, Hyers-Ulam-Rassias constant is

$$C_\varphi = (Lp(\xi) + 1)\Omega^{-1} (\Omega(1) + \lambda k_1)$$

We consider equation (4.654) in the next theorem.

Theorem 4.63:

Suppose all the conditions of Theorem 4.59 remain valid. Equation (4.654) is Hyers-Ulam-Rassias stable and

$$C_\varphi = L(\lambda a(\xi) + \lambda r(\alpha)N + \lambda p(\eta) + 1)\Omega^{-1} \left(\Omega(1) + \lambda^{n+1} k_2\right)$$

Proof:

By evaluating (4.657) we have

$$\begin{aligned} -\varphi \leq (a(t)u'(t))'' + r(t)u''(t) + p(t)u'(t) + q(t)f(u(t)) \\ -P(t, u(t), u'(t)) \leq \varphi(t), \end{aligned} \tag{4.710}$$

Multiplying by $u'(t)$, to have

$$\begin{aligned} -u'(t)\varphi \leq (a(t)u'(t))''u'(t) + r(t)u''(t)u'(t) + p(t)u'(t)u'(t) \\ +q(t)f(u(t))u'(t) - P(t, u(t), u'(t))u'(t) \leq u'(t)\varphi(t), \end{aligned} \tag{4.711}$$

Integrating trice, using Lemma 1.1 together with equation (4.183)

$$\begin{aligned} & u'(t) \int_{t_0}^{t} a(s)u'(s)ds + u'(t)\frac{t^2}{2} \int_{t_0}^{t} r(s)u''(s)ds \\ & +\frac{t^2}{2} \int_{t_0}^{t} p(s)(u'(s))^2 ds + \frac{t^2}{2} \int_{t_0}^{t} q(s)\frac{d}{ds}F(u(s))ds \\ & -\frac{t^2}{2} \int_{t_0}^{t} P(s, u(s), u'(s))u'(s)ds \leq \frac{t^2}{2} \int_{t_0}^{t} u'(s)\varphi(s)ds \end{aligned} \tag{4.712}$$

Integrating by part equation (4.713), $q(t)$ a nondecreasing, then $q'(t) \geq 0$, we have

$$\begin{aligned} & u'(t) \int_{t_0}^{t} a(s)u'(s)ds + u'(t)\frac{t^2}{2} \int_{t_0}^{t} r(s)u''(s)ds \\ & +\frac{t^2}{2} \int_{t_0}^{t} p(s)(u'(s))^2 ds + \frac{t^2}{2}F(u(t)) \\ & -\frac{t^2}{2} \int_{t_0}^{t} P(s, u(s), u'(s))u'(s)ds \leq \frac{t^2}{2} \int_{t_0}^{t} u'(s)\varphi(s)ds \end{aligned} \tag{4.713}$$

Multiplying by $\frac{1}{t^2}$, applying Theorem 1.1 there exist $\xi, \alpha, \eta \in [t_0, t]$ ∍

$$\begin{aligned} \frac{1}{t^2} q(t)F(u(s)) \leq \int_{t_0}^{t} u'(s)\varphi(s)ds \\ -u'(t)a(\xi)\int_{t_0}^{t} u'(s)ds - u'(t)r(\alpha)\int_{t_0}^{t} u''(s)ds \\ -u'(t)p(\eta)\int_{t_0}^{t} u'(s)ds + \int_{t_0}^{t} P(s, u(s), u'(s))u'(s)ds \end{aligned} \tag{4.714}$$

Using conditions (i) and (ii) of Theorem 4.59, to obtain

$$\begin{aligned} \frac{1}{t^3} q(t)F(|u(s)|) \leq \int_{t_0}^{t} |u'(s)|\varphi(s)ds + |u'(t)|a(\xi)L \\ +|u'(t)|r(\alpha)N + |u'(t)|p(\eta)L + \int_{t_0}^{t} \phi(t)\varpi(|u(s)|)(|u'(s)|)^{n+1}ds \end{aligned} \tag{4.715}$$

Since $r(t)$ is nondecreasing, nonnegative continuous function on $\mathbf{R}_+$ there exists $\delta > 0$ such that $\frac{r(t)}{t^2} \geq \delta$ the, $\delta F(|u(t)|) \geq |u(t)|$ Setting $|u'(t)| \leq \lambda$, to have

$$\begin{aligned} |u(s)| \leq L(a(\xi) + r(\alpha)N \\ +p(\eta) + 1)\lambda \int_{t_0}^{t} \varphi(s)ds + \lambda^{n+1}\int_{t_0}^{t} \phi(s)\varpi(|u(s)|)ds \end{aligned} \tag{4.716}$$

With the application of Theorem 2.9 equals

$$\begin{aligned} |u(t)| \leq L(a(\xi) + r(\alpha)N + p(\eta) + 1)\lambda \int_{t_0}^{t} \varphi(s)ds \\ \Omega^{-1}\left(\Omega(1) + \lambda^{n+1}\int_{t_0}^{t} \phi(s)ds\right) \end{aligned} \tag{4.717}$$

Using the conditions (iii) and (iv) of Theorem 4.59 to give

$$|u(t)| \leq L(a(\xi) + \lambda r(\alpha)N + p(\eta) + 1)\Omega^{-1}\left(\Omega(1) + \lambda^{n+1}k_2\right)\varphi(t). \tag{4.718}$$

Therefore, Hyers-Ulam-Rassias constant is

$C_\varphi = L(\lambda a(\xi) + \lambda r(\alpha)N + \lambda p(\eta) + 1)\Omega^{-1}\left(\Omega(1) + \lambda^{n+1}k_2\right)$

Equation (4.655) is considered in the next theorem.

Theorem 4.64:

Let all the conditions of Theorem 4.59 remain valid. Equation (4.655) is H-U-R stable with constant

$C_\varphi = L(\lambda a(\xi) + \lambda p(\eta) + 1)\Omega^{-1}\left(\Omega(1) + \lambda^{n+1}k_2\right)$

Proof:

By evaluating (4.657), since $r(t)u"(t) = 0$ then, it follows that

$$\begin{aligned} -\varphi \leq (a(t)u'(t))'' + p(t)u'(t) + q(t)f(u(t)) \\ -P(t, u(t), u'(t)) \leq \varphi(t). \end{aligned} \tag{4.719}$$

Multiplying by $u'(t)$, to have

$$-u'(t)\varphi \leq (a(t)u'(t))''u'(t) + p(t)u'(t)u'(t)$$
$$+q(t)f(u(t))u'(t) - P(t,u(t),u'(t))u'(t) \leq u'(t)\varphi(t). \tag{4.720}$$

Using equation (4.183) to arrive

$$u'(t)\int_{t_0}^{t} a(s)u'(s)ds + u'(t)\frac{t^2}{2}\int_{t_0}^{t} r(s)u''(s)ds$$
$$+\frac{t^2}{2}\int_{t_0}^{t} p(s)(u'(s))^2 ds + \frac{t^2}{2}\int_{t_0}^{t} q(s)\frac{d}{ds}F(u(s))ds \tag{4.721}$$
$$-\frac{t^2}{2}\int_{t_0}^{t} P(s,u(s),u'(s))u'(s)ds \leq \frac{t^2}{2}\int_{t_0}^{t} u'(s)\varphi(s)ds$$

Integrating by part, using the property that $q'(t) \geq 0$, when $q(t)$ a nondecreasing, multiplying by $\frac{1}{t^2}$, applying Theorem 1.1 there exist $\xi, \alpha, \eta \in [t_0, t]$ such that

$$\frac{1}{t^2}q(t)F(u(s)) \leq \int_{t_0}^{t} u'(s)\varphi(s)ds$$
$$-u'(t)a(\xi)\int_{t_0}^{t} u'(s)ds \tag{4.722}$$
$$-u'(t)p(\eta)\frac{1}{t}\int_{t_0}^{t} u'(s)ds + \int_{t_0}^{t} P(s,u(s),u'(s))u'(s)ds$$

By absolute value property and employing the conditions (i) and (ii)

of Theorem 4.59, there exists $\delta > 0$ such that

$\frac{r(t)}{t^2} \geq \delta$ then $\frac{1}{t^3}q(t)F(|u(t)|) \geq |u(t)|$ and setting $|u'(t)| \leq \lambda$ to obtain

$$|u(s)| \leq L(a(\xi) + p(\eta) + 1)\lambda\frac{1}{t}\int_{t_0}^{t} \varphi(s)ds + \lambda^{n+1}\int_{t_0}^{t} \phi(s)\varpi(|u(s)|)ds \tag{4.723}$$

By applying Theorem (2.2), one concludes

$$|u(t)| \leq L(a(\xi) + p(\eta) + 1)\lambda\int_{t_0}^{t} \varphi(s)ds$$
$$\Omega^{-1}\left(\Omega(1) + \lambda^{n+1}\int_{t_0}^{t} \phi(s)ds\right) \tag{4.724}$$

Using the conditions (iii) and (iv) of Theorem 4.59 we obtain

$$|u(t)| \leq L(a(\xi) + p(\eta) + 1)\Omega^{-1}\left(\Omega(1) + \lambda^{n+1}k_2\right)\varphi(t). \tag{4.725}$$

By this, the constant is given as

$C_\varphi = L(a(\xi) + \lambda p(\eta) + 1)\Omega^{-1}\left(\Omega(1) + \lambda^{n+1}k_2\right)$

If $P(t,u(t),u'(t)) = 0$ then, equation (4.654) is reduced to

$$(a(t)u'(t))'' + r(t)u''(t) + p(t)u'(t) + q(t)f(u(t)) = 0 \tag{4.726}$$

Theorem 4.65:

Let all the conditions of Theorem 4.59 remain valid.

Equation (4.726) is Hyers-Ulam-Rassias stable with Hyers-Ulam-Rassias constant

$C_\varphi = (r(\xi)N + p(\eta)L^2 + 1)\varphi(t)\Omega^{-1}(\Omega(1) + d\lambda)$

Proof:

From equation(4.657) with $P(t, u(t), u'(t)) = 0$, we have

$$-\varphi \leq (a(t)u'(t))'' + p(t)u'(t) + q(t)f(u(t)) \leq \varphi(t). \tag{4.727}$$

Multiplying by $u'(t)$, to have

$$\begin{aligned}-u'(t)\varphi \leq (a(t)u'(t))''u'(t) + p(t)u'(t)u'(t)\\ +q(t)f(u(t))u'(t) \leq u'(t)\varphi(t).\end{aligned} \tag{4.728}$$

Integrating trice, applying Lemma 1.1 we get

$$\begin{aligned}u'(t)\int_{t_0}^{t} a(s)u'(s)ds + u'(t)\frac{t^2}{2}\int_{t_0}^{t} r(s)u''(s)ds\\ +\frac{t^2}{2}\int_{t_0}^{t} p(s)(u'(s))^2 ds + \frac{t^2}{2}\int_{t_0}^{t} q(s)f(u(s))u'(s)ds \leq \frac{t^2}{2}\int_{t_0}^{t} u'(s)\varphi(s)ds\end{aligned} \tag{4.729}$$

Integrating by part, since $a(t)$ is nondecreasing, then $a'(t) \geq 0$, multiplying by $\frac{2}{t^2}$ for $t > 0$, by applying Theorem 1.1 there exist $\xi, \alpha, \eta \in [t_0, t]$ such that

$$\begin{aligned}\frac{1}{t^2}u'(t)a(t)u(t) + u'(t)r(\xi)\int_{t_0}^{t} u''(s)ds\\ +p(\eta)\int_{t_0}^{t}(u'(s))^2 ds + u'(t)\int_{t_0}^{t} q(s)f(u(s))ds \leq \frac{1}{t}\int_{t_0}^{t} u'(s)\varphi(s)ds\end{aligned} \tag{4.730}$$

By making use absolute value property and condition (i) of Theorem 4.59 to obtain

$$\begin{aligned}\frac{1}{t^2}a(t)|u'(t|)|u(t)| \leq \int_{t_0}^{t} |u'(s)|\varphi(s)ds + |u'(t)|r(\alpha)N + |u'(t)|p(\eta)L^2\\ +|u'(t)|\int_{t_0}^{t} q(s)f(u(s))ds\end{aligned} \tag{4.731}$$

Setting $|u'(t)| \leq \lambda$, it follows that

$$\frac{1}{t^2}a(t)\lambda|u(t)| \leq (r(\xi)N + p(\eta)L^2 + 1)\lambda\int_{t_0}^{t}\varphi(s)ds + \lambda\int_{t_0}^{t} q(s)f(u(s))ds \tag{4.732}$$

Since $a(t)$ is nonnegative and nondecreasing, there exists $\delta > 0$ such that $\frac{r(t)}{t^2} \geq \delta$ then, setting $\delta|u'(t|)|u(t)| \geq |u(t)|$ and using Theorem 2.9 to get

$$|u(t)| \leq (r(\xi)N + p(\eta)L^2 + 1)\lambda\int_{t_0}^{t}\varphi(s)ds\Omega^{-1}\left(\Omega(1) + \lambda\int_{t_0}^{t} q(s)ds\right) \tag{4.733}$$

Letting $\lim_{t\to\infty}\int_{t_0}^{t} q(s)ds \leq d$, where $d > 0$ and $\frac{1}{t}\int_{t_0}^{t}\varphi(s)ds \leq \varphi(t)$

$$|u(t)| \leq (r(\xi)N + p(\eta)L^2 + 1)\lambda\varphi(t)\Omega^{-1}(\Omega(1) + d\lambda) \tag{4.734}$$

Therefore, H-U-R constant is

$C_\varphi = (r(\xi)N + p(\eta)L^2 + 1)\lambda\varphi(t)\Omega^{-1}(\Omega(1) + d\lambda)$

The set of last equations to be considered are damped third order nonlinear differential equations:

$$u'''(t) + r(t)u''(t) + nf(t)u'(t) + q(t)u(t) + Q(t, u(t)) = P(t, u(t), u'(t)) \tag{4.735}$$

$$u'''(t) + nf(t)u'(t) + q(t)u(t) + Q(t, u(t)) = P(t, u(t), u'(t)) \tag{4.736}$$

initial conditions $u(t_0) = u'(t_0) = 0$ for $n \in \mathbf{N}$, $f, r, q \in C(\mathbf{R})$, $Q \in (\mathbf{R}, \mathbf{R})$ and $P \in (\mathbf{I} \times \mathbf{R^2}, \mathbf{R})$ are presented for consideration using previous methods.

Definition 4.29:

If there exists $u(t) \in C^3(\mathbf{I}, \mathbf{R})$ satisfying

$$\begin{aligned}|u'''(t) + r(t)u''(t) + nf(t)u'(t) + q(t)u(t) + Q(t, u(t))\\ -P(t, u(t), u'(t))| \leq \varphi(t)\end{aligned} \tag{4.737}$$

and also there exists any solution $u_0(t) \in C^3(\mathbf{I}, \mathbf{R})$ of the equation (4.735) for which

$|u(t) - u_0(t)| \leq C_\varphi \varphi(t)$,

therefore equation (4.735) has Hyers-Ulam-Rassias stability and

Hyers-Ulam-Rassias constant is denoted as C_φ.

Theorem 4.66:

The equation (4.735) together with its initial conditions is H-U-R stable provided undermentioned are satisfied:

(i) If $\int_{t_0}^{\infty} |u'(s)|ds \leq L$, $\int_{t_0}^{\infty} |u''(s)|ds \leq N$, for $N, L > 0$. In addition, $|Q(t, u(t))| \leq p(t)\gamma(|u(t)|)$

(ii) $\exists$ a positive function $\phi(t) \in C(\mathbf{I}, \mathbf{R}_+)$

$\ni |P(t, u(t), u'(t))| \leq \phi(t)\varpi(|u(t)|)(|u'(t)|)^n$ where $n \in \mathbf{N}$

(iii)$\lim_{t\to\infty} \int_{t_0}^{t} p(s)ds = k_1 < \infty$ and $\lim_{t\to\infty} \int_{t_0}^{t} q(s)ds = k_2 < \infty$, where $k_1, k_2 > 0$

(iv) $\lambda > 0$ so that $|u'(t)| \leq \lambda$, and $1\lambda \int_{t_0}^{t} \varphi(s)ds \leq \varphi(t)$ for $t \in \mathbf{I}$

hence, Hyers-Ulam-R assias constant

$$\begin{aligned}C_\varphi = (\lambda + nf(\xi)L^2 + |u'''(\eta)|L + r(\alpha)N + 1)\\ \Omega^{-1}\left(\Omega(1) + \lambda^{n+1}k_1\varpi\left(F^{-1}\left(F(1) + \lambda k_2\right)\right) ds\right)\\ F^{-1}\left(F(1) + \lambda k_2\right) \quad t \in \mathbf{I}\end{aligned} \tag{4.738}$$

Proof:

Evaluate (4.737)

$$\begin{aligned}-\varphi(t) \leq u'''(t) + r(t)u''(t) + nf(t)u'(t) + q(t)u(t) + Q(t, u(t))\\ -P(t, u(t), u'(t)) \leq \varphi(t)\end{aligned} \tag{4.739}$$

Multiplying by $u'(t)$, to have

$$-u'(t)\varphi(t) \leq u'''(t) + r(t)u''(t)u'(t) + nf(t)(u'(t))^2 + q(t)u(t) + Q(t,u(t))u'(t) \\ -P(t,u(t),u'(t))u'(t) \leq u'(t)\varphi(t) \tag{4.740}$$

Integrating and using equation (4.20) we have

$$\int_{t_0}^{t} u'''(s)u'(s)ds + \int_{t_0}^{t} r(t)u''(s)u'(s)ds + n\int_{t_0}^{t} f(s)(u'(s))^2 ds \\ + \int_{t_0}^{t} q(s)\frac{d}{ds}\mathbb{G}(u(s))ds + \int_{t_0}^{t} Q(s,u(s))u'(s)ds \\ - \int_{t_0}^{t} P(s,u(s),u'(s))u'(s)ds \leq \int_{t_0}^{t} u'(s)\varphi(s)ds \tag{4.741}$$

Integrating by part and using Theorem 1.1, there exist $t_0 \leq \eta, \xi, \alpha \leq t$ such that

$$u'''(\eta)\int_{t_0}^{t} u'(s)ds + r(\alpha)u'(t)\int_{t_0}^{t} u''(s)ds \\ +nf(\xi)\int_{t_0}^{t} (u'(s))^2 ds + q(t)\mathbb{G}(u(t)) + \int_{t_0}^{t} Q(s,u(s))u'(s)ds \\ - \int_{t_0}^{t} P(s,u(s),u'(s))u'(s)ds \leq \int_{t_0}^{t} u'(s)\varphi(s)ds \tag{4.742}$$

Taking absolute value and using conditions (i) and (ii) and

setting $q(t)|\mathbb{G}(u(t))| \geq |u(t)|$ and setting $|u'(t)| \leq \lambda$,

$$|u(t)| \leq (\lambda + nf(\xi)L^2 + |u'''(\eta)|L + r(\alpha)N + 1)\lambda\int_{t_0}^{t} \varphi(s)ds \\ +\lambda\int_{t_0}^{t} p(s)\gamma(u(s))ds + \lambda^{n+1}\int_{t_0}^{t} \phi(t)\varpi(|u(t)|)ds \tag{4.743}$$

By making use of Corollary 3.1 the resulting inequality is

$$|u(t)| \leq (\lambda + nf(\xi)L^2 + |u'''(\eta)|L + r(\alpha)N + 1)\lambda\int_{t_0}^{t} \varphi(s)ds \\ \Omega^{-1}\left(\Omega(1) + \lambda^{n+1}\int_{t_0}^{t} \phi(s)\varpi\left(F^{-1}\left(F(1)\right.\right.\right. \\ \left.\left.\left. +\lambda\int_{t_0}^{t} p(\delta)d\delta\right)\right)ds\right)F^{-1}\left(F(1) + \lambda\int_{t_0}^{t} p(s)ds\right) \quad t \in \mathbf{I} \tag{4.744}$$

By conditions (iii) and (iv) we arrive at

$$|u(t)| \leq (\lambda^2 + nf(\xi)L^2 + |u'''(\eta)|L + r(\alpha)N + 1) \\ \Omega^{-1}\left(\Omega(1) + \lambda^{n+1}k_1\varpi\left(F^{-1}\left(F(1)\right.\right.\right. \\ \left.\left.\left. +\lambda k_2\right)\right)ds\right)F^{-1}\left(F(1) + \lambda k_2\right)\varphi(t) \quad t \in \mathbf{I} \tag{4.745}$$

Therefore, Hyer-Ulam-Rassias constant is

$$C_\varphi = (\lambda + nf(\xi)L^2 + |u'''(\eta)|L + r(\alpha)N + 1) \\ \Omega^{-1}\left(\Omega(1) + \lambda^{n+1}k_1\varpi\left(F^{-1}\left(F(1) + \lambda k_2\right)\right)ds\right) \\ F^{-1}\left(F(1) + \lambda k_2\right) \quad t \in \mathbf{I}$$

Theorem 4.67:

Equation (4.736) is Hyers-Ulam-Rassias stable provided the prescribed conditions of Theorem 4.66 remained valid. Then, Hyers-Ulam-Rassias constant is given as

$$C_\varphi = (\lambda + nf(\xi)L^2 + |u'''(\eta)|L + 1)\lambda$$
$$\Omega^{-1}\left(\Omega(1) + \lambda^{n+1}k_1\varpi\left(F^{-1}\left(F(1) + \lambda k_2\right)\right)ds\right)F^{-1}\left(F(1) + \lambda k_2\right) \tag{4.746}$$

Proof:

From equation(4.737), if $r(t)u''(t) = 0$, we obtain

$$-\varphi(t) \leq u'''(t) + nf(t)u'(t) + q(t)u(t) + Q(t, u(t))$$
$$-P(t, u(t), u'(t)) \leq \varphi(t) \tag{4.747}$$

Multiplying by $u'(t)$, to have

$$-u'(t)\varphi(t) \leq u'''(t) + nf(t)(u'(t))^2 + q(t)u(t) + Q(t, u(t))u'(t)$$
$$-P(t, u(t), u'(t))u'(t) \leq u'(t)\varphi(t). \tag{4.748}$$

Using equation (4.20), we obtain

$$\int_{t_0}^{t} u'''(s)u'(s)ds + n\int_{t_0}^{t} f(s)(u'(s))^2 ds + \int_{t_0}^{t} q(s)\frac{d}{ds}\mathbb{G}(u(s))ds$$
$$+\int_{t_0}^{t} Q(s, u(s))u'(s)ds - \int_{t_0}^{t} P(s, u(s), u'(s))u'(s)ds \leq \int_{t_0}^{t} u'(s)\varphi(s)ds \tag{4.749}$$

Integrating by part the equation (4.749) and by applying Theorem 1.1, there exist $t_0 \leq \eta, \xi, \alpha \leq t$ such that

$$u'''(\eta)\int_{t_0}^{t} u'(s)ds + nf(\xi)\int_{t_0}^{t}(u'(s))^2 ds + q(t)\mathbb{G}(u(t))$$
$$+\int_{t_0}^{t} Q(s, u(s))u'(s)ds - \int_{t_0}^{t} P(s, u(s), u'(s))u'(s)ds \leq \int_{t_0}^{t} u'(s)\varphi(s)ds \tag{4.750}$$

Making use of absolute and using conditions (i) and (ii) and setting $q(t)|\mathbb{G}(u(t))| \geq |u(t)|$ and $|u'(t)| \leq \lambda$, we get

$$|u(t)| \leq (\lambda + nf(\xi)L^2 + |u'''(\eta)|L + 1)\lambda\int_{t_0}^{t}\varphi(s)ds$$
$$+\lambda\int_{t_0}^{t} p(s)\gamma(u(s))ds + \lambda^{n+1}\int_{t_0}^{t}\phi(t)\varpi(|u(t)|)ds \tag{4.751}$$

By applying Corollary 3.1 we obtain

$$|u(t)| \leq (\lambda + nf(\xi)L^2 + |u'''(\eta)|L + 1)\lambda\int_{t_0}^{t}\varphi(s)ds$$
$$\Omega^{-1}\left(\Omega(1) + \lambda^{n+1}\int_{t_0}^{t}\phi(s)\varpi\left(F^{-1}\left(F(1)\right.\right.\right. \tag{4.752}$$
$$\left.\left.\left.+\lambda\int_{t_0}^{t} p(\delta)d\delta\right)\right)ds\right)F^{-1}\left(F(1) + \lambda\int_{t_0}^{t} p(s)ds\right) \quad t \in \mathbf{I}$$

Using the conditions (iii) and (iv) we obtain

$$|u(t)| \leq (\lambda + nf(\xi)L^2 + |u'''(\eta)|L + 1)$$
$$\Omega^{-1}\left(\Omega(1) + \lambda^{n+1}k_1\varpi\left(F^{-1}\left(F(1) + \lambda k_2\right)\right)ds\right)F^{-1}\left(F(1) + \lambda k_2\right)\varphi(t) \tag{4.753}$$

Therefore, Hyers-Ulam-Rassias constant is

$$C_\varphi = (\lambda + nf(\xi)L^2 + |u'''(\eta)|L + 1)\lambda$$

$$\Omega^{-1}\left(\Omega(1) + \lambda^{n+1}k_1\varpi\left(F^{-1}\left(F(1)+\lambda k_2\right)\right)ds\right)F^{-1}\left(F(1)+\lambda k_2\right)$$

Lastly, we consider Hyers-Ulam-Rassias stability of equation

$$u'''(t) + r(t)u''(t) + nf(t)u'(t) + q(t)u(t) + Q(t, u(t)) = 0 \tag{4.754}$$

with initial conditions $u(t_0) = u'(t_0) = u"(t_0) = 0$.

Theorem 4.68:

Equation (4.754) together with initial conditions be Hyers-Ulam-Rassias stable. If the conditions of Theorem 4.66 remained valid and given Hyers-Ulam-Rassias constant as

$$\begin{gathered} C_\varphi = (\lambda + nf(\xi)L^2 + |u'''(\eta)|L + r(\alpha)N + 1)\lambda \\ \Omega^{-1}\left(\Omega(1) + \lambda^{n+1}k_1\varpi\left(F^{-1}\left(F(1)+\lambda k_2\right)\right)ds\right)F^{-1}\left(F(1)+\lambda k_2\right) \end{gathered} \tag{4.755}$$

Proof:

From equation (4.737), if $P(t, u(t), u'(t)) = 0$, we have

$$-\varphi(t) \le u'''(t) + r(t)u''(t) + nf(t)u'(t) + q(t)u(t) + Q(t, u(t)) \le \varphi(t) \tag{4.756}$$

Multiplying by $u'(t)$, to have

$$\begin{aligned} -u'(t)\varphi(t) \le u'''(t) + r(t)u''(t)u'(t) + nf(t)(u'(t))^2 + q(t)u(t) \\ +Q(t, u(t))u'(t) \le u'(t)\varphi(t) \end{aligned} \tag{4.757}$$

Multiplying by $\frac{1}{t}$ for $t > 0$ and using equation (4.20) we get

$$\begin{aligned} &\int_{t_0}^t u'''(s)u'(s)ds + \int_{t_0}^t r(t)u''(s)u'(s)ds + n\int_{t_0}^t f(s)(u'(s))^2 ds \\ &+\int_{t_0}^t q(s)\frac{d}{ds}\mathbb{G}(u(s))ds + \int_{t_0}^t Q(s, u(s))u'(s)ds \le \int_{t_0}^t u'(s)\varphi(s)ds \end{aligned} \tag{4.758}$$

Integrating (4.758) by part and applying Theorem 1.1, there exist

$t_0 \le \eta, \xi, \alpha \le t$ such that

$$\begin{aligned} u'''(\eta)\int_{t_0}^t u'(s)ds + r(\alpha)u'(t)\frac{1}{t}\int_{t_0}^t u''(s)ds + nf(\xi)\int_{t_0}^t (u'(s))^2 ds \\ +q(t)\mathbb{G}(u(t)) + \int_{t_0}^t Q(s, u(s))u'(s)ds \le \int_{t_0}^t u'(s)\varphi(s)ds \end{aligned} \tag{4.759}$$

By taking the absolute value of both sides, by conditions(i) and (ii) and setting $\frac{1}{t}q(t)|\mathbb{G}(u(t))| \ge |u(t)|$, and $|u'(t)| \le \lambda$, we have

$$\begin{aligned} |u(t)| \le (\lambda + nf(\xi)L^2 + |u'''(\eta)|L + r(\alpha)N + 1)\lambda\int_{t_0}^t \varphi(s)ds \\ +\lambda\frac{1}{t}\int_{t_0}^t p(s)\gamma(u(s))ds \end{aligned} \tag{4.760}$$

By applying Theorem 2.9 we obtain

$$|u(t)| \leq (\lambda + nf(\xi)L^2 + |u'''(\eta)|L + r(\alpha)N + 1)\lambda \int_{t_0}^{t} \varphi(s)ds \\ \Omega^{-1}\left(\Omega(1) + \lambda \int_{t_0}^{t} p(s)\right) \quad t \in \mathbf{I} \tag{4.761}$$

Using the conditions (iii) and (iv), we obtain

$$|u(t)| \leq (\lambda + nf(\xi)L^2 + |u'''(\eta)|L + r(\alpha)N + 1)\lambda\varphi(t) \\ \Omega^{-1}\left(\Omega(1) + k_1\lambda\right), \quad t \in \mathbf{I}. \tag{4.762}$$

Therefore, Hyers-Ulam-Rassias constant is

$$C_{\varphi} = (\lambda^2 + nf(\xi)L^2 + |u'''(\eta)|L + r(\alpha)\lambda N + \lambda)\Omega^{-1}\left(\Omega(1) + k_1\lambda\right). \tag{4.763}$$

Finally, all the results obtained in chapter four extended the results of the following researchers Qarawani (2012), Algfiary and Jung (2014). Algfiary and Jung (2014) used Gronwall lemma while we used Gronwall-Bellman-Bihari to address more difficult situations than them.

CHAPTER FIVE

SUMMARY, CONCLUSION AND RECOMMENDATIONS

5.1 Summary

In this work, it has been shown that integral inequalities in chapter three played an important role in investigating stability of nonlinear differential equations of second and third order in the sense of Hyers-Ulam and Hyers-lamU-Rassias stability. First, second, third and nth order linear differential equations were reasonably considered by some others researchers, while the situation is not the same on the area of nonlinear differential equation because of methods employed by these researchers were inadequate to examine the equations which were more advanced. Considering the results of this work, it observed that they extend the ones existing in the literature. Development of Gronwall-Bellman-Bihari type inequality has tremendously helped us to overcome most of the problems encountered by authors during investigation of stability of second and third order nonlinear differential equation.

5.2 Conclusion and Recommendations

Gronwall-Bellman-Bihari type inequalities are majorly used to achieve Hyers-Ulam and Hyers-Ulam-Rassias stability of all perturbed and nonperturbed nonlinear second and third order differential equation whose stability are not common in the literature. Several second and third order nonlinear ordinary differential equations such as: damped equation, Euler type equation, Lienard equation and an host of order, were considered in this researched work without encountering with any problem and attained the desired results. It is therefore suggested that this kind of method employed to consider stability need to be embraced by researchers in order to establish the stability of many perturbed nonlinear ordinary equations that may appear in setting of model for biological situation of a nation. This can also be used to ascertain the stability of mathematical models of most dynamic processes in engineering, physical and biological sciences which often conveniently expressed by nonlinear ordinary differential equation before it is used.

5.3 Further Research

Researches are still on going on the Hyers-Ulam and Hyers-Ulam-Rassias stability via Gronwall-Bellman-Bihari type inequalities of second, third and higher order of perturbed nonlinear differential equation, delay differential equation, fractional differential and stochastic differential equation.

5.4 Contributions to Knowledge

Based on this research work the following are the additional knowledge to the existing ones:

(1) This work has opened ways for developing the different extensions of Gronwall-Bellman-Bihari type inequalities and establishing more of its applications in the field of mathematics.

(2) It enhances the investigation of stability of perturbed nonlinear second and third order differential equation by means of Hyers-Ulam and Hyers-Ulam-Rassias stabilities.

(3) Variants of nonlinear second and third order ordinary differential equation which stabilities are seemed to be difficult to consider are dealt with by methods established in this research work.

(4) Techniques to obtain Hyers-Ulam and Hyers-Ulam-Rassias constants which are not common are employed to confirm the veracity of the stability which is second and third order differential equation.

(5) Appropriate Gronwall-Bellman-Bihari type inequality can be developed to achieve the stability of the other nonlinear differential equation in the sense of Hyers-Ulam and Hyers-Ulam-Rassias stability.

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