# CHARACTERISING SMOOTHNESS OF TYPE A SCHUBERT VARIETY USING PALINDROMIC POINCARÉ POLYNOMIAL AND PLÜCKER COORDINATE METHODS.

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BY

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# Certification

I certify that this work was carried out by P. Afinotan in the Department of Mathematics, University of Ibadan.

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# Dedication

This thesis is dedicated to God Almighty, for his infinite mercies, wisdom and guardiance through out this research work.

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AFINOTAN Patience.

# Abstract

Schubert varieties are subvarieties of the flag variety $\mathcal{F}\ell_n(\mathbb{C})$, a smooth complex projective variety consisting of sequences of sublinear subspaces of an n-dimensional complex vector space, ordered by inclusion. They are indexed by permutation matrices and studied in various types with important roles in algebraic geometry due to their combinatorial structures. The smoothness and singularity of Schubert variety have been characterised by various methods using the elements of the n-dimensional symmetric group. However, characterising smoothness using the exponents of the monomial of the Schubert variety and Plücker coordinate which uniquely and clearly identifies the symmetry of the Poincaré polynomial have not been established. Hence this research aims at establishing smoothness and singularity of type A Schubert varieties using the exponents of the monomials of the Schubert variety and the Jacobian criterion on the equations of the ideals of the Schubert variety obtained via the Plücker embedding.

For the Schubert varieties $X_\sigma$, the cohomology of the flag varieties $f : H_{n-k}(\mathcal{F}\ell_n(\mathbb{C});\mathbb{Z}) \to H^k(\mathcal{F}\ell_n(\mathbb{C});\mathbb{Z})$ defined by $f[X_\sigma] = [X^\sigma] \in H^k(\mathcal{F}\ell_n(\mathbb{C}))$ was considered, to obtain its monomials. The Poincaré polynomial was determined in order to compute the symmetry of the Schubert varieties. The flag varieties are embedded into the product of Grassmanians which is also embedded into the product of projective spaces given by the embedding map $\mathcal{F}\ell_n(\mathbb{C}) = X_\sigma \hookrightarrow \prod_{k=1}^{n-1} Gr(k,n) \hookrightarrow \prod_{k=1}^{n-1} \mathbb{P}^{\begin{pmatrix} n \\ k \end{pmatrix}-1}$. defined by $A \mapsto [P_{12}, P_{13}, \cdots, P_{(n-1)n}]$ , with $P_{ij}, 1 \leq i < j \leq n$ being the $\begin{pmatrix} n \\ k \end{pmatrix}$ minors for $A_{k,n}$ in $Gr(k,n)$. The equations of the ideal of the Schubert varieties were obtained by taking all the minors of the matrix Schubert varieties. The rank of the Jacobian matrix and the co-dimension of the Schubert varieties were determined.

The Schubert classes forms additive $\mathbb{Z}$ basis that generates the cohomology ring $H^k(\mathcal{F}\ell_4(\mathbb{C});\mathbb{Z})$. The basis for the cohomology ring are the geometric and algebraic basis. The algebraic basic classes $x_1^{i_1} x_2^{i_2} \cdots, x_m^{i_m}$ with exponents $i_j = m - j$ forms $Z$ basis for the cohomology ring and these basic classes are the monomials. The

Poincaré polynomial $P_\sigma(t) = \sum_{v \leq \sigma} t^{l(v)}$ , defined with respect to the length function and via the Bruhat order, $v \leq \sigma \implies l(v) \leq l(\sigma)$ shows that the symmetry $P_\sigma(t) = t^r P_\sigma(t^{-1})$ Of the Poincaré polynomial is palindromic or not palindromic. The rank of the Jacobian matrix obtained using the equations of the ideal $I(X_\sigma)$ derived through the embedding map is found to be equal to the co-dimension of the varieties which indicates smoothness.

The exponent of the monomials $x_1^{i_1} x_2^{i_2} \cdots x_m^{i_m}$ of the Schubert variety $X_\sigma$ have uniquely satisfied the symmetry of its Poincaré polynomial for smooth Schubert varieties and have successfully extended the underlying group from $S_n$ to $Z_+^n$. Smoothness has successfully been generalised in terms of the differential equations using the equations defining the ideals, of the Schubert varieties through the Plücker coordinates.

Keywords: Flag varieties, Cohomology, Bruhat order, Monomials exponent.

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# Chapter 1
# INTRODUCTION

## 1.1 Background of the study

Schubert varieties are certain subvarieties of Grassmann varieties. They are usually singular points. We review and extend smooothness of type $A$ Schubert varieties in terms of the exponents of the monomials of the varieties and the equations defining the ideals of the Schubert varieties by means of the Poincaré palindromic polynomials and the Plücker coordinate methods.

Schubert varieties are both combinatorial and algebraic varieties. Combinatorial varieties are the different arrangements of different objects that gives a set of solutions while algebraic varieties are sets of solutions of a polynomial equation over the real or complex numbers. They are subvarieties of the flag varieties and are studied in various types by the means of linear algebra with the $Gl(n, \mathbb{C})$ as the underlying group for type A Schubert varieties. The flag varieties are $G$-varieties, due to their transitive group actions. They are also seen as homogenoeus and compact homogenoeus spaces because they can be identified with the quotient group $G/B$ and $Gl(n, \mathbb{C})$ which is locally compact contains compact subgroups such as $U(n, \mathbb{C})$ that also act transitively on the flag varieties by left multiplication, giving the dimension of the flag to be $\frac{n(n-1)}{2}$.

The flag varieties are also seen from the angle of the $T$-fixed points. These are $n$ factorial flags associated to permutation matrices. The elements of $\mathcal{F}\ell_n(\mathbb{C})^T$ embeds in $\mathcal{F}\ell_n(\mathbb{C})$ as the set of the T-fixed points, $\mathcal{F}\ell_n(\mathbb{C})^T \cong W \cong S_n$ . The elements of $W$ index $B$-orbits $n!$ flag varieties $G/B$ and together they form the Bruhat decomposition Theorem. The flag varieties are partitioned into cells arising from double Cosets, that is ,

$$\mathcal{F}\ell_n(\mathbb{C}) = G/B = \coprod_{\sigma \in S_n} B\sigma B/B = \coprod_{\sigma \in S_n} C_\sigma. \tag{1.1}$$

called the Bruhat cells (Schubert cells) that is isomorphic to affine space of dimension $l(\sigma)$. The closure of these cells is called the Schubert variety. The classes of the closure forms additive $\mathbb{Z}$ basis that generates the cohomology ring with basis classes called the Schubert classes. The basis for the cohomology ring are the geometric

and algebraic basis. The algebraic classes are the monomials. The exponents of these monomials and the equations used to define the ideals are then used to show smoothness of the varieties respectively.
Chapter two contains the basic definitions needed to aid proper understanding of this work. It also provides some conceptual reviews of the connected literatures in the area which helps to establish a mathematical background for understanding the concepts of smoothness of the Schubert varieties. Chapter three discusses the methodology adopted from the literatures in establishing smoothness. The applications of the methodology to the exponents of the monomials of the Schubert varieties and the equations defining the ideals of the Schubert varieties provides answers to the research problems.

## 1.2 Statement of the Problem

Lakshmibai & Seshadri (1984) showed that $X_\sigma$ is smooth at $v \in S_n$ if and only if $\dim T_v(X_\sigma) := \sharp\{(i < j) : vt_{ij} \leq \sigma\} = l(\sigma)$ which is also equivalent to $\sharp\{(i < j) : v < vt_{ij} \leq \sigma\} = l(\sigma) - l(v)$. This gave rise to the Theorem of Lakshmibai & Seshadri (1984) that for $v \leq \sigma \in S_n$, the tangent space of $X_\sigma$ at $v$ is given by $\dim T_v(X_\sigma) = \sharp\{(i < j) : vt_{ij} \leq \sigma\}$.

Lakshmibai & Sandhya (1990) gave a criterion for a $X_\sigma$ to be singular, they stated that $X_\sigma$ is singular iff $\sigma$ contains the 3412 or 4231 permutation pattern otherwise it is smooth. Also Carrell (1994) gave a criterion for computing the smooth and singular Schubert varieties in terms of any permutation $\sigma \in S_n$, then $X_\sigma$ is smooth if the Poincaré polynomial is palindromic.

We show smoothness and singularity of type $A$, Schubert varieties using the exponents of the monomials of the Schubert varieties. The problem is presented in the following Theorem:

**Theorem 1.2.1.** *Let $\sigma \in \mathbb{Z}_+^n$ be the monomial exponent of the $X_\sigma$, then the following are equivalent:*

1. *The Schubert variety $X_\sigma$ is rationally smooth at every point (since smoothness in type $A$ is equivalent to rational smoothness).*
2. *The Poincaré polynomial $P_\sigma(t)$ is Palindromic (Symmetric) .*
3. *The Bruhat graph $\Gamma(id, \sigma)$ is regular, that is every vertex has the same number of edges, $l(\sigma)$.*

Smoothness and singularities of Schubert varieties are determined in type A, by means of the Jacobian criteria on the defining equations of the ideal of the Schubert varieties. as given in the following Theorem:

**Theorem 1.2.2.** *Let $S_n$ be the symmetric group of $n$ letters with $\sigma, v \in S_n$ such that $\sigma$ is of maximal length. Then the Schubert variety $X_\sigma$ is smooth if*

$$R(J(I(X\sigma))) = l(\sigma) - l(v). \tag{1.2}$$

## 1.3 Aims

Lakshmibai & Seshadri (1984), determined the singularity of Schubert varieties by computing the set of points for which the Schubert varieties are singular. Smoothness and singularity of Schubert varieties were computed by Lakshmibai & Sandhya (1990) using permutation pattern avoidance, for the elements of the symmetric group. They described this as the 4231 and 3412 permutation pattern avoidance. Carrell (1994) described smoothness and singularity of Schubert varieties through the Poincaré polynomials of the Schubert varieties. He stated that the Schubert varieties are smooth iff their Poincaré polynomials are Palindromic. Oh et al. (2008) worked on the fact that $P_\sigma(q) = R_\sigma(q)$ iff the Schubert variety $X_\sigma$ is smooth and also Woo & Yong (2008) formulated a new combinatorial notion which generalised pattern avoidance and it was called the interval pattern avoidance,

The main purpose for this study is to evaluate smoothness and singularity of Schubert varieties using the exponents of the monomials and the equations defining the ideal of the Schubert varieties.

## 1.4 Objectives of the Study

The objectives of this study are to:

- evaluate smoothness and singularity of Schubert varieties using the exponents of the monomials of $X_\sigma$.
- establish that the equation defining the ideal of $X_\sigma$ is always smooth at the identity.
- characterise singularity of Schubert varieties using the equations defining the ideal of the Schubert varieties.
- compare the defining equations for the ideal of the Schubert varieties, $(X_\sigma)$ with the equation of the ideal obtained through the essential set for $X_\sigma$.

## 1.5 Motivation of the Study

Motivated by the results of Lakshmibai & Seshadri (1984) , Carrell (1994), and the recent work of Oh et al. (2008), it is natural to ask the following questions:

*How do we characterise smoothness of type A Schubert varieties using the:*

- *exponents of the monomials of the Schubert varieties*?
- *equations defining the ideal of the Schubert varieties*?

## 1.6 Justification

Schubert varieties are singular algebraic varieties. They are subvarieties of the smooth complex projective varieties consisting of sequences of an $n$-dimensional complex vector space ordered by inclusion. The smoothness and singularities of Schubert varieties have been characterised by various methods using the elements of the n-dimensional symmetric group.

However, characterising smoothness using the exponents of the monomials of the Schubert varieties have not been given much attention by authors in this area of research. Hence, this work establishes smoothness of Schubert varieties using the exponents of the monomials of the Schubert varieties. This extend the result of Carrell (1994) to the positive finite intergers.

The smoothness of type A Schubert varieties using the defining equations of the ideal of the Schubert varieties is seen to be equivalent to smoothness in differential equations. The present study has applications in the area of graph theory, networking, permutation patterns and reduced words .

## 1.7 Significance of the Study

This research work gives details on the smoothness and singularity of Schubert varieties. It reviews and extends the work of Carrell (1994). In addition it shows that smoothness in algebraic geometry is same as that of differential equations.

## 1.8 Scope of Coverage

This work comprises of many aspects of group theory, linear algebra, topology, representation theory and algebraic geometry among others.

## 1.9 Organisation of the Thesis

This research work is organised as follows: Chapter One contains a comprehensive and general introductory perception to the main work in type A. In the same chapter the motivation for undertaking this work is stated, the aims, objectives and the problems that we intend to provide answers to are provided. Chapter Two centers mainly on the basic definitions and general review of the literature materials based on the concept of our interest.

In Chapter Three the methodology used to carry out the research is described. While Chapter Four discusses the main results obtained. Chapter Five contains the summary of findings, conclusion, contributions to knowledge and area of further work.

# Chapter 2
# LITERATURE REVIEW

## 2.1 Preamble

This chapter reviews various concepts and results that are found in the literatures needed in this area of research.

## 2.2 Flag Varieties

Schubert varieties are combinatorial subvarieties of the flag varieties, hence we begin this session by considering flag varieties and their properties.

**Definition 2.2.1.** *Let $V = \mathbb{C}^n$, which denotes a complex vector space of dimension $n$, A flag $V_\bullet$ in $\mathbb{C}^n$ is a sequence of ordered subspaces,*

$$V_\bullet : V_0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_n = V \tag{2.1}$$

$\ni dim_{\mathbb{C}} V_i = i$ *where* $0 \leq i \leq n$.

**Remark 2.2.2.** *The set of all such flags forms a smooth complex projective variety called the full flag variety denoted by $\mathcal{F}\ell_n(\mathbb{C})$.*

**Remark 2.2.3.** *The flag varieties are smooth complex projective varieties because they can be embedded into the products of the grassmannians which are embedded into the products of higher dimensional projective spaces by means of the plücker embedding map.*

$$\mathcal{F}\ell_n(\mathbb{C}) \hookrightarrow \prod_{k=1}^{n-1} Gr(k,n) \hookrightarrow \prod_{k=1}^{n-1} \mathbb{P}^{\binom{n}{k}-1}. \tag{2.2}$$

**Definition 2.2.4.** *Let*

$$V_\bullet : V_0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_n = \mathbb{C}^n \tag{2.3}$$

*then the standard basis for* $V = \langle e_1, e_2, \cdots, e_n \rangle$ *and the standard flag for the flag* $V_\bullet \in V$ *is given by*

$$V_\bullet = \{\} \subsetneq \langle e_1 \rangle \subsetneq \langle e_1, e_2 \rangle \subsetneq \langle e_1, e_2, e_3 \rangle \subsetneq \cdots \subsetneq \langle e_1, e_2, e_3, \cdots, e_n \rangle . \tag{2.4}$$

### 2.2.1 Algebraic Description of a Flag

Let $G = Gl(n, \mathbb{C}) = \{M_{n\times n} \in \mathbb{C}^n\}$ be non singular.

Given a flag

$$V_\bullet : V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_n = \mathbb{C}^n \tag{2.5}$$

where $V_1$ is a line spanned by a vector, $V_2$ is a plane containing a line and so on, hence (2.5) is spanned by the vectors

$$\langle g_1 \rangle \subsetneq \langle g_1, g_2 \rangle \subsetneq \langle g_1, g_2, g_3 \rangle \subsetneq \cdots \langle g_1, g_2, g_3, \cdots, g_n \rangle. \tag{2.6}$$

The matrix $(g = g_1, g_2, g_3, \cdots, g_n)$ represents a flag.

$\mathcal{F}\ell_n(\mathbb{C})$ is described algebraically by considering $G = GL_n(\mathbb{C})$, and $B$ the Borel subgroup of $G$, with

$B = \{a_{ij} \in GL_n(\mathbb{C}) \ni a_{ij} = 0, i > j\}$.

$\mathcal{F}\ell_n(\mathbb{C})$ are $G$-varieties, since they admits a transitive group action of $GL_n(\mathbb{C})$, $G$ acts transitively on the set of all flags by left multiplication.

$$GL_n(\mathbb{C}) \times \mathcal{F}\ell_n(\mathbb{C}) \to \mathcal{F}\ell_n(\mathbb{C}) \tag{2.7}$$

defined by

$$(g, V_\bullet) \mapsto gV_\bullet = V'_\bullet. \tag{2.8}$$

### 2.2.2 The Flag Satisfies the Properties of an Equivalence Relation

Let $G$ be a group with identity $e$ and $\mathcal{F}\ell_n(\mathbb{C})$ be the set of all flags. Let $\star : G \times \mathcal{F}\ell_n(\mathbb{C}) \to \mathcal{F}\ell_n(\mathbb{C})$ be a group action. Let $R_G$ be the relation induced by $G$ that is $V_\bullet R_G V'_\bullet$ implies $V'_\bullet \in Orb(V_\bullet)$ where $Orb(V_\bullet)$ denotes the Orbit of $V_\bullet \in \mathcal{F}\ell_n(\mathbb{C})$

Then $R_G$ is an equivalence relation .

To show equivalence relation, we must show that the relation $R_G$ is reflexive, symmetric and transitive.

Let $V_\bullet R_G V_\bullet'$ implies $V_\bullet' \in Orb(V_\bullet)$ where the $Orbit(V_\bullet) = \{gV_\bullet : g \in Gl_n(\mathbb{C})\}$
$V_\bullet' \in Orb(V_\bullet)$ implies $gV_\bullet = V_\bullet' \forall g \in Gl_n(\mathbb{C})$

Reflexive property:
$V_\bullet = V_o \star V_\bullet$ implies $V_\bullet \in Orb(V_\bullet)$. Therefore $R_G$ is reflexive.

Symmetric Property:
$V_\bullet' \in Orb(V_\bullet)$ implies there exist a $g \in G : V_\bullet' = g \star V_\bullet$
implies that $g^{-1} \star (g \star V_\bullet) = g^{-1} \star V_\bullet'$, therefore $V_\bullet = g^{-1} \star V_\bullet'$
there exist a $g^{-1} \in G : V_\bullet = g^{-1} \star V_\bullet'$ which implies $V_\bullet \in Orb(V_\bullet')$
Therfore $R_G$ is symmetric

Transitive property:
$V_\bullet' \in Orb(V_\bullet)$ and $V_\bullet'' \in Orb(V_\bullet')$ then there exist $g_1 \in G : V_\bullet' = g_1 V_\bullet$
and $g_2 \in G : V_\bullet'' = g_1 V_\bullet'$
$V_\bullet'' = g_2 \star (g_1 V_\bullet)$ and $V_\bullet'' = (g_2 g_1) \star V_\bullet)$ , which implies $V_\bullet'' \in Orb(V_\bullet)$
Thus $R_G$ is transitive.
Hence, the relation is equivalence.

### 2.2.3 Flag Varieties as a Homogeneous Space

$Gl_n(\mathbb{C})$ acts transitively on the set of all flags $\mathcal{F}\ell_n(\mathbb{C})$ and $B$ is the Borel subgroup of $G$, the stabilizer of the standard flag. $B$ is the subset of an $n \times n$ non singular upper triangular matrices, $gB$ gives the same flag as $g$. Hence, the flag variety

$$\mathcal{F}\ell_n(\mathbb{C}) = Gl_n(\mathbb{C})/B = \{gB : g \in G\} \tag{2.9}$$

where each flag is a coset of the right action of $B$ on $G$.

The flag varieties are seen to be associated to $G/B$, hence it is a homogeneous space, since for any $V_\bullet \in \mathcal{F}\ell_n(\mathbb{C})$ and $g \in Gl_n(\mathbb{C}) \ni gV_\bullet = V_\bullet' \in \mathcal{F}\ell_n(\mathbb{C})$.

### 2.2.4 Flag Varieties as a Compact Homogeneous Space

The flag varieties $\mathcal{F}\ell_n(\mathbb{C})$ can also be seen as a compact homogeneous space, since there is an action of the closed compact subgroup of $Gl_n(\mathbb{C})$ which is the Unitary group $U_n(\mathbb{C})$ on $\mathcal{F}\ell_n(\mathbb{C})$.

The general linear group which is locally compact, contains compact subgroups such as the unitary group, given by

$$U_n(\mathbb{C}) = \{A \in Gl_n(\mathbb{C}) : AA^* = I_n\} \tag{2.10}$$

with $T$ (Toroidal group) as the stabilizer of points. The unitary group acts transitively on the flag,

$$\mathcal{F}\ell_n(\mathbb{C}) = U_n(\mathbb{C})/T^n \tag{2.11}$$

and this action results in $\mathcal{F}\ell_n(\mathbb{C})$ becoming a compact homogeneous space with dimension $\frac{n(n-1)}{2}$.

## 2.3 $T$-Fixed Points

In this session we define the flag varieties in terms of the $T$-fixed points and also show that the elements of $W \cong S_n$ index $B$-orbit $n!$ flag varieties and together they form the Bruhat decomposition theorem .

**Definition 2.3.1.** ***($T$-fixed points)*** *The $T$-fixed points are flags associated to permutation matrices.*

**Definition 2.3.2.** *Given that $\sigma$ is a permutation in $S_n$, then the $T$-fixed points of the flag $V_\bullet$ is*

$$V_\bullet^\sigma = \langle e_{\sigma(1)} \rangle \subset \langle e_{\sigma(1)} e_{\sigma(2)} \rangle \subset \cdots \langle e_{\sigma(1)} e_{\sigma(2)} \cdots e_{\sigma(n)} \rangle \tag{2.12}$$

*defined by*

$$V_\bullet^\sigma \mapsto \sigma B = \{\sigma B : \sigma \in G\}, \tag{2.13}$$

*where $\sigma$ is a permutation matrix. There are $n!$ of these permutation matrices.*

**Example 2.3.3.** *Let $\sigma = 2413$ where $\sigma \in S_n$. The $T$-fixed point of the flag $V_0^\sigma$ where $\sigma = 2413$ is*

$$V_\bullet^\sigma = \langle e_{\sigma(2)} \rangle \subset \langle e_{\sigma(2)} e_{\sigma(4)} \rangle \subset \langle e_{\sigma(2)} e_{\sigma(4)} e_{\sigma(1)} \rangle \subset \langle e_{\sigma(2)} e_{\sigma(4)} e_{\sigma(1)} e_{\sigma(3)} \rangle. \tag{2.14}$$

**Remark 2.3.4.** *The elements of $\mathcal{F}\ell_n(\mathbb{C})^T$ embeds in $\mathcal{F}\ell_n(\mathbb{C})$ as the set of the $T$-fixed points. $\mathcal{F}\ell_n(\mathbb{C})^T \cong W \cong S_n$ where $W = N_G(T)$ is the normalizer of $T$ on $G$*

*and $N_G(T)/T$ consist of the monomial matrices with only one non-zero entry in each row and each column.*

The elements of $W$ index $B$-orbits $n!$ flag variety $G/B$ and together they form the Bruhat decomposition theorem.

**Theorem 2.3.5.** *[Curtis (1964)]*

*The general linear group $G = Gl_n(\mathbb{C})$ is a disjoint union $G = \coprod_{\sigma \in W} B\sigma B$.*

The flag varieties are partitioned into cells arising from double Cosets, that is

$$\mathcal{F}\ell_n(\mathbb{C}) = G/B = \coprod_{\sigma \in S_n} B\sigma B/B = \coprod_{\sigma \in S_n} C_\sigma \tag{2.15}$$

called the Bruhat cell. Each Bruhat cell $C_\sigma \cong \mathbb{C}^{l(\sigma)}$ where $\mathbb{C}^{l(\sigma)}$ is the affine space and $l(\sigma)$ is the length of $\sigma$. The length of $\sigma$ is given by the number of inversions.

**Definition 2.3.6.** *The **inversion number** of $\sigma$ is a pair*

$$(i, j) = \sharp\{1 \leq i < j \leq n \ni \sigma(i) > \sigma^{-1}(j)\}. \tag{2.16}$$

## 2.4 Schubert Varieties

This session comprises of the defintions and examples of the Schubert cell, Schubert varieties and their duals. It also gives the properties of the Schubert varieties.

**Definition 2.4.1.** ***Schubert cell***

*The Schubert cell is defined by*

$$C_\sigma = \{g \in G : pos(g) = \sigma\}. \tag{2.17}$$

.

**Definition 2.4.2.** *The geometric definition of the Schubert cell $C_\sigma$ is given by*

$$C_\sigma = \{V_0 \in \mathcal{F}\ell_n(\mathbb{C}) \mid dim(W_p \bigcap V_q) = r_\sigma(p, q), 1 \leq p, q \geq n\}. \tag{2.18}$$

$$\{V_0 \in \mathcal{F}\ell_n(\mathbb{C}) \mid dim(W_p \bigcap V_q) = \sharp\{i \leq p : \sigma(i) \leq q\} for 1 \leq p, q \leq n\}. \tag{2.19}$$

Let $\sigma = 3425167$ where $\sigma(1) = 3, \sigma(2) = 4, \sigma(3) = 2, \sigma(4) = 5, \sigma(5) = 1, \sigma(6) = 6, \sigma(7) = 7$ the Schubert cell is given by the matrix ,

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} * & * & 1 & 0 & 0 & 0 & 0 \\ * & * & 0 & 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The diagram above is the diagram of the $C_\sigma$ for $\sigma = 3425167$. where a hook of zero's are drawn downwards and to the left of a 1-entry. The number of stars gives the length of $\sigma$.

Therefore $\sigma = 3425167, L(3425167) = 6. C_\sigma \cong \mathbb{C}^{l(\sigma)}, C_{3425167} \cong \mathbb{C}^6$.

**Definition 2.4.3.** *The opposite Schubert cell denoted by $C^\sigma$ is given by*

$$C^\sigma = B^- \sigma B/B \tag{2.20}$$

*where $B^-$ is the subgroup of lower triangular matrices.*

Let $\sigma = 3425167$ where $\sigma(1) = 3$, $\sigma(2) = 4$, $\sigma(3) = 2$, $\sigma(4) = 5$, $\sigma(5) = 1$, $\sigma(6) = 6$, $\sigma(7) = 7$ the opposite Schubert cell is given by the matrix,

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & * & * & * & * \\ 0 & 0 & 0 & 1 & * & * & * \\ 0 & 1 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * \\ 1 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The diagram above gives the opposite $C_\sigma$ for $\sigma = 3425167$ . where a hook of zero's are drawn downwards and to the right of a 1-entry and the number of stars gives the length of $\sigma$.

Therefore $\sigma = 3425167, L(3425167) = 15. C_\sigma \cong \mathbb{C}^{l(\sigma)}, C_{3425167} \cong \mathbb{C}^{15}$ .

**Definition 2.4.4.** ***Schubert varieties***

*The Schubert varieties are the closure of the Schubert cells , they are denoted* *by*

$$X_\sigma = \bar{C_\sigma} = \bigcup_{v \leq \sigma} C_v. \tag{2.21}$$

*where* $v \leq \sigma$ *and the* $l(v) \leq l(\sigma)$ *.*

**Definition 2.4.5.** ***Dual Schubert varieties***

*The dual Schubert varieties are the closures of the dual Schubert cells and they are given by*

$$X^\sigma = \bar{C^\sigma} = \bigcup_{v \geq \sigma} C^v \tag{2.22}$$

*where the* $l(v) \geq l(\sigma)$*.*

**Remark 2.4.6.** *The Schubert varieties* $X_\sigma$ *and* $X^\sigma$ *are irreducible subvarieties of the flag varieties* $\mathcal{F}\ell_n(\mathbb{C})$ *of dimension* $l(\sigma)$ *and* $n - l(\sigma)$ *.*

**Lemma 2.4.7.** *[Fulton & Fulton (1997)]*

*The dimension of the flag variety is related to the dimension of* $X_\sigma$ *and* $X^\sigma$ *by* $dim(X_\sigma + X^\sigma) = dim\mathcal{F}\ell_n(\mathbb{C})$*.*

## 2.5 The Partial Flag Varieties

This session comprises of the definition of the partial flag varieties with examples. It gives details on the derivation of the equations defining the ideals of the Schubert varieties by means of the Plücker coordinates and the Plücker embedding map.

**Definition 2.5.1.** *A partial flag of type* $(i_1, i_2, \cdots, i_k)$ *in* $\mathbb{C}^n$ *is a sequence of ordered subspaces,*

$$\{\} \subsetneq V_{i_1} \subsetneq V_{i_2} \subsetneq \cdots \subsetneq V_{i_k} = \mathbb{C}^n \tag{2.23}$$

*such that the* $dimV_{ij} = i_j$*, where* $0 \leq j \leq k$*.*

**Remark 2.5.2.** *The set of all partial flags of type* $(i_1, i_2, \cdots, i_k) \in \mathbb{C}^n$ *forms a smooth compact complex algebraic varieties called the partial flag varieties denoted by.* $\mathcal{F}\ell(i_1, i_2, \cdots, i_k; \mathbb{C})$*.*

### 2.5.1 The Grassmannian Varieties

The Grassmann varieties are the set of all $k$-dimensional subspaces of an $n$-dimensional vector space $V$. They are denoted by $Gr(k, n)$.

**Remark 2.5.3.** • *The Grammann varieties has the structures of smooth projective varieties, homogeneous spaces and complex compact manifolds.*

• *The Grassmann varieties are algebraic varieties, identified with the $k$- dimensional projective space $\mathbb{P}(\wedge^k \vee)$.*

Let $\{v_1, v_2. \cdots, v_n\}$ be column vectors and let $U \subseteq V$ be the span of these columns vectors. Let $\{u_1, u_2, \cdots, u_k\} \in U$ be ordered basis for $U \in Gr(k,n)$. Thus $u_i$ can be written as a linear combination

$$u_i = \sum_{j=1}^{n} x_{ij} v_j. \tag{2.24}$$

This definition describes a $k \times n$ matrix $A$ of rank $k$ as,

$$\begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ x_{k1} & x_{k2} & \cdots & x_{kn} \end{bmatrix}. \tag{2.25}$$

For any sequence $I : 1 \leq i_1 \leq \cdots \leq i_k \leq n$ . The determinant of the maximal minor corresponding to columns in I is called the plucker coordinate $P_I$. The matrix A has maximal rank therefore, at least one of the coordinate is non-zero. Changing the basis of $U$ has the effect of multiplying $U$ on the left by a $k \times k$ non singular matrix say $B$ which implies each $P_I$ is multiplied by det(B). Therefore, we define a map ,

$$\pi : G(k,n) \to \mathbb{P}^{\begin{pmatrix} n \\ k \end{pmatrix}^{-1}} = \mathbb{P}^N \tag{2.26}$$

by sending $U$ to its collection of Plücker coordinates

$$\pi :< U > \to [P_{12\cdots k, \cdots,} P_I, \cdots]. \tag{2.27}$$

**Definition 2.5.4.** *The Plücker embedding is the map $\pi : Gr(k,n) \hookrightarrow \mathbb{P}^{\binom{n}{k}-1}$ defined by $A \mapsto [P_{1,2}, P_{1,3} \cdots P_{n-1,n}] = P \in \mathbb{P}^{n-1}$, with $P_{i,j},\ 1 \leq i < j \leq n$ are the $\binom{n}{k}$ minors for $M_{k,n}$ in $Gr(k,n)$.*

### 2.5.2 Equation Defining the Ideal of Schubert Varieties through the Plücker Embedding Map

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In this section the equation defining the ideal of the Schubert varieties embedded in the product of the Grassmannians and also in the product of the projective spaces is computed through the Plücker embedding map.

For $n = 3$, the Schubert variety $X_{321} = \mathcal{F}\ell_3(\mathbb{C})$, since the dimension is complete. The equation defining the ideal of the Schubert varieties embedded in the Grassmannians and also embedded in the product of the Projective space is computed as follows.

$$\mathcal{F}\ell_3(\mathbb{C}) = X_{321} \hookrightarrow \prod_{k=1}^{n-1} Gr(k,n) \hookrightarrow \prod_{k=1}^{n-1} \mathbb{P}^{\binom{n}{k}-1}. \tag{2.28}$$

$$\mathcal{F}\ell_3(\mathbb{C}) = X_{321} \hookrightarrow \prod_{k=1}^{3-1} Gr(k,3) \hookrightarrow \prod_{k=1}^{3-1} \mathbb{P}^{\binom{3}{k}-1}. \tag{2.29}$$

$$\mathcal{F}\ell_3(\mathbb{C}) = X_{321} \hookrightarrow \prod_{k=1}^{2} Gr(k,3) \hookrightarrow \prod_{k=1}^{2} \mathbb{P}^{\binom{3}{k}-1} = \mathbb{P}^2 \times \mathbb{P}^2. \tag{2.30}$$

Taking all the minors of the matrix Schubert variety, we have

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} \to [p_1 : p_2 : p_3 : p_{12} : p_{13} : p_{23}]. \tag{2.31}$$

where

$$\begin{gathered} p_1 = x_{11}, p_2 = x_{12}, p_3 = x_{13}, p_{12} = \det \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}, \\ p_{13} = \det \begin{pmatrix} x_{11} & x_{13} \\ x_{21} & x_{23} \end{pmatrix}, p_{23} = \det \begin{pmatrix} x_{12} & x_{13} \\ x_{22} & x_{23} \end{pmatrix}. \end{gathered} \tag{2.32}$$

Also by expressing the embedding of the matrix representation of $X_{321}$ as a product of the representation in $\prod_{k=1}^{2} Gr(k,3)$, we have $Gr(1,3)$ and $Gr(2,3)$

hence, we have the equation ,

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} \hookrightarrow \left( \begin{bmatrix} x_{11} & x_{12} & x_{13} \end{bmatrix}, \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{bmatrix} \right). \tag{2.33}$$

By appending the matrix $Gr(1, 3)$ and $Gr(2, 3)$, we obtain a $3 \times 3$ matrix with determinant variables as

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{bmatrix} = x_{11} \begin{vmatrix} x_{12} & x_{13} \\ x_{22} & x_{23} \end{vmatrix} - x_{12} \begin{vmatrix} x_{11} & x_{13} \\ x_{21} & x_{23} \end{vmatrix} + x_{13} \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix} = 0. \tag{2.34}$$

which is equal to

$$x_{11}(x_{12}x_{23} - x_{13}x_{22}) - x_{12}(x_{11}x_{23} - x_{13}x_{21}) + x_{13}(x_{11}x_{22} - x_{12}x_{21}) = 0. \tag{2.35}$$

Therefore, the equation defining the ideal of $X_{321}$ is

$$0 = p_1p_{23} - p_2p_{13} + p_3p_{12}. \tag{2.36}$$

For n = 4, the equation defining the ideal of the Schubert varieties embedded in the Grassmannians and also embedded in the product of the Projective spaces is computed as follows.

$$\mathcal{F}\ell_4(\mathbb{C}) = X_{4321} \hookrightarrow \prod_{k=1}^{n-1} Gr(k, n) \hookrightarrow \prod_{k=1}^{n-1} \mathbb{P}^{\binom{n}{k}-1}. \tag{2.37}$$

$$\mathcal{F}\ell_4(\mathbb{C}) = X_{4321} \hookrightarrow \prod_{k=1}^{4-1} Gr(k, 4) \hookrightarrow \prod_{k=1}^{4-1} \mathbb{P}^{\binom{4}{k}-1}. \tag{2.38}$$

$$\mathcal{F}\ell_4(\mathbb{C}) = X_{4321} \hookrightarrow \prod_{k=1}^{3} Gr(k, 4) \hookrightarrow \prod_{k=1}^{3} \mathbb{P}^{\binom{4}{k}-1} = \mathbb{P}^3 \times \mathbb{P}^5 \times \mathbb{P}^3. \tag{2.39}$$

Taking all the minors of the matrix Schubert variety, we have,

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix} \to \begin{matrix}[p_1 : p_2 : p_3 : p_4 : p_{12} : p_{13}; p_{14} : p_{23} \\ \\ : p_{24} : p_{34} : p_{123} : p_{124} : p_{134} : p_{234}]\end{matrix}. \tag{2.40}$$

where

$$\begin{gathered} p_1 = x_{11}, p_2 = x_{12}, p_3 = x_{13}, p_4 = x_{14}, p_{12} = \det \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}, \\ p_{13} = \det \begin{pmatrix} x_{11} & x_{13} \\ x_{21} & x_{23} \end{pmatrix}, p_{14} = \det \begin{pmatrix} x_{11} & x_{14} \\ x_{21} & x_{24} \end{pmatrix}, p_{23} = \det \begin{pmatrix} x_{12} & x_{13} \\ x_{22} & x_{23} \end{pmatrix}, \\ p_{24} = \det \begin{pmatrix} x_{12} & x_{14} \\ x_{22} & x_{24} \end{pmatrix}, p_{34} = \det \begin{pmatrix} x_{13} & x_{14} \\ x_{23} & x_{24} \end{pmatrix}, p_{123} = \det \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}, \\ p_{124} = \det \begin{pmatrix} x_{11} & x_{12} & x_{14} \\ x_{21} & x_{22} & x_{24} \\ x_{31} & x_{32} & x_{34} \end{pmatrix}, p_{234} = \det \begin{pmatrix} x_{12} & x_{13} & x_{14} \\ x_{22} & x_{23} & x_{24} \\ x_{32} & x_{33} & x_{34} \end{pmatrix} \end{gathered} \tag{2.41}$$

Also by expressing the embedding of the matrix representation of $X_{4321}$ as a product of the representation in $\prod_{k=1}^{3} Gr(k,4)$, we have $Gr(1,4), Gr(2,4), Gr(3,4)$, hence we have the equation

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix} \hookrightarrow \left( \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \end{bmatrix}, \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \end{bmatrix} \right). \tag{2.42}$$

By appending the matrix $Gr(1,4)$ to $Gr(3,4)$ we obtain a $4 \times 4$ matrix with determinant variables as

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \end{bmatrix} = x_{11} \begin{vmatrix} x_{12} & x_{13} & x_{14} \\ x_{22} & x_{23} & x_{24} \\ x_{32} & x_{33} & x_{34} \end{vmatrix} - x_{12} \begin{vmatrix} x_{11} & x_{13} & x_{14} \\ x_{21} & x_{23} & x_{24} \\ x_{31} & x_{33} & x_{34} \end{vmatrix} + . \tag{2.43}$$

$$x_{13} \begin{vmatrix} x_{11} & x_{12} & x_{14} \\ x_{21} & x_{22} & x_{24} \\ x_{31} & x_{32} & x_{34} \end{vmatrix} - x_{14} \begin{vmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix} = 0.$$

$$\begin{aligned} p_1 p_{234} &= x_{11}[x_{12}[x_{23}x_{34} - x_{24}x_{33}] - x_{13}[x_{22}x_{34} - x_{24}x_{32}] + x_{14}[x_{22}x_{33} - x_{23}x_{32}]] \\ -p_2 P_{134} &= x_{12}[x_{11}[x_{23}x_{34} - x_{24}x_{33}] - x_{13}[x_{21}x_{34} - x_{24}x_{31}] + x_{14}[x_{21}x_{33} - x_{23}x_{31}]] \\ p_3 p_{124} &= x_{13}[x_{11}[x_{22}x_{34} - x_{24}x_{32}] - x_{12}[x_{21}x_{34} - x_{24}x_{31}] + x_{14}[x_{21}x_{32} - x_{22}x_{31}]] \\ -p_4 p_{123} &= x_{14}[x_{11}[x_{22}x_{33} - x_{23}x_{32}] - x_{12}[x_{21}x_{33} - x_{23}x_{31}] + x_{13}[x_{21}x_{32} - x_{22}x_{31}]]. \end{aligned} \tag{2.44}$$

which is equal to

$$p_1 p_{234} - p_2 p_{134} + p_3 p_{124} - p_4 p_{123} = 0. \tag{2.45}$$

Next, append the matrix $Gr(1,4), Gr(1,4)$ and $Gr(2,4)$ which gives the $4 \times 4$ matrix in equation 2.43 and then we pick the $3 \times 3$ minors

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{11} & x_{12} & x_{13} & x_{14} \\ x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \end{bmatrix} \hookrightarrow \left( \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \end{bmatrix}, \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \end{bmatrix} \right). \tag{2.46}$$

picking the $3 \times 3$ minors gives,

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{11} & x_{12} & x_{13} & x_{14} \\ x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \end{bmatrix} = \begin{aligned} p_{234} &= \begin{bmatrix} x_{12} & x_{13} & x_{14} \\ x_{12} & x_{13} & x_{14} \\ x_{22} & x_{23} & x_{24} \end{bmatrix} = p_2 p_{34} - p_3 p_{24} + p_4 p_{23} = 0 \\ p_{134} &= \begin{bmatrix} x_{11} & x_{13} & x_{14} \\ x_{11} & x_{13} & x_{14} \\ x_{21} & x_{23} & x_{24} \end{bmatrix} = p_1 p_{24} - p_3 p_{14} + p_4 p_{13} = 0 \\ p_{124} &= \begin{bmatrix} x_{11} & x_{12} & x_{14} \\ x_{11} & x_{12} & x_{14} \\ x_{21} & x_{22} & x_{24} \end{bmatrix} = p_1 p_{24} - p_2 p_{14} + p_4 p_{12} = 0 \\ p_{123} &= \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{bmatrix} = p_1 p_{23} - p_2 p_{13} + p_3 p_{12} = 0 \end{aligned} .$$

Next, we append the matrix $Gr(2,4)$ to $Gr(2,4)$ which gives a $4 \times 4$ matrix and then pick the $2 \times 2$ minors ,

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \end{bmatrix} = \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix} \begin{vmatrix} x_{13} & x_{14} \\ x_{23} & x_{24} \end{vmatrix} - \begin{vmatrix} x_{11} & x_{13} \\ x_{21} & x_{23} \end{vmatrix} \begin{vmatrix} x_{12} & x_{14} \\ x_{22} & x_{24} \end{vmatrix} + \begin{vmatrix} x_{11} & x_{14} \\ x_{21} & x_{24} \end{vmatrix} \begin{vmatrix} x_{12} & x_{13} \\ x_{22} & x_{23} \end{vmatrix} . \tag{2.48}$$

$$= p_{12} p_{34} - p_{13} p_{24} + p_{14} p_{23} = 0. \tag{2.49}$$

Next we append the matrix $Gr(2,4)$ to $Gr(3,4)$ which gives a $5 \times 4$ matrix

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \end{bmatrix} \hookrightarrow \left( \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \end{bmatrix} \right) . \tag{2.50}$$

An extra column is added to the matrix to make a square matrix such that the determinants can be determined. The extra column is obtained from any of the $4 \times 5$ matrix and the process is repeated for all the columns of the matrix. Obtaining 4 copies of a $5 \times 5$ matrix and then take the block determinant.

$$\begin{vmatrix} x_{11} & x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{21} & x_{22} & x_{23} & x_{24} \\ x_{11} & x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{31} & x_{32} & x_{33} & x_{34} \end{vmatrix} = \begin{vmatrix} x_{11} & x_{11} \\ x_{21} & x_{21} \end{vmatrix} \begin{vmatrix} x_{12} & x_{13} & x_{14} \\ x_{22} & x_{23} & x_{24} \\ x_{32} & x_{33} & x_{34} \end{vmatrix} + \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix} \begin{vmatrix} x_{11} & x_{13} & x_{14} \\ x_{21} & x_{23} & x_{24} \\ x_{31} & x_{33} & x_{34} \end{vmatrix}.$$

$$- \begin{vmatrix} x_{11} & x_{13} \\ x_{21} & x_{23} \end{vmatrix} \begin{vmatrix} x_{11} & x_{12} & x_{14} \\ x_{21} & x_{22} & x_{24} \\ x_{31} & x_{32} & x_{34} \end{vmatrix} + \begin{vmatrix} x_{11} & x_{14} \\ x_{21} & x_{24} \end{vmatrix} \begin{vmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix} = 0. \tag{2.51}$$

$$= p_{12}p_{134} - p_{13}p_{124} + p_{14}p_{123} = 0. \tag{2.52}$$

$$\begin{vmatrix} x_{11} & x_{12} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{22} & x_{23} & x_{24} \\ x_{11} & x_{12} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{32} & x_{33} & x_{34} \end{vmatrix} = - \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix} \begin{vmatrix} x_{12} & x_{13} & x_{14} \\ x_{22} & x_{23} & x_{24} \\ x_{32} & x_{33} & x_{34} \end{vmatrix} - \begin{vmatrix} x_{12} & x_{12} \\ x_{22} & x_{22} \end{vmatrix} \begin{vmatrix} x_{11} & x_{13} & x_{14} \\ x_{21} & x_{23} & x_{24} \\ x_{31} & x_{33} & x_{34} \end{vmatrix}$$

$$- \begin{vmatrix} x_{12} & x_{13} \\ x_{22} & x_{23} \end{vmatrix} \begin{vmatrix} x_{11} & x_{12} & x_{14} \\ x_{21} & x_{22} & x_{24} \\ x_{31} & x_{32} & x_{34} \end{vmatrix} - \begin{vmatrix} x_{12} & x_{14} \\ x_{22} & x_{24} \end{vmatrix} \begin{vmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix} = 0. \tag{2.53}$$

$$= p_{12}p_{234} - p_{23}p_{124} + p_{14}p_{123} = 0. \tag{2.54}$$

$$\begin{vmatrix} x_{11} & x_{12} & x_{13} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{23} & x_{24} \\ x_{11} & x_{12} & x_{13} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{33} & x_{34} \end{vmatrix} = - \begin{vmatrix} x_{11} & x_{13} \\ x_{21} & x_{23} \end{vmatrix} \begin{vmatrix} x_{12} & x_{13} & x_{14} \\ x_{22} & x_{23} & x_{24} \\ x_{32} & x_{33} & x_{34} \end{vmatrix} - \begin{vmatrix} x_{12} & x_{13} \\ x_{22} & x_{23} \end{vmatrix} \begin{vmatrix} x_{11} & x_{13} & x_{14} \\ x_{21} & x_{23} & x_{24} \\ x_{31} & x_{33} & x_{34} \end{vmatrix}.$$

$$- \begin{vmatrix} x_{13} & x_{13} \\ x_{23} & x_{23} \end{vmatrix} \begin{vmatrix} x_{11} & x_{12} & x_{14} \\ x_{21} & x_{22} & x_{24} \\ x_{31} & x_{32} & x_{34} \end{vmatrix} - \begin{vmatrix} x_{13} & x_{14} \\ x_{23} & x_{24} \end{vmatrix} \begin{vmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix} = 0. \tag{2.55}$$

$$= p_{13}p_{234} - p_{23}p_{134} + p_{34}p_{123} = 0. \tag{2.56}$$

$$\begin{vmatrix} x_{11} & x_{12} & x_{13} & x_{14} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} & x_{24} \\ x_{11} & x_{12} & x_{13} & x_{14} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} & x_{34} \end{vmatrix} = - \begin{vmatrix} x_{11} & x_{14} \\ x_{21} & x_{24} \end{vmatrix} \begin{vmatrix} x_{12} & x_{13} & x_{14} \\ x_{22} & x_{23} & x_{24} \\ x_{32} & x_{33} & x_{34} \end{vmatrix} - \begin{vmatrix} x_{12} & x_{14} \\ x_{22} & x_{24} \end{vmatrix} \begin{vmatrix} x_{11} & x_{13} & x_{14} \\ x_{21} & x_{23} & x_{24} \\ x_{31} & x_{33} & x_{34} \end{vmatrix}$$
$$- \begin{vmatrix} x_{12} & x_{14} \\ x_{22} & x_{24} \end{vmatrix} \begin{vmatrix} x_{11} & x_{12} & x_{14} \\ x_{21} & x_{22} & x_{24} \\ x_{31} & x_{32} & x_{34} \end{vmatrix} - \begin{vmatrix} x_{14} & x_{14} \\ x_{24} & x_{24} \end{vmatrix} \begin{vmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix} = 0. \tag{2.57}$$

$$= p_{14}p_{234} - p_{24}p_{134} + p_{34}p_{124} = 0. \tag{2.58}$$

Therefore, the equations defining $\mathcal{F}\ell_4(\mathbb{C}) = X_{4321}$ for n = 4 is determined by equating the sum of all the minors to zero.

$p_1p_{234} - p_2p_{134} + p_3p_{124} - p_4p_{123} = 0$
$p_2p_{34} - p_3p_{24} + p_4p_{23} = 0$
$p_1p_{24} - p_3p_{14} + p_4p_{13} = 0$
$p_1p_{24} - p_2p_{14} + p_4p_{12} = 0$
$p_1p_{23} - p_2p_{13} + p_3p_{12} = 0$
$p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0$
$p_{12}p_{134} - p_{13}p_{124} + p_{14}p_{123} = 0$
$-p_{12}p_{234} + p_{23}p_{124} - p_{14}p_{123} = 0$
$p_{13}p_{234} - p_{23}p_{134} + p_{34}p_{123} = 0$
$p_{14}p_{234} - p_{24}p_{134} + p_{34}p_{124} = 0.$

## 2.6 Bruhat Order

The Bruhat order is a partial order relation defined on the elements of $S_n$ with respect to the length function.

**Remark 2.6.1.** *The Bruhat graph is the transitive closure of the partial order relation defined on the elements of $W$ with respect to the length function.*

**Definition 2.6.2.** *For any $\sigma \in S_n$, the Bruhat graph is the graph with vertex set equal to $\{v \in S_n : v \leq \sigma\} = [id, \sigma]$ where there exist an edge between $v$ and $vt_{ij}$ if $v, vt_{ij} \leq \sigma$ and $t$ is the transposition.*

**Definition 2.6.3.** *Vertex*

*The vertex is said to be the point that two or more straight lines meets .*

**Definition 2.6.4.** *Edge*

*An edge is the line segment between faces.*

**Definition 2.6.5.** *Degree*

*The degree of a permutation $v$ is the number of edges connected to $v$ on the Bruhat graph for $\sigma$ and it is equal to the dimension of $T_v(X_\sigma)$.*

**Theorem 2.6.6.** *[Lakshmibai & Sandhya (1990)]*

*Let $(W, S)$ be an arbitrary Coxeter system. For $v \leq y \leq \sigma$*

$$\sharp\{r \in R | v \leq ry \leq \sigma\} \geq l(\sigma) - l(v). \tag{2.59}$$

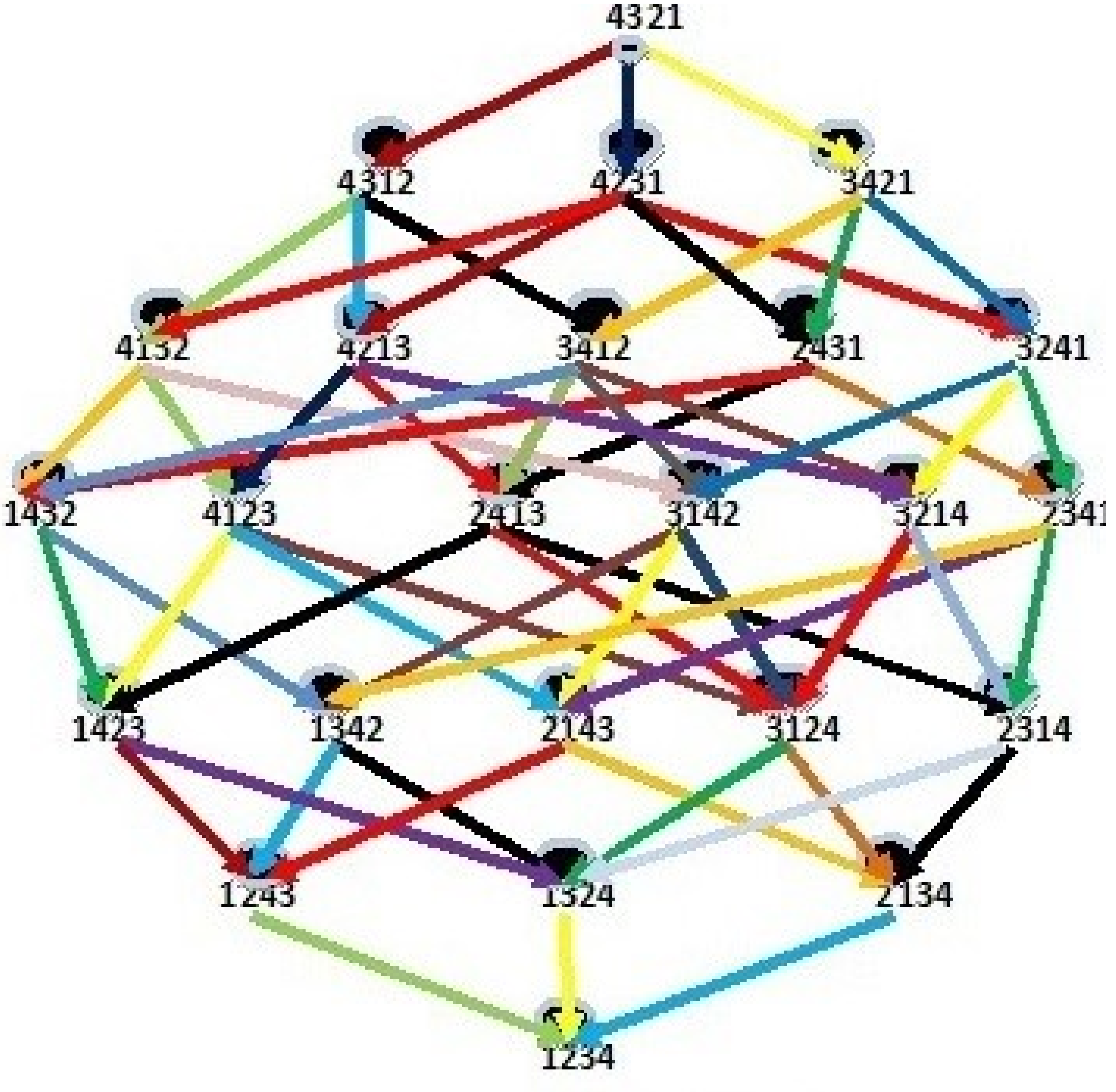


Figure 2.1: The Bruhat graph for $S_4$.

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Source: [ Abe & Billey (2016)]

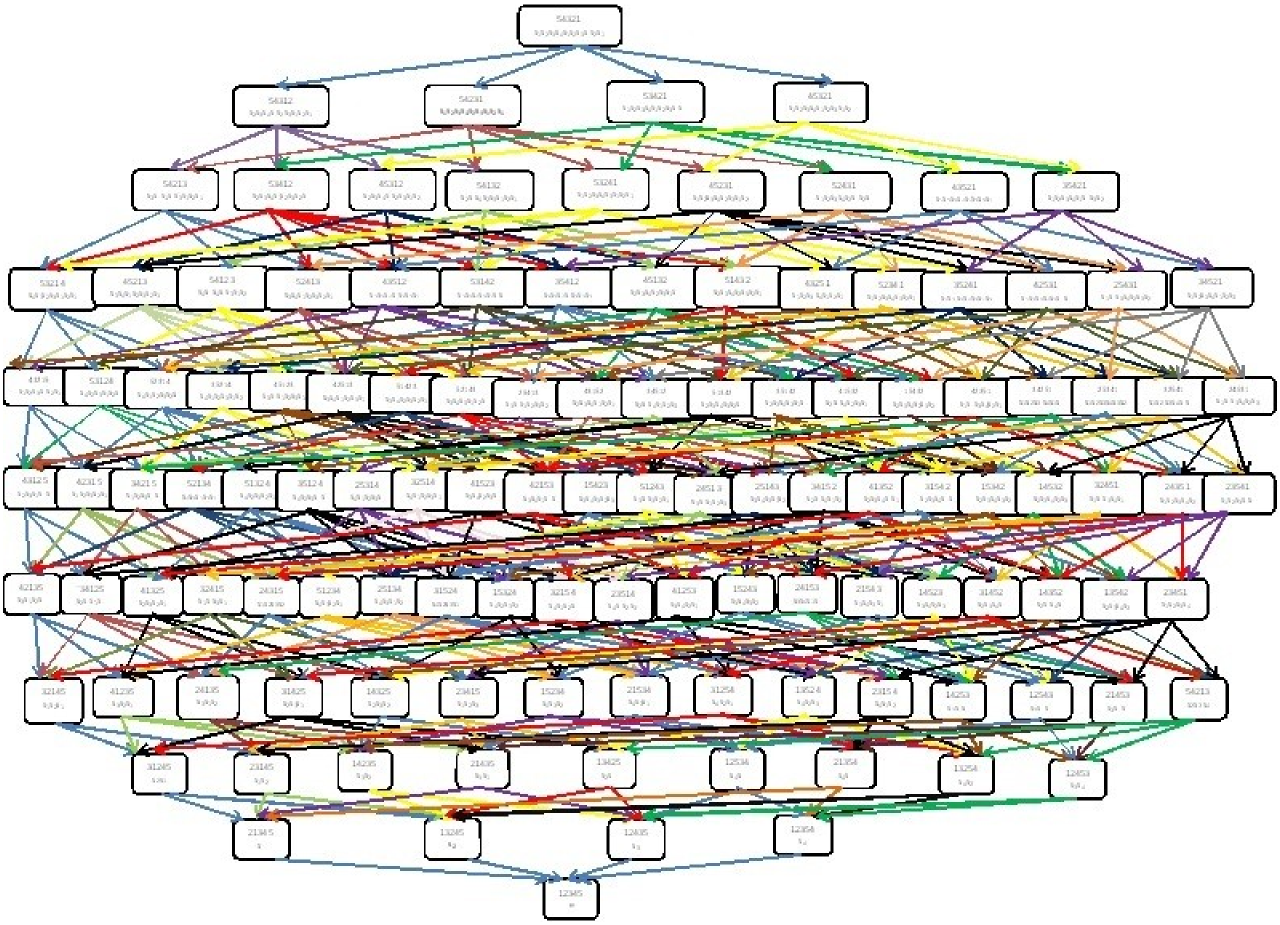

Figure 2.2: The Bruhat graph for $S_5$

Source: [ Abe & Billey (2016)]

## 2.7 Singular locus of Schubert Varieties

This session gives the definition of the smooth and singular Schubert varieties with examples. We note that the singular locus of the Schubert varieties are closed sets of points where the Schubert varieties are not smooth.

### 2.7.1 Smooth Schubert Varieties

**Definition 2.7.1.** *The Schubert varieties $X_\sigma = G/B$ are smooth manifold if each point has a dimension of $\frac{n(n-1)}{2}$ and the dimension of the tangent space at each point is $\frac{n(n-1)}{2}$.*

**Corollary 2.7.2.** *[Abe & Billey (2016)]*

*$X_\sigma$ is smooth iff $X_\sigma$ is smooth at $v = id$.*

Let $G/B = X_{\sigma_0} = C_{\sigma_0} \bigcup_{v<\sigma_0} C_v$. $C_{\sigma_0}$ is an affine neighborhood of $\sigma_0$.
For instance when $n = 5, \sigma_0 = 54321$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \in \begin{pmatrix} * & * & * & * & 1 \\ * & * & * & 1 & 0 \\ * & * & 1 & 0 & 0 \\ * & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} = C_{\sigma_0}.$$

This neighborhood can be moved around to contain the identity by left multiplication by the matrix $\sigma_0$.

$$\sigma_0 C_{\sigma_0} = \sigma_0 B \sigma_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} * & * & * & * & 1 \\ * & * & * & 1 & 0 \\ * & * & 1 & 0 & 0 \\ * & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ x_{21} & 1 & 0 & 0 & 0 \\ x_{31} & x_{32} & 1 & 0 & 0 \\ x_{41} & x_{42} & x_{43} & 1 & 0 \\ x_{51} & x_{52} & x_{53} & x_{54} & 1 \end{pmatrix}.$$

The Matrix of $\sigma_0 \in C_{\sigma_0}$.

Where the stars in the matrix on the right are replaced with affine coordinates.

### 2.7.2 Singular Schubert Varieties

**Definition 2.7.3.** *A point $p \in C_v \subset X_\sigma$ is singular in $X_\sigma$ iff every point in $C_v$ is singular in $X_\sigma$,*

## 2.8 Pattern Avoidance

This section discusses the classical permutation patterns .

### 2.8.1 Permutation Patterns

**Definition 2.8.1.** *Permutation*

*A permutation of length n is a one to one mapping from n- elements set to itself.*

**Definition 2.8.2.** *Permutation pattern*

*A permutation pattern is a subpermutation of a longer permutation. An element $\sigma \in S_n$ contains the pattern $v \in S_k$ if whenever $\sigma$ is expressed in one-line notation, it contains a subword of length k whose entries are in the same relative order as the entries of v, if $\sigma$ does not contain the pattern v then $\sigma$ avoids v.*

### 2.8.2 Classical Permutation Patterns

A permutation pattern is classically defined if there exist an occurrence of a permutation $\tau$ in $\sigma$ as a subsequence in $\sigma$ and of the same length as $\tau$ whose letters are in the same relative order as those in $\tau$.

### 2.8.3 Interval Pattern Avoidance

For $m \leq n$, Let $v \in S_m$ and $\sigma \in S_n$ be two permutations such that $v$ embeds in $\sigma$ then there exist integers $1 \leq \tau_1 < \tau_2 < \tau_3 < \cdots < \tau_m \leq n$ such that $\sigma(\tau_1) < \sigma(\tau_2) < \sigma(\tau_3) < \cdots < \sigma(\tau_m)$ are in the same relative order as $v(1), v(2), \cdots, v(m)$. $\sigma$ avoid v if no such embedding occurs.

### 2.8.4 The 3412 Pattern

If $\sigma \in S_n$ and $(i_1, i_2, i_3, i_4)$ be integers we have a 3412 pattern of $\sigma$ if $1 \leq i_1 < i_2 < i_3 < i_4 \leq n$ and $\sigma(i_3) < \sigma(i_4) < \sigma(i_1) < \sigma(i_2)$. The set of all 3412 patterns of $\sigma$ is given by $P_{3412}(\sigma)$. If $\sigma$ contains 3412 pattern then $P_{3412}(\sigma) \neq \emptyset$.

### 2.8.5 The 4231 pattern

If $\sigma \in S_n$ and $(i_1, i_2, i_3, i_4)$ be integers we have a 4231 pattern of $\sigma$ if $1 \leq i_1 < i_2 < i_3 < i_4 \leq n$ and $\sigma(i_4) < \sigma(i_2) < \sigma(i_3) < \sigma(i_1)$. The set of all 4231 patterns of $\sigma$ is given by $P_{4231}(\sigma)$. If $\sigma$ contains 4231 pattern then $P_{4231}(\sigma) \neq \emptyset$.

## 2.9 Cohomology of the Flag Varieties

This session discusses the cohomology of the flag varieties and the computation of the algebraic additive $\mathbb{Z}$ basis which are the monomials.

The classes of the closure of the Schubert cells forms additive basis for the cohomology of $\mathcal{F}\ell_n(\mathbb{C})$. The homology of the flag varieties does not have a ring structure but since the flag varieties $\mathcal{F}\ell_n(\mathbb{C})$ satisfies Poincare duality, this implies that there exist an isomorphism from the homology to the cohomology of $\mathcal{F}\ell_n(\mathbb{C})$ given by the map ,

$$f : H_{n-k}(\mathcal{F}\ell_n(\mathbb{C}); \mathbb{Z}) \to H^k(\mathcal{F}\ell_n(\mathbb{C}); \mathbb{Z}). \tag{2.60}$$

and defined by

$$f[X_\sigma] = [X^\sigma] \in H^k(\mathcal{F}\ell_n(\mathbb{C})). \tag{2.61}$$

called the Schubert class.

The Poincaré map f enables one to identify each graded piece of the cohomology ring $H^k(\mathcal{F}\ell_n(\mathbb{C}); \mathbb{Z})$ with the homology group $H_{n-k}(\mathcal{F}\ell_n(\mathbb{C}); \mathbb{Z})$. The Schubert classes forms additive $\mathbb{Z}$ basis that generates the cohomology ring $H^k(\mathcal{F}\ell_n(\mathbb{C}); \mathbb{Z})$. The basis for the cohomology ring are the geometric basis and the algebraic basis.

The degree of $[X_\sigma]$ is $2\dim[X_\sigma] = 2l(\sigma)$ .

**Definition 2.9.1.** *The $k^{th}-$ Betti number, $b_k = \dim^{2k}(\mathcal{F}\ell_n(\mathbb{C}); \mathbb{Z})$, $0 \leq k \leq dim\mathcal{F}\ell_n(\mathbb{C})$.*

That is the number of generators of each of the graded piece of the cohomology ring $\mathcal{F}\ell_n(\mathbb{C})$ gives $b_k$.

The algebraic basis for the cohomology of the ring $\mathcal{F}\ell_n(\mathbb{C})$ is described as follows:

**Definition 2.9.2.** *A Symmetric function of a polynomial ring $\mathbb{Z}[x_1, x_2, \cdots, x_n]$ in $x_1, x_2, \cdots, x_n$ variable over an integral domain $\mathbb{Z}$ is symmetric if it is invariant for every permutation $e_i \in S_n$.*

**Proposition 2.9.3.** *[Fulton & Fulton (1997)] The cohomology ring $H^{2l(\sigma)}(\mathcal{F}\ell_n(\mathbb{C});\mathbb{Z})$ is generated by the basic classes $x_1, \cdots, x_n$ subject to the relations $e_i(x_1,\cdots,x_n) = 0$ for $1 \leq i \leq n$. The classes $x_1^{i_1}x_2^{i_2}\cdots x_m^{i_m}$ with exponents $i_j \leq m-j$ form a $\mathbb{Z}$ basis for $H^{2l(\sigma)}(\mathcal{F}\ell_n(\mathbb{C});\mathbb{Z})$.*

**Example 2.9.4.** *The $H^{2l(\sigma)}(\mathcal{F}\ell_n(\mathbb{C});\mathbb{Z}) \cong \mathbb{Z}[x_1,x_2,\cdots,x_n]/I$, for $I = \langle e_i(x_1,\cdots,x_n)\rangle$, where $1 \leq i \leq n$ and $e_i$ is the ith elementary symmetric function For $\mathcal{F}\ell_n(\mathbb{C}) = V_6$, $H^{2l(\sigma)}(\mathcal{F}\ell_4(\mathbb{C});\mathbb{Z}) \cong \mathbb{Z}[x_1,x_2,x_3,x_4]/I = \langle e_1,e_2,e_3,e_4\rangle$ since the cohomology ring is a graded ring it implies that,*

$$H^{2k}(\mathcal{F}\ell_4(\mathbb{C});\mathbb{Z}) = \bigoplus_{k=0}^{n} H^{2k}(\mathcal{F}\ell_4(\mathbb{C});\mathbb{Z}). \tag{2.62}$$

*Where $0 \leq k \leq 6$.*

- *For $k=0$, $H^{2k}(\mathcal{F}\ell_4(\mathbb{C});\mathbb{Z}) = H^{2.0} = 1$.*
- *For $k=1$, $H^{2k}(\mathcal{F}\ell_4(\mathbb{C});\mathbb{Z}) = H^{2.1} = \langle x_1,x_2,x_3\rangle$.*
- *For $k=2$, $H^{2k}(\mathcal{F}\ell_4(\mathbb{C});\mathbb{Z}) = H^{2.2} = \langle x_1^2, x_2^2, x_1x_3, x_1x_2, x_2x_3\rangle$.*
- *For $k=3$, $H^{2k}(\mathcal{F}\ell_4(\mathbb{C});\mathbb{Z}) = H^{2.3} = \langle x_1^3, x_1^2x_2, x_2^2x_1, x_1x_2x_3, x_1^2x_3, x_2^2x_3\rangle$.*
- *For $k=4$, $H^{2k}(\mathcal{F}\ell_4(\mathbb{C});\mathbb{Z}) = H^{2.4} = \langle x_1^3x_2, x_1^3x_3, x_1^2x_2^2, x_1^2x_2x_3, x_1x_2^2x_3\rangle$.*
- *For $k=5$, $H^{2k}(\mathcal{F}\ell_4(\mathbb{C});\mathbb{Z}) = H^{2.5} = \langle x_1^3x_2^2, x_1^3x_2x_3, x_1^2x_2^2x_3\rangle$.*
- *For $k=6$, $H^{2k}(\mathcal{F}\ell_4(\mathbb{C});\mathbb{Z}) = H^{2.6} = \langle x_1^3x_2^2x_3\rangle$ .*

Therefore the flag varieties are generated by the basic classes with generators $x_1, x_2, x_3, x_4$

**Example 2.9.5.** *The $H^{2l(\sigma)}(\mathcal{F}\ell_n(\mathbb{C});\mathbb{Z}) \cong \mathbb{Z}[x_1,x_2,\cdots,x_n]/I$ for $I = \langle e_i(x_1,\cdots,x_n)\rangle$ , where $1 \leq i \leq n$ and $e_i$ is the i-th elementary symmetric functions For $\mathcal{F}\ell_n(\mathbb{C}) = V_{10}$, $H^{2l(\sigma)}(\mathcal{F}\ell_5(\mathbb{C});\mathbb{Z}) \cong \mathbb{Z}[x_1,x_2,x_3,x_4,x_5]/I = \langle e_1,e_2,e_3,e_4,e_5\rangle$ since the cohomology ring is a graded ring it implies that,*

$$H^{2k}(\mathcal{F}\ell_5(\mathbb{C});\mathbb{Z}) = \bigoplus_{k=0}^{n} H^{2k}(\mathcal{F}\ell_5(\mathbb{C});\mathbb{Z}).$$

*Where $0 \leq k \leq 10$.*

- *For* $k = 0, H^{2k}(\mathcal{F}\ell_5(\mathbb{C});\mathbb{Z}) = H^{2.0} = 1.$

- *For* $k = 1, H^{2k}(\mathcal{F}\ell_5(\mathbb{C});\mathbb{Z}) = H^{2.1} = \langle x_1, x_2, x_3, x_4\rangle.$

- *For* $k = 2, H^{2k}(\mathcal{F}\ell_5(\mathbb{C});\mathbb{Z}) = H^{2.2} = \langle x_1^2, x_2^2, x_3^2, x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_2x_4, x_3x_4\rangle.$

- *For* $k = 3, H^{2k}(\mathcal{F}\ell_5(\mathbb{C});\mathbb{Z}) = H^{2.3} = \langle x_1^3, x_2^3, x_1^2x_2, x_1^2x_3, x_1^2x_4, x_1x_2x_3, x_1x_2x_4, x_2^2x_3,$ $x_2^2x_4, x_2^2x_1, x_3^2x_2, x_3^2x_4, x_3^2x_1, x_1x_3x_4, x_2x_3x_4\rangle.$

- *For* $k = 4, H^{2k}(\mathcal{F}\ell_5(\mathbb{C});\mathbb{Z}) = H^{2.4} = \langle x_1^4, x_1^3x_2, x_1^3x_3, x_1^3x_4, x_1^2x_2^2, x_1^2x_3^2, x_1^2x_2x_3, x_1^2x_2x_3,$ $x_1^2x_2x_4, x_1^2x_3x_4, x_1x_2x_3x_4, x_2^3x_1, x_2^3x_3, x_2^3x_4, x_1x_2^2x_3, x_2^2x_3x_4, x_2^2x_3^2, x_3^2x_1x_2,$ $x_3^2x_2x_4, x_3^2x_1^2, x_2^2x_1x_4\rangle.$

- *For* $k = 5, H^{2k}(\mathcal{F}\ell_5(\mathbb{C});\mathbb{Z}) = H^{2.5} = \langle x_1^4x_2, x_1^4x_3, x_1^4x_4, x_1^3x_2^2, x_1^3x_3^2, x_1^3x_2x_3, x_1^3x_2x_4,$ $x_1^3x_3x_4, x_1^2x_2^3, x_1x_2^3x_4, x_1^2x_2^2x_3, x_1^2x_2^2x_4, x_1^2x_3^2x_4, x_2^2x_3^2x_4, x_1x_2^3x_3, x_1x_2^2x_3^2,$ $x_1x_2^2x_3x_4, x_1x_2x_3^2x_4, x_1^2x_2x_3x_4, x_2^3x_3x_4, x_3^2x_2x_1x_4, x_3^2x_1x_2x_4\rangle.$

- *For* $k = 6, H^{2k}(\mathcal{F}\ell_5(\mathbb{C});\mathbb{Z}) = H^{2.6} = \langle x_1x_2^2x_3^2x_4, x_1^2x_2x_3^2x_4, x_1x_2^3x_3x_4, x_1^2x_2^2x_3x_4, x_1^3x_2^3,$ $x_2^3x_3^2x_4, x_1^3x_2^2x_4, x_1^2x_2^3x_4, x_1^4x_3x_4, x_1^2x_2^2x_3^2, x_1^3x_2^2x_4, x_1x_2^3x_3^2, x_1^4x_2x_4, x_1^4x_3^2, x_1^4x_2^2,$ $x_1^3x_2x_3^2, x_1^3x_2x_3^2x_4, x_1^2x_2^3x_3, x_1^4x_2x_3, x_1^3x_2^2x_3\rangle.$

- *For* $k = 7, H^{2k}(\mathcal{F}\ell_5(\mathbb{C});\mathbb{Z}) = H^{2.7} = \langle x_1^2x_2^2x_3^2x_4, x_1x_2^3x_3^2x_4, x_1^3x_2x_3^2x_4, x_1^2x_2^3x_3x_4, x_1^4x_2^3,$ $x_1^4x_2x_3x_4, x_1^3x_2^2x_3x_4, x_1^4x_2^2x_4, x_1^3x_2^3x_4, x_1^2x_2^3x_3^2, x_1^4x_2^2x_4, x_1^3x_2^2x_3^2, x_1^4x_2x_3^2, x_1^3x_2^3x_3,$ $x_1^4x_2^2x_3, \rangle.$

- *For* $k = 8, H^{2k}(\mathcal{F}\ell_5(\mathbb{C});\mathbb{Z}) = H^{2.8} = \langle x_1^2x_2^3x_3^2x_4, x_1^3x_2^2x_3^2x_4, x_1^4x_2x_3^2x_4, x_1^3x_2^3x_3x_4, x_1^4x_2^3x_3,$ $x_1^4x_2^2x_3x_4, x_1^4x_2^3x_4, x_1^3x_2^3x_3^2, x_1^4x_2^2x_2, \rangle.$

- *For* $k = 9, H^{2k}(\mathcal{F}\ell_5(\mathbb{C});\mathbb{Z}) = H^{2.9} = \langle x_1^3x_2^3x_3^2x_4, x_1^4x_2^2x_3^2x_4, x_1^4x_2^3x_3x_4, x_1^4x_2^3x_3^2, \rangle.$

- *For* $k = 10, H^{2k}(\mathcal{F}\ell_5(\mathbb{C});\mathbb{Z}) = H^{2.10} = \langle x_1^4x_2^3x_3^2x_4, \rangle.$

Therefore the flag varieties are generated by the basic classes with generators $x_1, x_2, x_3, x_4, x_5$

**Example 2.9.6.** *The* $H^{2l(\sigma)}(\mathcal{F}\ell_n(\mathbb{C});\mathbb{Z}) \cong \mathbb{Z}[x_1, x_2, \cdots, x_n]/I$ *, for* $I = \langle e_i(x_1, \cdots, x_n)\rangle$ *where* $1 \leq i \leq n$ *and* $e_i$ *is the ith elementary symmetric function For* $\mathcal{F}\ell_n(\mathbb{C}) = V_{15}$*,*

$H^{2l(\sigma)}(\mathcal{F}\ell_6(\mathbb{C});\mathbb{Z}) \cong \mathbb{Z}[x_1,x_2,x_3,x_4,x_5,x_6]/I = \langle e_1,e_2,e_3,e_4,e_5,e_6\rangle$ *since the cohomology ring is a graded ring it implies that ,*

$$H^{2k}(\mathcal{F}\ell_6(\mathbb{C});\mathbb{Z}) = \bigoplus_{k=0}^{n} H^{2k}(\mathcal{F}\ell_6(\mathbb{C});\mathbb{Z}).$$

*Where* $0 \leq k \leq 15$.

- *For* $k=0, H^{2k}(\mathcal{F}\ell_6(\mathbb{C});\mathbb{Z}) = H^{2.0} = 1.$

- *For* $k=1, H^{2k}(\mathcal{F}\ell_6(\mathbb{C});\mathbb{Z}) = H^{2.1} = \langle x_1,x_2,x_3,x_4,x_5\rangle.$

- *For* $k=2, H^{2k}(\mathcal{F}\ell_6(\mathbb{C});\mathbb{Z}) = H^{2.2} = \langle x_1^2,x_2^2,x_3^2,x_4^2,x_1x_2,x_1x_3,x_1x_4,x_1x_5,x_2x_3,$ $x_2x_4,x_2x_5,x_3x_4,x_3x_5,x_4x_5\rangle.$

- *For* $k=3, H^{2k}(\mathcal{F}\ell_6(\mathbb{C});\mathbb{Z}) = H^{2.3} = \langle x_1^3,x_2^3,x_3^3,x_1^2x_2,x_1^2x_3,x_1^2x_4,x_1^2x_5,x_1x_2x_3,$ $x_1x_2x_4,x_1x_2x_5,x_2^2x_3,x_2^2x_4,x_2^2x_5,x_2^2x_1,x_3^2x_2,x_3^2x_4,x_3^2x_1,x_3^2x_5,x_4^2x_1,x_4^2x_2,x_4^2x_3,$ $x_4^2x_5,x_1x_3x_4,x_1x_3x_5,x_2x_3x_5,x_2x_3x_4,x_2x_4x_5,x_3x_4x_5\rangle/$

- *For* $k=4, H^{2k}(\mathcal{F}\ell_6(\mathbb{C});\mathbb{Z}) = H^{2.4} = \langle x_1^4,x_2^4,x_1^3x_2,x_1^3x_3,x_1^3x_4,x_1^2x_2^2,x_1^2x_3^2,x_1^3x_5,$ $x_2^3x_5,x_3^3x_1,x_3^3x_2,x_3^3x_4,x_3^3x_5,x_1^2x_4^2,x_2^2x_4^2,x_3^2x_4^2,,x_1^2x_2x_3,x_4^2x_2x_3,x_1^2x_2x_4,x_1^2x_3x_4,$ $x_1x_2^2x_5,x_1^2x_2x_5,x_1^2x_3x_5,x_1^2x_4x_5,x_3^2x_4x_5,x_4^2x_3x_5,x_4^2x_1x_2,x_1x_2x_3x_4,x_2^3x_1,x_2^3x_3,$ $x_2^3x_4,x_1x_2^2x_3,x_2^2x_3x_4,x_2^2x_3^2,x_3^2x_1x_2,x_3^2x_2x_4,x_3^2x_1^2,x_2^2x_1x_4,x_1x_2x_3x_5,x_1x_2x_4x_5,$ $x_1x_3x_4x_5\rangle.$

- *For* $k=5, H^{2k}(\mathcal{F}\ell_6(\mathbb{C});\mathbb{Z}) = H^{2.5} = \langle x_1^5,x_1^4x_2,x_1^4x_3,x_1^4x_4,x_1^4x_5,x_1^3x_2^2,x_1^3x_3^2,x_1^3x_4^2,$ $x_1^2x_2^3,x_1^2x_3^3,x_1^3x_2x_3,x_1^3x_3x_4,x_1^3x_4x_5,x_1^3x_2x_4,x_1^3x_2x_5,x_1^3x_3x_5,x_1^2x_2^2x_3,x_1^2x_2^2x_4,x_1^2x_2^2x_5,$ $x_2^2x_3^3,x_2^3x_3^2,x_1^2x_3^2x_5,x_1^2x_3^2x_4,x_1^2x_2x_3^2,x_1^2x_3x_4^2,x_1x_3x_4^2x_5,x_1x_2^2x_4x_5,x_1x_3^2x_4x_5,$ $x_1x_2x_3x_4x_5,x_1x_2^3x_3,x_1x_2^2x_3^2,x_2^3x_3x_4,x_2^3x_4x_5,x_3^3x_4x_5,x_3^3x_4^2,x_2^3x_4^2,x_2^3x_3x_5,x_1x_2^4,$ $x_2^2x_3^2x_4,x_3^2x_4^2x_5,x_3^3x_4^2x_5,x_2x_3x_4^2x_5,x_2^2x_4^2x_5,x_1^2x_4^2x_5,x_1x_2^3x_3,x_1x_2^3x_4,x_1x_2^3x_5,$ $x_1x_3^3x_5,x_1x_2x_3^3,x_1x_3^3x_4,x_1x_2x_3^2x_4,x_2x_3^2x_4x_5\rangle.$

- *For* $k=6, H^{2k}(\mathcal{F}\ell_6(\mathbb{C});\mathbb{Z}) = H^{2.6} = \langle x_1^5x_2,x_1^5x_3,x_1^5x_4,x_1^5x_5,x_1^3x_2^3,x_1^4x_2^2,x_1^4x_2x_3'$ $x_1^4x_2x_4,x_1^4x_2x_5,x_1^4x_3x_4,x_1^4x_3x_5,x_1^4x_4x_5,x_1^4x_3^2,x_1^4x_4^2,x_1^3x_2^2x_3,x_1^3x_2^2x_4,x_1^3x_2^2x_5,x_1^3x_2x_3^2,$ $x_1^3x_2x_4^2,x_1^3x_2x_4x_5,x_1^3x_2x_3x_5,x_1^3x_2x_3x_4,x_1^3x_2^3,x_1^3x_2^2x_4,x_1^3x_3^2x_5,x_1^3x_4^2x_5,x_1^3x_3x_4^2,$ $x_1^3x_3x_4x_5,x_1^2x_2^4,x_1^2x_2^3x_3,x_1^2x_2^3x_4,x_1^2x_2^3x_5,x_1^2x_2^2x_3^2,x_1^2x_2^2x_4^2,x_1^2x_2^2x_3x_4,x_1^2x_2^2x_3x_5,$ $x_1^2x_2^2x_4x_5,x_1^2x_2x_3^3,x_1^2x_2x_3^2x_4,x_1^2x_2x_3^2x_5,x_1^2x_2x_4^2x_5,x_1^2x_2x_3x_4x_5,x_1^2x_3^3x_4,x_1^2x_3^3x_5,$

$x_1^2x_3^2x_4^2, x_1^2x_3^2x_4x_5, x_1^2x_3x_4^2x_5, x_1x_2^4x_3, x_1x_2^4x_4, x_1x_2^4x_5, x_1x_2^3x_3^2, x_1x_2^3x_4^2, x_1x_2^3x_4x_5,$

$x_1x_2^3x_3x_4, x_1x_2^3x_3x_5, x_1x_2^2x_3^3, x_1x_2^2x_3^2x_4, x_1x_2^2x_3^2x_5, x_1x_2^2x_4^2x_5, x_1x_2^2x_3x_4x_5, x_2^4x_4^2,$

$x_1x_2x_3^3x_4, x_1x_2x_3^2x_4^2, x_1x_2x_3^2x_4x_5, x_1x_2x_3x_4^2x_5, x_1x_3^3x_4^2, x_1x_3^3x_4x_5, x_1x_3^2x_4^2x_5, x_2^4x_3^2,$

$x_2^4x_3x_5, x_2^3x_4^3, x_2^3x_3^2x_4, x_2^3x_3^2x_5, x_2^3x_3x_4^2, x_2^3x_4^2x_5, x_2^2x_3^3x_4, x_2^2x_3^3x_5, x_2^2x_3^2x_4^2, x_2^4x_4x_5,$

$x_2^4x_3x_4, x_2^2x_3^2x_4x_5, x_2^2x_3x_4^2x_5, x_2x_3^3x_4^2, x_2x_3^2x_4^2x_5, x_3^3x_4^2x_5, x_1x_2x_3^3x_5, \rangle.$

- *For* $k = 7, H^{2k}(\mathcal{F}\ell_6(\mathbb{C}); \mathbb{Z}) = H^{2.7} = \langle x_1^5x_2^2, x_1^5x_2x_3, x_1^5x_2x_4, x_1^5x_2x_5, x_1^5x_3^2, x_1^5x_3x_4,$

$x_1^5x_4x_5, x_1^5x_4^2, x_1^4x_2^3, x_1^4x_2^2x_3, x_1^4x_2^2x_4, x_1^4x_2^2x_5, x_1^4x_2x_3^2, x_1^4x_2x_3x_4, x_1^4x_2x_3x_5, x_1^3x_2^2x_4^2,$

$x_1^4x_2x_4x_5, x_1^4x_2x_4^2, x_1^3x_2^4, x_1^3x_2^3x_3, x_1^3x_2^3x_4, x_1^3x_2^3x_5, x_1^3x_2^2x_3^2, x_1^3x_2^2x_3x_4, x_1^3x_2^2x_3x_5,$

$x_1^3x_2^2x_4x_5, x_1^3x_2^2x_4x_5, x_1^3x_2^2x_3x_4, x_1^3x_2^2x_3x_5, x_1^3x_2^2x_4x_5, x_1^3x_2x_3^3, x_1^3x_2x_3^2x_4, x_1^3x_2x_3^2x_5,$

$x_1^3x_2x_3x_4^2, x_1^3x_2x_3x_4x_5, x_1^3x_2x_4^2x_5, x_1^3x_3^3x_4, x_1^3x_3^2x_4^2, x_1^3x_3^2x_4x_5, x_1^2x_2^4x_3, x_1^2x_2^4x_4, x_2^4x_3^3,$

$x_1^2x_2^4x_5, x_1^2x_2^3x_3^2, x_1^2x_2^3x_3x_4, x_1^2x_2^3x_3x_5, x_1^2x_2^3x_4x_5, x_1^2x_2^3x_4^2, x_1^2x_2^2x_3^3, x_1^2x_2^2x_3^2x_4, x_2^4x_3x_4^2,$

$x_1^2x_2^2x_3^2x_5, x_1^2x_2^2x_3x_4^2, x_1^2x_2^2x_4^2x_5, x_1^2x_2^2x_3x_4x_5, x_1^2x_2x_3^3x_4, x_1^2x_2x_3^3x_5, x_1^2x_2x_3^2x_4^2, x_2^4x_3^2x_4,$

$x_1^2x_2x_3^2x_4x_5, x_1^2x_2x_3x_4^2x_5, x_1^2x_3^2x_4^2x_5, x_1x_2^4x_3^2, x_1x_2^4x_4^2, x_1x_2^4x_3x_4, x_1x_2^4x_3x_5, x_1x_2^3x_3^3,$

$x_1x_2^4x_4x_5, x_1x_2^3x_3^2x_4, x_1x_2^3x_3^2x_5, x_1x_2^3x_3x_4^2, x_1x_2^3x_3x_4x_5, x_1x_2^2x_3^2x_4^2, x_1x_2^2x_3^2x_4x_5,$

$x_1x_2^2x_3x_4^2x_5, x_1x_2^2x_3^3x_4, x_1x_2^2x_3^3x_5, x_1x_2^3x_4^2x_5, x_1x_2x_3^3x_4^2, x_1x_2x_3^3x_4x_5, x_1x_3^3x_4^2x_5,$

$x_2^4x_3^2x_5, x_2^4x_3x_4x_5, x_2^3x_3^3x_4, x_2^3x_3^3x_5, x_2^3x_3^2x_4^2, x_2^3x_3^2x_4^2x_5, x_2^3x_3^2x_4x_5, x_2^2x_3^3x_4^2, x_2^2x_3^3x_4x_5,$

$x_2^2x_3^2x_4^2x_5, x_2x_3^3x_4^2x_5, x_1^5x_3x_5, \rangle.$

- *For* $k = 8, H^{2k}(\mathcal{F}\ell_6(\mathbb{C}); \mathbb{Z}) = H^{2.8} = \langle x_1^5x_2^3, x_1^5x_2^2x_3, x_1^5x_2^2x_4, x_1^5x_2^2x_5, x_1^5x_2x_3^2, x_1^5x_2x_4^2,$

$x_1^5x_2x_3x_4, x_1^5x_2x_3x_5, x_1^5x_2x_4x_5, x_1^5x_3^2x_5, x_1^5x_3^2x_4, x_1^5x_3^3, x_1^5x_3x_4^2, x_1^5x_4^2x_5, x_1^5x_3x_4x_5,$

$x_1^4x_2^3x_3, x_1^4x_2^2x_3^2, x_1^4x_2^3x_4, x_1^4x_2^3x_5, x_1^4x_2^2x_3x_4, x_1^4x_2^2x_4^2, x_1^4x_2^2x_3x_5, x_1^4x_2^2x_4x_5, x_1^4x_2x_3^3,$

$x_1^4x_3^2x_4^2, x_1^4x_2x_3^2x_4, x_1^4x_2x_3^2x_5, x_1^4x_2x_4^2x_5, x_1^4x_2x_3x_4^2, x_1^4x_2x_3x_4x_5, x_1^4x_3^3x_5, x_1^4x_3^3x_4,$

$x_1^4x_3x_4^2x_5, x_1^3x_2^4x_3, x_1^3x_2^4x_4, x_1^3x_2^4x_5, x_1^3x_2^3x_3x_5, x_1^3x_2^3x_4x_5, x_1^3x_2^3x_3^2, x_1^3x_2^3x_3x_4, x_1^3x_2^2x_3^3,$

$x_1^4x_3^2x_4x_5, x_1^3x_2^2x_3^2x_4, x_1^3x_2^2x_3x_4^2, x_1^3x_2^2x_3^2x_5, x_1^3x_2^2x_4^2x_5, x_1^3x_2^2x_3x_4x_5, x_1^3x_2x_3^3x_4, x_2^4x_3^3x_5,$

$x_1^3x_2x_3^2x_4^2, x_1^3x_2x_3^2x_4x_5, x_1^3x_2x_3x_4^2x_5, x_1^3x_3^3x_4^2, x_1^3x_3^3x_4x_5, x_1^3x_3^2x_4^2x_5, x_1^2x_2^4x_3^2, x_1^2x_2^4x_4^2,$

$x_1^2x_2^4x_3x_4, x_1^2x_2^4x_3x_5, x_1^2x_2^4x_4x_5, x_1^2x_2^3x_3^3, x_1^2x_2^3x_3^2x_4, x_1^2x_2^3x_3^2x_5, x_1^2x_2^3x_4^2x_5, x_1^2x_2^3x_3x_4^2,$

$x_1^2x_2^2x_3^3x_4, x_1^2x_2^2x_3^3x_5, x_1^2x_2^2x_3^2x_4^2, x_1^2x_2^2x_3^2x_4x_5, x_1^2x_2^2x_3x_4^2x_5, x_1^2x_2x_3^3x_4^2, x_1^2x_2x_3^3x_4x_5,$

$x_1^2x_2x_3^2x_4^2x_5, x_1^2x_3^3x4^2x_5, x_1x_2^4x_3^3, x_1x_2^4x_3^2x_4, x_1x_2^4x_3^2x_5, x_1x_2^4x_3x_4^2, x_1x_2^4x_3x_4x_5,$

$x_1x_2^4x_4^2x_5, x_2^4x_3^3x_4, x_2^4x_3^2x_4^2, x_2^4x_3^2x_4x_5, x_2^4x_3x_4^2x_5, x_2^3x_3^3x_4^2, x_2^3x_3^3x_4x_5, x_2^3x_3^2x_4^2x_5,$

$x_2^2x_3^3x_4^2x_5, x_1^2x_2^3x_3x_4x_5, x_1^3x_2x_3^3x_5, x_1^4x_2^4, \rangle.$

- *For* $k = 9, H^{2k}(\mathcal{F}\ell_6(\mathbb{C}); \mathbb{Z}) = H^{2.9} = \langle x_1^5x_2^4, x_1^5x_2^3x_3, x_1^5x_2^3x_4, x_1^5x_2^2x_3^2, x_1^5x_2^2x_3x_4,$

$x_1^5x_2^2x_4x_5, x_1^5x_2x_3^2x_4, x_1^5x_2x_3^2x_5, x_1^5x_2x_3x_4x_5, x_1^5x_2x_3x_4^2, x_1^5x_3^2x_4x_5, x_1^5x_3^3x_4, x_1^5x_3^3x_5,$

$x_1^5x_3^2x_4^2, x_1^5x_3^2x_4x_5, x_1^4x_2^4x_3, x_1^4x_2^4x_4, x_1^4x_2^4x_5, x_1^4x_2^3x_3^2, x_1^4x_2^3x_4^2, x_1^4x_2^3x_3x_4, x_1^5x_2^2x_4^2,$
$x_1^5x_2^2x_3x_4, x_1^5x_2^2x_4x_5, x_1^5x_2x_3^3, x_1^5x_2x_3^2x_4, x_1^5x_2x_3^2x_5, x_1^5x_2x_3x_4^2, x_1^5x_2x_4^2x_5, x_1^4x_2^3x_4^2,$
$x_1^4x_2^3x_3x_4, x_1^5x_3^2x_4^2, x_1^5x_3^2x_4x_5, x_1^5x_3x_4^2x_5, x_1^4x_2^4x_3, x_1^4x_2^4x_4, x_1^4x_2^4x_5, x_1^4x_2^3x_3^2,$
$x_1^5x_3^3x_5, x_1^4x_2^3x_3x_5, x_1^4x_2^3x_4^2, x_1^4x_2^3x_4x_5, x_1^4x_2^2x_3^3, x_1^4x_2^2x_3^2x_4, x_1^4x_2^2x_3^2x_5, x_1^4x_2^2x_3x_4^2,$
$x_1^4x_2x_3^3x_4, x_1^4x_2x_3^3x_5, x_1^4x_2x_3^2x_4^2, x_1^4x_2x_3^2x_4x_5, x_1^4x_2^2x_3x_4x_5, x_1^4x_2x_3^3x_4, x_1^4x_2x_3^3x_5,$
$x_1^4x_2x_3^2x_4x_5, x_1^4x_2x_3x_4^2x_5, x_1^4x_3^3x_4^2, x_1^4x_3^2x_4^2x_5, x_1^4x_3^3x_4x_5, x_1^3x_2^4x_3^2, x_1^3x_2^4x_4^2, x_1^3x_2^4x_4x_5,$
$x_1^3x_2^4x_3x_5, x_1^3x_2^3x_3^3, x_1^3x_2^3x_3^2x_4, x_1^3x_2^3x_3^2x_5, x_1^3x_2^3x_3x_4^2, x_1^3x_2^3x_3x_4x_5, x_1^3x_2^2x_3^3x_4, x_1^3x_2^2x_3^3x_5,$
$x_1^3x_2^2x_3^2x_4^2, x_1^3x_2^2x_3^2x_4x_5, x_1^3x_2^2x_3x_4^2x_5, x_1^3x_2x_3^3x_4^2, x_1^3x_2x_3^3x_4x_5, x_1^3x_2x_3^2x_4^2x_5, x_1^3x_3^3x_4^2x_5,$
$x_1^2x_2^4x_3^2x_4, x_1^2x_2^4x_3^2x_5, x_1^2x_2^4x_3x_4x_5, x_1^2x_2^3x_3x_4^2x_5, x_1^2x_2^3x_3^2x_4^2, x_1^2x_2^3x_3^2x_4x_5, x_1^2x_2^2x_3^3x_4^2,$
$x_1^2x_2^2x_3^2x_4^2x_5, x_1^2x_2x_3^3x_4^2x_5, x_1x_2^4x_3^3x_4, x_1x_2^4x_3^3x_5, x_1x_2^4x_3^2x_4^2, x_1x_2^4x_3x_4^2x_5, x_1x_2^3x_3^2x_4^2x_5,$
$x_1x_2^3x_3^3x_4x_5, x_1x_2^2x_3^3x_4^2x_5, x_2^4x_3^3x_4^2, x_2^4x_3^3x_4x_5, x_2^4x_3^2x_4^2x_5, x_2^3x_3^3x_4^2x_5, x_1^5x_2^3x_5, x_1^5x_2x_3^3,$
$x_1^5x_3^3x_4, x_1^5x_2^2x_4^2, x_1^5x_2^2x_3^2, x_1^5x_2x_3x_4x_5, x_1^4x_2^2x_4^2x_5, x_1^4x_2x_3^2x_4^2, x_1^3x_2^4x_3x_4, x_1^2x_2^4x_3^3,$
$x_1^2x_2^2x_3^3x_4x_5, x_1x_2^3x_3^3x_4^2, x_1^5x_2^2x_3x_5, x_1^5x_2^2x_3x_5, \rangle.$

- *For* $k = 10, H^{2k}(\mathcal{F}\ell_6(\mathbb{C});\mathbb{Z}) = H^{2.10} = \langle x_1^5x_2^4x_3, x_1^5x_2^4x_4, x_1^5x_2^4x_5, x_1^5x_2^3x_3^2, x_1^5x_2^3x_4^2,$
$x_1^5x_2^3x_3x_5, x_1^5x_2^2x_3^2x_4, x_1^5x_2^2x_3^3, x_1^5x_2^2x_3x_4^2, x_1^5x_2^2x_4^2x_5, x_1^5x_2^2x_3x_4x_5, x_1^5x_2^2x_3^2x_5, x_1^5x_2x_3^3x_4,$
$x_1^5x_2x_3^3x_4, x_1^5x_2x_3^3x_5, x_1^5x_2x_3^2x_4^2, x_1^5x_2x_3^2x_4x_5, x_1^5x_2x_3x_4^2x_5, x_1^5x_3^3x_4^2, x_1^5x_3^2x_4^2x_5, x_1^4x_2^4x_3^2,$
$x_1^4x_2^4x_4x_5, x_1^4x_2^4x_3x_5, x_1^4x_2^4x_4^2, x_1^4x_2^3x_3^3, x_1^4x_2^4x_3^2x_4, x_1^4x_2^4x_3^2x_5, x_1^4x_2^3x_4^2x_5, x_1^4x_2^3x_4^2x_5,$
$x_1^4x_2^3x_3x_4^2, x_1^4x_2^3x_3^2x_4, x_1^4x_2^2x_3^2x_4^2, x_1^4x_2^2x_3^3x_4, x_1^4x_2^2x_3^2x_4x_5, x_1^4x_2^2x_3x_4^2x_5, x_1^4x_2x_3^2x_4^2x_5,$
$x_1^4x_2x_3^3x_4x_5, x_1^4x_3^3x_4^2x_5, x_1^3x_2^4x_3^2x_4, x_1^3x_2^4x_3^2x_5, x_1^3x_2^4x_3^3, x_1^3x_2^4x_3x_4x_5, x_1^3x_2^3x_3^3x_4,$
$x_1^3x_2^3x_3^2x_4^2, x_1^3x_2^3x_3^2x_4x_5, x_1^3x_2^2x_3^3x_4^2, x_1^3x_2^2x_3^3x_4x_5, x_1^3x_2^2x_3^2x_4^2x_5, x_1^3x_2^3x_3^1x_4^2x_5, x_1^5x_2^3x_3x_4,$
$x_1^2x_2^4x_3^3x_5, x_1^2x_2^4x_3^2x_4x_5, x_1^2x_2^4x_3x_4^2x_5, x_1^2x_2^3x_3^2x_4^2x_5, x_1^2x_2^3x_3^3x_4^2, x_1^2x_2^3x_3^3x_4x_5,$
$x_1x_2^4x_3^3x_4x_5, x_1x_2^4x_3^2x_4^2x_5, x_1x_2^3x_3^3x_4^2x_5, x_2^4x_3^3x_4^2x_5, x_1^5x_2^3x_4x_5, x_1^5x_2^2x_3^2x_4, x_1^4x_2^4x_3x_4,$
$x_1^4x_2^3x_3x_4x_5, x_1^4x_2x_3^3x_4^2x_5, x_1^3x_2^3x_3^3x_5, x_1^3x_2x_3^3x_4^2x_5, x_1^2x_2^4x_3^3x_4, x_1^2x_2^2x_3^3x_4^2x_5, x_1x_2^4x_3^3x_4^2, \rangle$
.

- *For* $k = 11, H^{2k}(\mathcal{F}\ell_6(\mathbb{C});\mathbb{Z}) = H^{2.11} = \langle x_1^5x_2^4x_3^2, x_1^5x_2^4x_3x_4, x_1^5x_2^3x_3^2x_4, x_1^5x_2^3x_3x_4x_5,$
$x_1^5x_2^2x_3^2x_4x_5, x_1^5x_2^2x_3x_4^2x_5, x_1^5x_2^1x_3^2x_4^2x_5, x_1^5x_3^3x_4^2x_5, x_1^4x_2^4x_3^3, x_1^4x_2^4x_3^2x_4, x_1^4x_2^4x_3^1x_4^2,$
$x_1^4x_2^4x_3^1x_4x_5, x_1^4x_2^3x_3^3x_4, x_1^4x_2^3x_3^3x_5, x_1^4x_2^4x_3^2x_5, x_1^4x_2^4x_3^2x_4, x_1^4x_2^3x_3^2x_4x_5, x_1^4x_2^3x_3x_4^2x_5,$
$x_1^4x_2^2x_3^2x_4^2x_5, x_1^4x_2^2x_3^3x_4x_5, x_1^4x_2x_3^3x_4^2x_5, x_1^4x_2^3x_3^3x_4^2, x_1^3x_2^3x_3^3x_4x_5, x_1^3x_2^3x_3^2x_4^2x_5,$
$x_1^3x_2^4x_3x_4^2x_5, x_1^3x_2^4x_3^2x_4x_5, x_1^3x_2^4x_3^2x_4^2, x_1^2x_2^4x_3^2x_4^2, x_1^2x_2^4x_3^3x_4x_5, x_1^4x_2^3x_3^2x_4^2, x_1^2x_2^4x_3^2x_4^2x_5,$
$x_1x_2^4x_3^3x_4^2x_5, x_1^4x_2^2x_3^3x_4^2, x_1^3x_2^2x_3^3x_4^2x_5, x_1^3x_2^4x_3^3x_4, x_1^3x_2^4x_3^3x_5, x_1^2x_2^3x_3^3x_4^2x_5, x_1^5x_2^2x_3^3x_4, \rangle.$

- *For* $k = 12, H^{2k}(\mathcal{F}\ell_6(\mathbb{C});\mathbb{Z}) = H^{2.12} = \langle x_1^5x_2^4x_3^3, x_1^5x_2^4x_3^2x_4, x_1^5x_2^4x_3x_4x_5, x_1^5x_2^4x_4^2x_5,$

$x_1^5x_2^3x_3^3x_4, x_1^5x_2^3x_3^2x_4^2, x_1^5x_2^3x_3^2x_4x_5, x_1^5x_2^3x_3x_4^2x_1, x_1^5x_2^3x_3^2x_4^2, x_1^5x_2^2x_3^3x_4^2, x_1^5x_2^2x_3^2x_4^2x_5,$
$x_1^5x_2^2x_3^3x_4x_5, x_1^5x_2^1x_3^3x_4^2x_5, x_1^4x_2^4x_3^3x_4, x_1^4x_2^4x_3^2x_4x_5, x_1^4x_2^4x_3^2x_4x_5, x_1^4x_2^3x_3^3x_4^2, x_1^3x_2^4x_3^3x_4^2,$
$x_1^4x_2^3x_3^2x_4^2x_5, x_1^4x_2^2x_3^3x_4^2x_1, x_1^3x_2^4x_3^3x_4x_5, x_1^3x_2^4x_3^2x_4^2x_5, x_1^3x_2^3x_3^3x_4^2x_5, x_1^4x_2^3x_3^3x_4x_5,$
$x_1^2x_2^4x_3^3x_4^2x_5, x_1^5x_2^4x_3x_4^2, x_1^5x_2^4x_3x_4^2, \rangle.$

- *For* $k = 13, H^{2k}(\mathcal{F}\ell_6(\mathbb{C}); \mathbb{Z}) = H^{2.13} = \langle x_1^5x_2^4x_3^3x_4, x_1^5x_2^4x_3^3x_5, x_1^5x_2^4x_3x_4^2x_5, x_1^5x_2^3x_3^2x_4^2x_5,$
  $x_1^5x_2^4x_3^2x_4x_5, x_1^5x_2^3x_3^3x_4x_5, x_1^5x_2^2x_3^3x_4^2x_5, x_1^4x_2^3x_3^3x_4^2x_5, x_1^4x_2^4x_3^2x_4^2x_5, x_1^4x_2^4x_3^3x_4x_5,$
  $x_1^3x_2^4x_3^3x_4^2x_5, x_1^4x_2^4x_3^3x_4^2, x_1^4x_2^4x_3^3x_4x_5, x_1^4x_2^4x_3^3x_4^2x_5, \rangle.$

- *For* $k = 14, H^{2k}(\mathcal{F}\ell_6(\mathbb{C}); \mathbb{Z}) = H^{2.14} = \langle x_1^5x_2^4x_3^3x_4^2, x_1^4x_2^4x_3^3x_4^2x_5, x_1^5x_2^3x_3^3x_4^2x_5, x_1^5x_2^4x_3^2x_4^2x_5,$
  $x_1^5x_2^4x_3^3x_4x_5\rangle.$

- *For* $k = 15, H^{2k}(\mathcal{F}\ell_6(\mathbb{C}); \mathbb{Z}) = H^{2.15} = \langle x_1^5x_2^4x_3^3x_4^2x_5\rangle.$

Therefore the flag varieties are generated by the basic classes with generators $x_1, x_2, x_3, x_4, x_5, x_6$

## 2.10 Schubert Polynomials

Schubert polynomials are representatives of cohomology classes in flag varieties. In $n$ variables they are indexed by permutations $\sigma \in S_n$. They also form a basis for the covariant of $S_n$ action on $\mathbb{Z}[x_1, x_2, \ldots], n < \infty$.

**Definition 2.10.1.** *Let* $S_n$ *be a group such that* $S_n = \{s_1, s_2, \ldots, s_{n-1}\}$ *with the following relations ,*

- $s_1^2 = e\forall, 1 \leq i \leq n-1.$
- $s_is_j = s_js_i$ *if,* $\mid i - j \mid \geq 2.$
- $s_is_{i+1}s_i = s_{i+1}s_is_{i+1}, 1 \leq i \leq n-1.$

  *where* $s_i = (i, i+1)$ *is a simple transposition and* $e$ *the identity element of* $S_n$*.*

**Definition 2.10.2.** *Given a permutation* $\sigma = s_{a_1}s_{a_2}\cdots s_{a_n}$ *where* $n = l(\sigma)$*. then* $\partial_{a_1}\partial_{a_2}\cdots\partial_{a_n}$ *are independent of the representation, hence we define the Schubert Polynomial* $\Omega_\sigma$ *for every permutation* $\sigma \in S_n$ *for every* $f \in R^n$ *by ,*

$$\Omega_\sigma = \partial_\sigma^{-1}\sigma_0 x_1^{n-1}x_2^{n-2}\cdots x_{n-1}^1 \tag{2.63}$$

.

**Lemma 2.10.3.** *[Fulton & Fulton (1997)]*

*For $\sigma_0 = n, n-1, \cdots, 2, 1,$ the permutation of longest length in $S_n$ is given by*

$$\Omega_{\sigma_0} = x_1^{n-1}.x_2^{n-2}.\cdots .x_{n-2}^2.x_{n-1}. \tag{2.64}$$

### 2.10.1 Properties of Schubert Polynomials

The Schubert polynomials has the following properties.

1. If $\sigma_0$ is the permutation of longest length in $S_n$, then $\Omega_{\sigma_0} = x_1^{n-1}x_2^{n-2}\cdots x_{n-1}^1$.
2. $\partial_i\Omega_\sigma = \Omega_\sigma s_i$ if $\sigma(i) > \sigma(i+1)$ where $s_i$ is the transposition $(i, i+1)$ .
3. $\Omega_{id} = 1$.
4. if $S_n$ is the transposition $(n, n+1)$ then $\Omega_{S_n} = x_1, + \cdots + x_n$.
5. Schubert polynomials have positive coefficient .

**Lemma 2.10.4.** *[Fulton & Fulton (1997)]*

1. *For any $i, \partial_i(\Omega_\sigma) = \Omega_{\sigma.s_i}$, if $\sigma(i) > \sigma(i+1)$. and $\partial_i(\Omega_\sigma) = 0$, otherwise.*
2. *$\Omega_{\sigma_0} = x_1^{n-1}x_2^{n-2}\cdots x_{n-2}^2 x_{n-1}$.*
3. *For each $i$, $\Omega_{s_i} = x_1 + x_2 + \cdots + x_i$.*

The Schubert polynomial for the symmetric group $S_n$ is derived by using the formula for the divided difference given by,

$$\partial_i(\Omega_\sigma) = \frac{(p - s_i p)}{x_i - x_{i+1}}. \tag{2.65}$$

### 2.10.2 Examples of Schubert Polynomials

**Example 2.10.5.** *Calculating the Schubert polynomials for $S_n$ where $n = 3$.*

*For $n = 3$ the permutations will be $S_3 = 6$ permutations .*

$$\sigma = \{123, 132, 213, 231, 312, 321\}.$$

$$\sigma_0 = 321 = x_1^2 x_2^1 x_3^0 = x_1^2 x_2^1,$$

*which is the permutation with the longest length.*

*The permutation* $\sigma_0 = 321 = x_1^2 x_2^1$, *using the formula for divided difference*

$$\partial_i(\Omega_\sigma) = \frac{(p - s_i p)}{x_i - x_{i+1}}.$$

1. $\Omega(312) = \partial_2(\Omega(321)) = \frac{x_1^2x_2 - x_1^2x_3}{x_2 - x_3} = \frac{x_1^2(x_2 - x_3)}{x_2 - x_3} = x_1^2.$
2. $\Omega(231) = \partial_1(\Omega(321) = \frac{x_1^2x_2 - x_1x_2^2}{x_1 - x_2} = \frac{x_1x_2(x_1 - x_2)}{x_1 - x_2} = x_1x_2.$
3. $\Omega(213) = \partial_2\partial_1(\Omega(321) = \frac{x_1^2x_2 - x_1x_2^2}{x_1 - x_2} = \frac{x_1x_2(x_1 - x_2)}{x_1 - x_2} = x_1x_2$

   $\partial_2(x_1x_2) = \frac{x_1x_2 - x_1x_3}{x_2 - x_3} = \frac{x_1(x_2 - x_3)}{x_2 - x_3} = x_1.$
4. $\Omega(132) = \partial_1\partial_2(\Omega(321) = \frac{x_1^2x_2 - x_1^2x_3}{x_2 - x_3} = \frac{x_1^2(x_2 - x_3)}{x_2 - x_3} = x_1^2$

   $\partial_1(x_1^2) = \frac{x_1^2 - x_2^2}{x_1 - x_2} = \frac{(x_1 + x_2)(x_1 - x_2)}{x_1 - x_2} = x_1 + x_2.$
5. $\Omega(123) = \partial_1\partial_2\partial_1(\Omega(321) = \frac{x_1^2x_2 - x_1x_2^2}{x_1 - x_2} = \frac{x_1x_2(x_1 - x_2)}{x_1 - x_2} = x_1x_2$

   $\partial_2(x_1x_2) = \frac{x_1x_2 - x_1x_3}{x_2 - x_3} = \frac{x_1(x_2 - x_3)}{x_2 - x_3} = x_1.$

   $\partial_1(x_1) = \frac{x_1 - x_2}{x_1 - x_2} = 1.$

**Example 2.10.6.** *Given the permutation* $w = (4132)$, *the Schubert polynomial is given by* $= x_1^3x_2 + x_1^3x_3$.

## 2.11 The Code of a Permutation

For any $\sigma$ in $S_n$ and for each $i \geq 1$, $c_i(\sigma) = card.(j : j > i, \sigma(j) < \sigma(i)) \in \mathbb{N}^n$ This is the number of points in the *ith* row of the diagram of $\sigma$. The code of the permutation $\sigma$ is the vector $c(\sigma) = (c_1(\sigma), \cdots, c_n(\sigma)) \in \mathbb{N}^n$.

Table 2.1: The Schubert Polynomials for the Permutations of $S_3$

| Permutations | Transpositions | Length | Schubert Polynomial |
|---|---|---|---|
| 123 | nil | 0 | 1 |
| 132 | $s_2$ | 1 | $x_1 + x_2$ |
| 213 | $s_1$ | 1 | $x_1$ |
| 231 | $s_1 s_2$ | 2 | $x_1 x_2$ |
| 312 | $s_2 s_1$ | 2 | $x_1^2$ |
| 321 | $s_1 s_2 s_1$ | 3 | $x_1^2 x_2$ |

Source: [Fulton & Fulton (1997)]

Table 2.2: The Schubert Polynomials for the Permutations of $S_4$

| $S/n$ | Permutatns | Length | $t_{ij}$ Products | $X_\sigma$ polynomials |
|---|---|---|---|---|
| 1 | 1234 | 0 | nil | 1 |
| 2 | 1324 | 1 | $s_2$ | $x_1 + x_2$ |
| 3 | 1342 | 2 | $s_2s_3$ | $x_1x_2 + x_3x_1 + x_3x_2$ |
| 4 | 1243 | 1 | $s_3$ | $x_2 + x_3$ |
| 5 | 1423 | 2 | $s_3s_2$ | $x_1^2 + x_2^2 + x_1x_2$ |
| 6 | 1432 | 3 | $s_2s_3s_2$ | $x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_2^2x_3 + x_1x_3^2$ |
| 7 | 2134 | 1 | $s_1$ | $x_1$ |
| 8 | 2314 | 2 | $s_1s_2$ | $x_1x_2$ |
| 9 | 2341 | 3 | $s_1s_2s_3$ | $x_1x_2x_3$ |
| 10 | 2143 | 2 | $s_3s_1$ | $x_1^2 + x_1x_2 + x_1x_3$ |
| 11 | 2413 | 3 | $s_3s_1s_2$ | $x_1^2x_2 + x_1x_2^2$ |
| 12 | 2431 | 4 | $s_1s_2s_3s_2$ | $x_1^2x_2x_3 + x_1x_2^2x_3$ |
| 13 | 3124 | 2 | $s_2s_1$ | $x_1^2$ |
| 14 | 3214 | 3 | $s_2s_1s_2$ | $x_1^2x_2$ |
| 15 | 3241 | 3 | $s_1s_2s_3s_1$ | $x_1^2x_2x_3$ |
| 16 | 3412 | 4 | $s_2s_3s_1s_2$ | $x_1^2x_2^2$ |
| 17 | 3421 | 5 | $s_1s_2s_3s_1s_2$ | $x_1^2x_2^2x_3$ |
| 18 | 3142 | 3 | $s_2s_3s_1$ | $x_1^2x_2 + x_1^2x_3$ |
| 19 | 4123 | 3 | $s_3s_2s_1$ | $x_1^3$ |
| 20 | 4132 | 4 | $s_3s_2s_3s_1$ | $x_1^3x_2 + x_1^3x_3$ |
| 21 | 4213 | 4 | $s_3s_1s_2s_1$ | $x_1^3x_2$ |
| 22 | 4312 | 5 | $s_3s_2s_3s_1s_2$ | $x_1^3x_2^2$ |
| 23 | 4231 | 5 | $s_1s_2s_3s_2s_1$ | $x_1^3x_2x_3$ |
| 24 | 4321 | 6 | $s_1s_2s_3s_1s_2s_1$ | $x_1^3x_2^2x_3$ |

Source: [Fulton & Fulton (1997)]

Table 2.3: The length and codes of the permutations of $S_4$

| s/n | Permutation | length | Code |
|---|---|---|---|
| 1 | 1234 | 0 | (0,0,0) |
| 2 | 2134 | 1 | (1,0,0) |
| 3 | 1324 | 1 | (0,1,0) |
| 4 | 1243 | 1 | (0,0,1) |
| 5 | 2314 | 2 | (1,1,0) |
| 6 | 2143 | 2 | (1,0,1) |
| 7 | 1342 | 2 | (0,1,1) |
| 8 | 3124 | 2 | (2,0,0) |
| 9 | 1423 | 2 | (0,2,0) |
| 10 | 2341 | 3 | (1,1,1) |
| 11 | 3214 | 3 | (2,1,0) |
| 12 | 3142 | 3 | (2,0,1) |
| 13 | 1432 | 3 | (0,2,1) |
| 14 | 2413 | 3 | (1,2,0) |
| 15 | 4123 | 3 | (3,0,0) |
| 16 | 3412 | 4 | (2,2,0) |
| 17 | 4213 | 4 | (3,1,0) |
| 18 | 4132 | 4 | (3,0,1) |
| 19 | 3241 | 4 | (2,1,1) |
| 20 | 2431 | 4 | (1,2,1) |
| 21 | 4312 | 5 | (3,2,0) |
| 22 | 4231 | 5 | (3,1,1) |
| 23 | 3421 | 5 | (2,2,1) |
| 24 | 4321 | 6 | (3,2,1) |

Source: [Fulton & Fulton (1997)]

## 2.12 Empirical Review

Schubert varieties are among the best studied classes of singular algebraic varieties. In 1874, Schubert calculus was named after Hermann Schubert, who initiated the study of the intersection theory on the Grassmannians in 1879 and Zeuthen continued this study in the $19^{th}$ century under the heading of enumerative geometry.

Kazhdan & Lusztig (1979) defined a condition called rational smoothness which is interpreted in terms of Kazhdan-Lusztig polynomials. Lakshmibai & Seshadri (1984) also determined smoothness and singularity by considering the set of points for which the Schubert varieties are singular.

Many authors have worked on the general properties of singularities of Schubert varieties, there are still many interesting unanswered questions about properties which not all Schubert varieties hold in common. The fundamental work of Ramanathan (1985), showed that all Schubert varieties are Cohen-Macaulay and Normal.

Deodhar (1985) worked on the local Poincaré duality and non singularity of Schubert varieties. he also established that smoothness in type A is same as rational smoothness.

Wolper (1989) presented a simple algorithm for deciding whether a Schubert variety in $G/P$ where $G = SL_n$ is singular. This led to a geometric characterisation of the non-singular Schubert varieties as sequences of Grassmannian bundles over Grassmannians.

Furthermore, Lakshmibai & Sandhya (1990), determined smoothness of the singular Schubert varieties in flag manifold using the method of pattern avoidance. Carrell (1994) showed that for $\sigma \in S_n$, the $X_\sigma$ is smooth if the poincaré polynomial is palindromic.

Brion (1999) worked on the generic singularities of certain Schubert varieties and then Gasharov (2001) worked on the sufficiency of the Lakshmibai-Sandhya singularity conditions for Schubert varieties.

Moreover, Billey & Postnikov (2005) presented a uniform approach to pattern avoidance in general terms of root systems and also extended the Lakshmibai-Sandhya criterion to the case of an arbitrary semi simple Lie group $G$. As a consequences of their main theorem, two additional criteria for (rational) smoothness in terms of root system embeddings and double parabolic factorisation were derived .

Woo (2010) determined which Schubert varieties are Gorenstein and also introduced a notion called Bruhat-restricted pattern. The interval pattern avoidance is a further generalisation which has the advantage of a geometric interpretation. The question of where non-Gorenstein Schubert varieties are Gorenstein was further pursued along with analogous questions for other local properties.

Billey & Postnikov (2005) published that the affine type A rationally smooth Schubert varieties are characteriesd using the $3412, 4231$ permutation pattern avoidance.

A new combinatorial notion was formulated by Woo & Yong (2008), used for characterising the singularity of Schubert varieties of flag manifolds and their local invariants. a uniform language was also provided to study semi continuously stable invariants of singularities . Also a number of authors have been able to answer the two most important questions about singularities of any given Schubert varietiy and the questions are :

- which Schubert varieties ($X_\sigma$) are singular ?
- where are the Schubert varieties ($X_\sigma$) singular ?

These questions were answered by a geometric characterisation by Ryan (1987). Beside, Oh et al. (2008) worked on the fact that $P_\sigma(q) = R_\sigma(q)$ iff the Schubert variety $X_\sigma$ is smooth with reference to Carrell (1994) which states that the Schubert variety $X_\sigma$ is smooth iff the Poincaré polynomial $P_\sigma(q)$ is Palindromic, that is if $P_\sigma(q) = q^{l(\sigma)} P_\sigma(q^{-1})$. if $X_\sigma$ is not smooth then the polynomial $P_\sigma(q)$ is not Palindromic but since the polynomial $R_\sigma(q)$ is always Palindromic then $P_\sigma(q) \neq R_\sigma(q)$ in this situation.

Furthermore Ulfarsson (2011) proved new connections between permutation patterns and singularities of Schubert varieties ($X_\sigma$) in the complete flag varieties $\mathcal{F}\ell_n(\mathbb{C})$, giving a new characterisation of factorial and Gorenstein varieties in terms of which bivincular patterns the permutation $\sigma$ avoids.

Billey & Crites (2012) studied the case when $\sigma$ is the affine weyl group of type A or the affine permutations and developed the notion of pattern avoidance for affine permutations. They also worked on the characterisation of the rational smooth Schubert varieties corresponding to affine permutations in terms of patterns 4231 and 3412 and the twisted spiral permutations

Recently, Abe & Billey (2016) presented analoques of Lakshmibai-Sandhya's theorem for determining if a given Schubert variety is smooth or not for all classical types $B_n, C_n$, and $D_n$. However, these constructions, including the definition of patterns depend on a particular way to represent elements in classical weyl groups as signed permutations. They also surveyed the many results and generalisation in the characterisation of Schubert varieties and showed the benefits of using pattern avoidance characterisation in terms of linear time algorithm.

Kim & Park (2018) Characterise standard embedding of smooth Schubert varieties in rational homogeneous manifolds of Picard number 1, by means of varieties of minimal rational tangents. They mainly considered non homogeneous smooth Schubert varieties in Symplectic Grassmannians . Gillespie (2019) provided an overview of many of the established combinatorial and algebraic tools of Schubert calculus. It is intended as a guide for readers with a combinatorial bent to understand the geometric and topological aspects of Schubert calculus.

More Recently, Cibotaru (2020) gave a complete list of smooth and rationally smooth normalised Schubert varieties in the twisted affine grassmannians associated with a tamely ramified group and a special vertex of its Brubat-Tits building. Besson & Hong (2022) introduced R-operators that are linked to positive roots which satisfies Braid relations.

Also, Gatto & Salehyan (2021) extended the Schubert derivatives to the infinite exterior power of a free $\mathbb{Z}-$ module of infinite rank. Huh (2022) showed that the intersection cohomology module of a matroid obeys poincaré duality . They also obtained proves for the nonnegativity of the Kazhdan-Lusztig polynomials for all matroids.

For any $\sigma \in S_n$ , Gaetz & Gao (2023) gave an exact equation for the least positive power in the kazhdan-Lusztig polynomial. The best possible upper bound on $h(\sigma)$ in simple laced types.

## 2.13 Theoretical Framework

This research work is based on some past results and theorems that has been proved by various authors in the literatures of singularities and smoothness of Schubert varieties. The following are some of these results:

**Proposition 2.13.1.** *[Carrell (1994)]*

*The following are equivalent:*

- $X_\sigma$ *is Smooth;*
- $X_\sigma$ *is smooth at id;*
- $|t_{ij} \leq \sigma| = l(\sigma)$;
- $\sigma$ *avoids 3412 and 4231;*
- *The Kazhdan-Lusztig Polynomial* $P_{id,\sigma}(q) = 1$;
- $P_{v,\sigma}(q) = 1 \forall v \leq \sigma$;
- *The Bruhat graph for* $\sigma$ *is regular;*

- *For* $\sigma, P_\sigma(t) = \sum_{v \leq \sigma} t^{l(v)}$ *is symmetrical;*
- *For* $\sigma, P_v(t) = \prod_{i=1}^{k}(1 + t + t^2 + \cdots + t^{e_i})$ *factors nicely.*

**Theorem 2.13.2.** *[Lakshmibai & Seshadri (1984)]*

*For* $v \leq \sigma \in S_n$*, the tangent space of* $X_\sigma$ *at* $v$ *is*

$$T_v(X_\sigma) \cong Span\{E_{v(j)}, v(i) : i < j, vt_{ij} \leq \sigma\} \text{ and}$$

$$dimT_v(X_\sigma) = \sharp\{(i < j) : vt_{ij} \leq \sigma\}.$$

**Theorem 2.13.3.** *[Carrell (1994)]*

*For any permutation* $\sigma \in S_n$*, the Schubert variety* $X_\sigma$ *is smooth iff the Poincaré polynomial is Symmetrical .*

# Chapter 3

# METHODOLOGY

## 3.1 Preamble

Schubert varieties are algebraic varieties studied in various types, where the type defines the underlying group. The smoothness and singularity of Schubert varieties in type A, has been characterised by different authors making use of different methods of characterisation such as ,

1. Tangent spaces method.
2. Permutation Pattern Avoidance method.
3. Poincaré Polynomial method.
4. The Essential set method.

In this chapter we adopt the Palindromic Poincaré polynomial and the essential set methods of characterising the smooth and singular Schubert varieties.

## 3.2 The Palindromic Poincaré Polynomial Method

The Poincaré polynomials are used to determine smoothness of Schubert varieties. This was first used by Carrell (1994) to determine smoothness and singularity of Schubert varieties by showing that the Poincaré polynomial is Palindromic.

### 3.2.1 Poincaré Polynomial

**Definition 3.2.1.** *[Deodhar (1985)] For a complex algebraic variety X, its Poincaré polynomial is given by*

$$P_x(t) = \sum_{i \geq 0} dim_{\mathbb{C}}(H^i(X))t^i. \tag{3.1}$$

*Where $H^i(X)$ is the singular homology of X.*

**Definition 3.2.2.** *The Poincaré polynomial of a Schubert variety ($X_\sigma$) is said to be the rank generating function for the interval $[id, \sigma]$, where the rank is the number of inversions $P_\sigma(t) = \sum_{v \leq \sigma} t^{l(v)}$ and the sum is over all elements $v \leq \sigma$ in the Bruhat-Chevalley order on W.*

**Definition 3.2.3.** *A Poincaré polynomial $p(t) = v_0 + v_1 t + \cdots + v_r t^r$ is Palindromic if defined with respect to the length function and via the Bruhat order, $v \leq \sigma \Leftrightarrow l(v) \leq l(\sigma)$ as $p(t) = t^r p(t^{-1})$.*

**Theorem 3.2.4.** *[Carrell (1994)]*

*For any permutation $\sigma \in S_n$ the Schubert variety $X_\sigma$ is smooth if and only if the Poincaré polynomial is Palindromic.*

**Example 3.2.5.** *For the Schubert variety $X_{4321}$ which is also a flag, Carrell (1994) showed that for any permutation $\sigma \in S_n$ where $n = 4$ we have the Bruhat order.*

| *Length* | *Permutations* |
|---|---|
| 6 | (4321) |
| 5 | (4312), (4231), (3421) |
| 4 | (4132), (4213), (3412), (2431), (3241) |
| 3 | (1432), (4123), (2413), (3142), (3214), (2341) |
| 2 | (1423), (1342), (2143), (3124), (2314) |
| 1 | (1243), (1324), (2134) |
| 0 | (1234) |

*Hence, the Poincaré polynomial of the Schubert variety $(X_{4321}) = \mathcal{F}\ell_4(\mathbb{C})$ for $n = 4$ with respect to the variable t is*

$$P_\sigma(\mathcal{F}\ell_4(\mathbb{C}), t) = t^6 + 3t^5 + 5t^4 + 6t^3 + 5t^2 + 3t + 1. \tag{3.2}$$

$$\begin{pmatrix} 1 & 3 & 5 & 6 & 5 & 3 & 1 \end{pmatrix}.$$

*Hence, $\mathcal{F}\ell_4(\mathbb{C})$ is smooth.*

**Example 3.2.6.** *For the Schubert variety $X_{3412}$ with permutation $\sigma = 3412 \in S_n$ where $n = 4$ we have the Bruhat order.*

| *Length* | *Permutations* |
|---|---|
| 4 | (3412) |
| 3 | (3142), (3214), (2341), (4123) |
| 2 | (1342), (3124), (2314), (1423), (2143) |
| 1 | (1324), (1243), (2134)) |
| 0 | (1234) |

$$P_\sigma((X_{3412}), t) = t^4 + 4t^3 + 5t^2 + 3t^1 + t^0 = t^4 + 4t^3 + 5t^2 + 3t + 1. \quad (3.3)$$

$$(1 \quad 4 \quad 5 \quad 3 \quad 1).$$

*Hence $(X_{3412})$ is singular since the Poincaré polynomial is not palindromic.*

## 3.3 The Essential set method

The essential set method uses the Jacobian criterion for determining smoothness and singularity of algebraic varieties. In this section we consider the diagram of a permutation, the essential sets of the permutation, the rank of the permutation and then the ideal defining the varieties using the essential sets.

### 3.3.1 Diagram of $\sigma$

The diagram of $\sigma$ denoted by $D'(\sigma)$ is given by

$$D'(\sigma) = \{(i, j) \in [n]^2 \ni \sigma(i) > j, \sigma^{-1}(j) < i\}. \quad (3.4)$$

**Remark 3.3.1.** *The number of elements in the $D^{'}(\sigma)$ is given by the codim$(X_\sigma)$ which is equal to $\begin{pmatrix} n \\ 2 \end{pmatrix} - l(\sigma)$.*

**Example 3.3.2.** *The diagram of $\sigma = 35142$ is $D'(\sigma) = \{(2, 3), (4, 1), (4, 3), (5, 1)\}$.*

**Example 3.3.3.** *The diagram of $\sigma = 51324$ is $D'(\sigma) = \{(3, 1), (4, 1), (5, 1), (5, 2), (5, 3)\}$.*

### 3.3.2 Essential Set

**Definition 3.3.4.** *The essential set of $\sigma$ is denoted by*

$$Ess'(\sigma) = \{(i, j) \in D'(\sigma) \ni (i-1, j), (i, j+1), (i-1, j+1) \notin D'(\sigma)\}. \tag{3.5}$$

**Remark 3.3.5.** *This set comprises of the north east corners of connected components in $D'(\sigma)$.*

**Example 3.3.6.** *The essential set of $\sigma = 35142$ is $Ess'(\sigma) = \{(2, 3), (4, 1), (4, 3)\}$.*

**Example 3.3.7.** *The essential set of $\sigma = 51324$ is $Ess'(\sigma) = \{(3, 1), (5, 2), (5, 3)\}$.*

**Remark 3.3.8.**

- *The $Ess^{'}(\sigma)$ is all on one row if and only if $\sigma$ has at most one ascent.*
- *All entries in $Ess^{'}(\sigma)$ are zero (0) entries in the canonical matrix form for $C_\sigma$.*

### 3.3.3 Rank Matrix Of The Permutation $\sigma$

To determine the rank matrix of $\sigma$, we recall the definition of the Schubert cell.

**Definition 3.3.9.** *The Schubert cell $C_\sigma$ is given by*

$$C_\sigma = \{V_0 \in \mathcal{F}\ell_n(\mathbb{C}) \mid dim(W_p \bigcap V_q) = r_\sigma(p, q), 1 \leq p, q \geq n\}. \tag{3.6}$$

$$\{V_0 \in \mathcal{F}\ell_n(\mathbb{C}) \mid dim(W_p \bigcap V_q) = \sharp\{i \leq p : \sigma(i) \leq q\} for 1 \leq p, q \leq n\}. \tag{3.7}$$

**Example 3.3.10.** *The rank matrix of the permutation $\sigma = 2413$ is*

$$R_\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 2 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

*The rank matrix of $\sigma = 2413$.*

**Example 3.3.11.** *The rank matrix of the permutation $\sigma = 35142$ is*

$$R_\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 2 & 3 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

*The rank matrix of* $\sigma = 35142$.

**Example 3.3.12.** *The rank matrix of the permutation* $\sigma = 51324$ *is*

$$R_\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 3 & 4 \\ 1 & 1 & 2 & 2 & 3 \\ 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

*The rank matrix of* $\sigma = 51324$.

### 3.3.4 Generating the Ideal of $\sigma$

**Definition 3.3.13.** *[Billey & Postnikov (2005)]*

*The Matrix Schubert variety is given by*

$$\{X \in Mat_{n\times n}(\mathbb{C}) : rk(X_{(i,j)}) \leq rk\sigma_{(i,j)}\ \forall i, j\}$$
$$= \left\{ X \in Mat_{n\times n}(\mathbb{C}) \;\middle|\; \begin{array}{l} rk\sigma_{(i,j)} + 1 \text{ minors vanish on} \\ \begin{pmatrix} x_{i1} & \cdots & x_{ij} \\ \vdots & & \vdots \\ x_{n1} & \cdots & xnj \end{pmatrix} \forall i, j \end{array} \right\}. \tag{3.8}$$

**Definition 3.3.14.** *The ideal of* $\sigma$ *determined by all* $[rk\sigma_{[i,j]} + 1]$ *minors of*

$$\begin{pmatrix} x_{i1} & \cdots & x_{ij} \\ \vdots & & \\ x_{n1} & \cdots & x_{nj} \end{pmatrix} \forall i, j. \tag{3.9}$$

**Proposition 3.3.15.** *Fulton & Fulton (1997) $I$ = ideal determined by the $[rk\sigma_{[i,j]}+1]$ minors of $X[i,j], \forall (i,j) \in Ess'(\sigma)$ Then $I_\sigma = I, \forall \sigma \in S_n$.*

**Example 3.3.16.** *The ideal of $\sigma = 35142$ is*

$$I_\sigma = \{\langle x_{41}, x_{51}, \begin{vmatrix} x_{41} & x_{42} \\ x_{51} & x_{52} \end{vmatrix}, \begin{vmatrix} x_{41} & x_{43} \\ x_{51} & x_{53} \end{vmatrix}, \begin{vmatrix} x_{42} & x_{43} \\ x_{52} & x_{53} \end{vmatrix}, \begin{vmatrix} x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \\ x_{41} & x_{42} & x_{43} \end{vmatrix}, \begin{vmatrix} x_{21} & x_{22} & x_{23} \\ x_{41} & x_{42} & x_{43} \\ x_{51} & x_{52} & x_{53} \end{vmatrix}, \begin{vmatrix} x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \\ x_{51} & x_{52} & x_{53} \end{vmatrix}, \begin{vmatrix} x_{31} & x_{32} & x_{33} \\ x_{41} & x_{42} & x_{43} \\ x_{51} & x_{52} & x_{53} \end{vmatrix} \rangle\}. \tag{3.10}$$

$$= \{\langle x_{41}, x_{51}, (x_{42}x_{53} - x_{52}x_{43})\rangle\}. \tag{3.11}$$

*Calculating the ideal for the permutation $\sigma = 35142$ . An element in $C_\sigma = B\sigma B$ has the form,*

$$\begin{bmatrix} * & * & 1 & 0 & 0 \\ * & * & 0 & * & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

*Given $M = x_{ij} \in B\sigma B$ and the essential points $x_{23}, x_{43}$, set to be equal to one (1).*

*Then the following equations are satisfied* $x_{41} = x_{51} = 0, \begin{vmatrix} x_{41} & x_{42} \\ x_{51} & x_{52} \end{vmatrix} = \begin{vmatrix} x_{41} & x_{43} \\ x_{51} & x_{53} \end{vmatrix} = 0$ , *and* $\begin{vmatrix} x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \\ x_{41} & x_{42} & x_{43} \end{vmatrix} = \begin{vmatrix} x_{21} & x_{22} & x_{23} \\ x_{41} & x_{42} & x_{43} \\ x_{51} & x_{52} & x_{53} \end{vmatrix} = \begin{vmatrix} x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \\ x_{51} & x_{52} & x_{53} \end{vmatrix} = \begin{vmatrix} x_{31} & x_{32} & x_{33} \\ x_{41} & x_{42} & x_{43} \\ x_{51} & x_{52} & x_{53} \end{vmatrix} = 0.$

*Hence,* $I_\sigma = I_{35142} = \{\langle x_{41}, x_{51}, (x_{42}x_{53} - x_{52}x_{43})\rangle\}$.

**Example 3.3.17.** *The ideal of the permutation $\sigma = 2413$ is*

$$I_\sigma = \left\langle x_{41}, x_{42}, \begin{vmatrix} x_{21} & x_{22} \\ x_{31} & x_{32} \end{vmatrix}, \begin{vmatrix} x_{21} & x_{22} \\ x_{41} & x_{42} \end{vmatrix}, \begin{vmatrix} x_{31} & x_{32} \\ x_{41} & x_{42} \end{vmatrix} \right\rangle = \left\langle x_{41}, x_{42}, \begin{vmatrix} x_{21} & x_{22} \\ x_{31} & x_{32} \end{vmatrix} \right\rangle. \tag{3.12}$$

$$= \langle x_{41}, x_{42}, x_{21}x_{32} - x_{22}x_{31} \rangle . \tag{3.13}$$

**Example 3.3.18.** *The ideal of $\sigma = 3412$ is*

$$I_\sigma = \left\langle x_{41}, \begin{vmatrix} x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \\ x_{41} & x_{42} & x_{43} \end{vmatrix} \right\rangle . \tag{3.14}$$

In order to solve for smoothness of the Schubert varieties, the Jacobian criterion is used on the equation defining the ideal of the Schubert varieties.

**Theorem 3.3.19.** *[Jacobian criterion]*

*Let $Y \in A^n$ given by $I(Y) = \{f_1, \cdots, f_r\}$ and $f_i = x_1, \cdots, x_n$. Then, $J(x_1, \cdots, x_n) = \left(\frac{\partial f_i}{\partial x_j}\right)$. For $p = (p_1, \cdots, p_n) \in A^n$ then,*

1. *$rkJ(p_1, \cdots, p_n) \leq codim_{A^n} Y = n - dimY$*

2. *$p$ is smooth $\in Y$ iff $rkJ(p_1, \cdots, p_n) = codim_{A^n} Y = n - dimY$ .*

**Example 3.3.20.** *Given that $\sigma = 35142$, Is $X_\sigma$ smooth ?*

*$X_\sigma$ is smooth everywhere iff it is smooth at $v = id$*

*The diagram of $\sigma = 35142$ is $D'(\sigma) = \{(2,3), (4,1), (4,3), (5,1)\}$.*

*The essential set of $\sigma = 35142$ is $Ess'(\sigma) = \{(2,3), (4,1), (4,3)\}$.*

*The ideal is generated for all $\sigma \in S_n$ by $rank(i,j)+1$ minors of $X(i,j), \forall (i,j) \in Ess'(\sigma)$.*

*The ideal for $\sigma = 35142$ is given by $I_{(35142)} = \{\langle x_{41}, x_{51}, (x_{42}x_{53} - x_{52}x_{43})\rangle\}$.*

$$J(x_1, \cdots, x_n) = \left(\frac{\partial f_i}{\partial x_j}\right) = \begin{bmatrix} \frac{\partial f_1}{\partial x_{41}} & \frac{\partial f_1}{\partial x_{42}} & \frac{\partial f_1}{\partial x_{43}} & \frac{\partial f_1}{\partial x_{51}} & \frac{\partial f_1}{\partial x_{52}} & \frac{\partial f_1}{\partial x_{53}} \\ \frac{\partial f_2}{\partial x_{41}} & \frac{\partial f_2}{\partial x_{42}} & \frac{\partial f_2}{\partial x_{43}} & \frac{\partial f_2}{\partial x_{51}} & \frac{\partial f_2}{\partial x_{52}} & \frac{\partial f_1}{\partial x_{53}} \\ \frac{\partial f_3}{\partial x_{41}} & \frac{\partial f_3}{\partial x_{42}} & \frac{\partial f_3}{\partial x_{43}} & \frac{\partial f_3}{\partial x_{51}} & \frac{\partial f_3}{\partial x_{52}} & \frac{\partial f_1}{\partial x_{53}} \end{bmatrix} .$$

*where $x_1 = x_{41}, x_2 = x_{42}, x_3 = x_{43}, x_4 = x_{51}, x_5 = x_{52}, x_6 = x_{53}$ and $f_1 = x_{41}, f_2 = x_{51}, f_3 = x_{42}x_{53} - x_{52}x_{43}$.*

$$J(x_{41}, x_{42}, x_{43}, x_{51}, x_{52}, x_{53}) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & x_{53} & -x_{52} & 0 & x_{43} & x_{42} \end{bmatrix} ,$$

*J(I) is obtained by setting all the variables $x_{ij}$ equal to 0*

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

*Rank of* $J(I) = 2$ *(number of non zero rows), codim*$X_\sigma = \begin{pmatrix} 5 \\ 2 \end{pmatrix} - 6 = 4$ .

*$X_{35142}$ is singular, since the rank of the Jacobian matrix of the equation defining the ideal is not equal to the co-dimension of the variety .*

**Example 3.3.21.** *Given that $\sigma = 51324$, Is $X_\sigma$ smooth?*

*$X_\sigma$ is smooth everywhere iff it is smooth at $v = id$.*

*The diagram of $\sigma = 51324$ is $D'(\sigma) = \{(3,1),(4,1),(5,1),(5,2),(5,3)\}$.*

*The essential set of $\sigma = 51324$ is $Ess'(\sigma) = \{(3,1),(5,2),(5,3)\}$.*

*The ideal is generated for all $\sigma \in S_n$ by $rank(i,j) + 1$ minors of $X(i,j)$ for all $(i,j) \in Ess'(\sigma)$.*

*The ideal for $\sigma = 51324$ is given by $I_{(51324)} = \{\langle x_{31}, x_{41}, x_{51}, x_{52}, x_{53} \rangle\}$.*

$$J(x_1, \cdots, x_n) = \left( \frac{\partial f_i}{\partial x_j} \right) = \begin{bmatrix} \frac{\partial f_1}{\partial x_{31}} & \frac{\partial f_1}{\partial x_{41}} & \frac{\partial f_1}{\partial x_{51}} & \frac{\partial f_1}{\partial x_{52}} & \frac{\partial f_1}{\partial x_{53}} \\ \frac{\partial f_2}{\partial x_{31}} & \frac{\partial f_2}{\partial x_{41}} & \frac{\partial f_2}{\partial x_{51}} & \frac{\partial f_2}{\partial x_{52}} & \frac{\partial f_2}{\partial x_{53}} \\ \frac{\partial f_3}{\partial x_{31}} & \frac{\partial f_3}{\partial x_{41}} & \frac{\partial f_3}{\partial x_{51}} & \frac{\partial f_3}{\partial x_{52}} & \frac{\partial f_3}{\partial x_{53}} \\ \frac{\partial f_4}{\partial x_{31}} & \frac{\partial f_4}{\partial x_{41}} & \frac{\partial f_4}{\partial x_{51}} & \frac{\partial f_4}{\partial x_{52}} & \frac{\partial f_4}{\partial x_{53}} \\ \frac{\partial f_5}{\partial x_{31}} & \frac{\partial f_5}{\partial x_{41}} & \frac{\partial f_5}{\partial x_{51}} & \frac{\partial f_5}{\partial x_{52}} & \frac{\partial f_5}{\partial x_{53}} \end{bmatrix}.$$

*where $x_1 = x_{31}, x_2 = x_{41}, x_3 = x_{51}, x_4 = x_{52}, x_5 = x_{53}$, and $f_1 = x_{31}, f_2 = x_{41}, f_3 = x_{51}, f_4 = x_{52}, f_5 = x_{53}$.*

$$J(x_{31}, x_{41}, x_{51}, x_{52}, x_{53}) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

*Rank of* $J(I) = 5$ *(number of non zero rows), codim*$X_\sigma = \begin{pmatrix} 5 \\ 2 \end{pmatrix} - 5 = 5$ .

*$X_{51324}$ is smooth, since the rank of the Jacobian matrix of the equation defining the ideal is equal to the co-dimension of the variety.*

# Chapter 4

# RESULTS AND DISCUSSION

This section comprises of the results obtained using the exponents of the monomials of the Schubert varieties and the equation defining the ideals of the Schubert variety via the Plucker coordinates to show smoothness. it also compares the equations derived through the plücker embedding map with that of the essential sets method.

## 4.1 Smoothness and Singularity of Schubert Varieties using the exponent of the monomials of the Schubert varieties

In this session smoothness of Schubert varieties using the exponents of the monomials through the Poincaré palindromic polynomial method is determined. The proof of the results and examples to support them are given.

### 4.1.1 Kazhdan-Lusztig Polynomials

**Definition 4.1.1.** *[Billey & Postnikov (2005)]*

*The Kazhdan-Lusztig polynomial is a polynomial in one variable that has the following properties:*

1. $P_{v,\sigma}(t) = 1$ *if* $v \leq \sigma$.

2. *The number of edges connected to* $P_{v,\sigma}(t)$ *is less or equal to* $\frac{1}{2}(l(\sigma) - l(v) - 1)$.

3. $P_{\sigma,\sigma}(t) = 1$.

4. $P_{v,\sigma}(t) \neq 0 \leftrightarrow v \leq \sigma$.

**Corollary 4.1.2.** *[Lakshmibai & Sandhya (1990)]*

*Let $\sigma \in W$. The i-th component (of the cohomology ring) $\mathbb{H}^i(X_\sigma) = 0$ for i odd. Furthermore*

$$\sum_i dim\mathbb{H}^i(X_\sigma)t^i = \sum_{v \leq \sigma} t^{l(v)} P_{v,\sigma}(t).$$

**Theorem 4.1.3.** *[Lakshmibai & Sandhya (1990)]*

*The following are equivalent for any $v \leq \sigma$ in $W$*

1. *$X_\sigma$ is rationally smooth at $e_v$.*

2. *$P_{x,\sigma}(t) = 1$ for all $v \leq x \leq \sigma$.*

**Theorem 4.1.4.** *[Billey & Postnikov (2005)]*

*Let $IH(\sigma)$ be the intersection cohomology sheaf of $X_\sigma$ with respect to middle perversity, then*

1. *$P_{v,\sigma}(t) = \sum dim(IH^{2i}(X_\sigma)v)q^i$ which implies that the coefficients of $P_{v,\sigma}(t)$ are nonnegative.*

2. *$P_{v,\sigma}(t)t^{l(v)} = \sum_{v \leq \sigma} dim(IH^{2i}(X_\sigma))q^i$ Which implies palindromic symmetric.*

3. *$P_{v,\sigma}(t) = 1$ for every $v \leq \sigma$ if and only if $X_\sigma$ is rationally smooth. and this will be taken to be the definition for rational smoothness.*

**Theorem 4.1.5.** *Let $\sigma \in \mathbb{Z}^n_+$ be the monomial exponent of the $X_\sigma$, then the following are equivalent:*

1. *The Schubert variety $X_\sigma$ is rationally smooth at every point.(since smoothness in type A is equivalent to rational smoothness);*

2. *The Poincaré polynomial $P_\sigma(t)$ is Palindromic;*

3. *The Bruhat graph $\Gamma(id, \sigma)$ is regular, that is every vertex has the same number of edges, $l(\sigma)$.;*

To prove Theorem 4.1.5, we must show that $1 \Rightarrow 2, 2 \Rightarrow 3$ and $3 \Rightarrow 1$.

*Proof.* For the case $1 \Rightarrow 2$

Suppose $X_\sigma$ is rationally smooth at every point then we must show that the Poincaré polynomial is symmetric.

As $X(\sigma)$ is rationally smooth,

$$P_{v,\sigma}(t) = 1, \forall, v \leq \sigma. \tag{4.1}$$

From the definition of the Poincaré polynomial of the Schubert variety we have

$$P_\sigma(t) = \sum_i dim H^{2i}(X(\sigma))t^i = \sum_{v \leq \sigma} t^{l(v)} P_{v,\sigma}(t). \tag{4.2}$$

which is a Palindromic polynomial.

Hence since $P_{v,\sigma}(t) = 1, \forall, v \leq \sigma$

$$P_\sigma(t) = \sum_{v \leq \sigma} t^{l(v)} P_{v,\sigma}(t) = \sum_{v \leq \sigma} t^{l(v)}. \tag{4.3}$$

is Palindromic.

Next we show that $2 \Rightarrow 3$

Assume $P_\sigma(t)$ is symmetric then we must show that every vertex has the same number of edges $l(\sigma)$.

Since $P_\sigma(t)$, is Palindromic, then

$$t^{l(\sigma)} P_\sigma(t^{-1}) = P_\sigma(t). \tag{4.4}$$

But

$$P_\sigma(t) = \sum_{v \leq \sigma} t^{l(v)}. \tag{4.5}$$

$$t^{l(\sigma)} \sum_{v \leq \sigma} t^{-l(v)} = \sum_{v \leq \sigma} t^{l(v)}. \tag{4.6}$$

$$\sum_{v \leq \sigma} (t^{l(\sigma)-l(v)} - t^{l(v)}) = 0. \tag{4.7}$$

Taking the derivative of (4.7), we have

$$\sum_{v \leq \sigma} [(l(\sigma) - l(v))t^{l(\sigma)-l(v)-1} - l(v)t^{l(v)-1}] = 0. \tag{4.8}$$

When $t = 1$ (4.8) becomes

$$\sum_{v \leq \sigma} [(l(\sigma) - l(v)) - l(v)] = 0 \tag{4.9}$$

i.e.

$$\sum_{v \leq \sigma} (l(\sigma) - l(v)) - \sum_{v \leq \sigma} l(v) = 0. \tag{4.10}$$

Thus

$$\sum_{v \leq \sigma} (l(\sigma) - l(v)) = \sum_{v \leq \sigma} l(v). \tag{4.11}$$

Let $v \in W$, by definition, $l(v) = \sharp\{r \in R, |rv < v\}$

i.e.

$$\sum_{v \leq \sigma} l(v) = \sum_{v \leq \sigma} \sharp\{r \in R, |rv < v\} = \sum_{v \leq \sigma} \sharp\{r \in R, |v < rv \leq \sigma\} \tag{4.12}$$

From Deodhar's Inequality, we have that

$\forall, x \leq y \leq \sigma$,

$$\sharp\{r \in R, |x \leq ry \leq \sigma\} \geq l(\sigma) - l(x). \tag{4.13}$$

In particular, if $x = y$,

$$\sharp\{r \in R, |y \leq ry \leq \sigma\} \geq l(\sigma) - l(y), \forall y \leq \sigma. \tag{4.14}$$

Thus (4.12) becomes

$$\sum_{v \leq \sigma} l(v) = \sum_{v \leq \sigma} \sharp\{r \in R, |v < rv \leq \sigma\} \geq \sum_{v \leq \sigma} l(\sigma) - l(v) = \sum_{v \leq \sigma} l(v). \tag{4.15}$$

Hence ,

$$\sum_{v \leq \sigma} l(\sigma) - l(v) = \sum_{v \leq \sigma} \sharp\{r \in R, |v < rv \leq \sigma\}. \tag{4.16}$$

i.e.

$$l(\sigma) - l(v) = \sharp\{r \in R, |v < rv \leq \sigma\}, \forall, v \leq \sigma. \tag{4.17}$$

i.e.

$$l(\sigma) = l(v) + \sharp\{r \in R, |v < rv \leq \sigma\}, \forall, v \leq \sigma. \tag{4.18}$$

$=$ number of edges of vertex $v \leq \sigma$.

Next, we show $3 \Rightarrow 1$

Suppose that every vertex of $\Gamma(id, \sigma)$ has the same number $l(\sigma)$ of edges then, we must show that $X_\sigma$ is rationally smooth at every point. That is $P_{v,\sigma}(t) = 1$ For $v \leq \sigma$

We show by induction on $l(\sigma) - l(v) = k$

$$\text{Let } k = 0,$$
$$\Longrightarrow l(\sigma) = \text{l(v)}$$
$$\Longrightarrow \sigma = v \ .$$

Therefore by definition

$$P_{\sigma,\sigma}(t) = 1$$
$$\text{Let } k = 1,$$
$$\Longrightarrow l(\sigma) - l(v) = 1$$
$$\Longrightarrow v < \sigma \ .$$

Hence by definition.

$P_{v,\sigma}$ has at most degree $\frac{1}{2}(l(\sigma) - l(v) - 1) = 0$ and $P_{v,\sigma}(0) = 1$

Thus $P_{v,\sigma}(t) = constant = 1, \forall, t$.

$$\text{Let } k = 2$$
$$\Longrightarrow l(\sigma) - l(v) = 2$$
$$\Longrightarrow v < \sigma \ .$$

Hence by definition.

$P_{v,\sigma}(t)$ has at most degree $\frac{1}{2}(l(\sigma) - l(v) - 1) = \frac{1}{2}$ and $P_{v,\sigma}(0) = 1$

Thus $P_{v,\sigma}(t) = constant = 1, \forall, t$.

$$\text{For } k = 3 \implies l(\sigma) - l(v) = 3$$
$$\Longrightarrow v < \sigma \ .$$

Hence by definition.

$P_{v,\sigma}$ has at most degree $\frac{1}{2}(l(\sigma) - l(v) - 1) = 1$ and $P_{v,\sigma}(0) = 1$

Thus $P_{v,\sigma}(t) = 1 + \alpha t$ for some $\alpha \in \mathbb{Z}_+$.

Hence by (4.4)

$$\frac{d}{dt}[t^{l(\sigma)-l(v)} P_{v,\sigma}(t^{-2})]_{t=1} = \sum_{r \in R | v < rv \leq \sigma} P_{rv,\sigma}(1) \tag{4.19}$$

i.e.

$$\frac{d}{dt}[t^3(1 + \frac{\alpha}{t^2})]_{t=1} = \sum_{r \in R | v < rv \leq \sigma} P_{rv,\sigma}(1) \tag{4.20}$$

i.e.

$$\frac{d}{dt}[t^3 + \alpha t]_{t=1} = \sum_{r \in R | v < rv \leq \sigma} P_{rv,\sigma}(1) \tag{4.21}$$

picking the left hand side of (4.21), we have

$$\frac{d}{dt}[t^3 + \alpha t]_{t=1} = [3t^2 + \alpha]_{t=1} = 3 + \alpha. \tag{4.22}$$

For $r \in R$ and $v < rv \leq \sigma$. Then

$$l(v) < l(rv)$$
$$l(v) \leq l(rv) - 1$$
$$-l(rv) \leq -l(v) - 1$$
$$l(\sigma) - l(rv) \leq l(\sigma) - l(v) - 1 = 3 - 1 = 2$$

Hence $P_{rv,\sigma}(t) = 1, \forall t$ by the definition

Therefore $P_{rv,\sigma}(t) = 1$, for $r \in R$ such that $v < rv \leq \sigma$ and so

$$\sum_{r \in R | v < rv \leq \sigma} P_{rv,\sigma}(1) = \sum_{r \in R | v < rv \leq \sigma} (1) = \sharp\{r \in R | v < rv \leq \sigma\} \tag{4.23}$$

$$= l(\sigma) - l(v) = 3. \tag{4.24}$$

Equation (4.21) now becomes (from the LHS of (4.22) and from the RHS of (4.24) )

$$3 + \alpha = 3, \tag{4.25}$$

$$\alpha = 0 \tag{4.26}$$

Hence

$$P_{v,\sigma}(t) = 1 + \alpha t = 1, \forall t. \tag{4.27}$$

Assume that $P_{v,\sigma} = 1$ is true for all $l(\sigma) - l(v) \leq k - 1$.

For some $k \geq 1$ we want to show that $P_{v,\sigma} = 1$.

For $l(\sigma) - l(v) = k$.

Let

$$f(t) = t^{l(\sigma)-l(v)}[P_{v,\sigma}(t^{-2}) - 1] \tag{4.28}$$

$l(\sigma) - l(v) = k \geq 1$, so $v < \sigma$ and thus , $P_{v,\sigma}(t)$ has degree $\frac{1}{2}(l(\sigma) - l(v) - 1)$, and $P_{v,\sigma}(0) = 1$

$$P_{v,\sigma}(t) = \sum_{i=0}^{\frac{1}{2}(l(\sigma)-l(v)-1)} \alpha_i t^i \tag{4.29}$$

with $\alpha_0 = P_{v,\sigma}(0) = 1$

So

$$f(t) = t^{l(\sigma)-l(v)}[\sum_{i=0}^{\frac{1}{2}(l(\sigma)-l(v)-1)} \alpha_i t^{-2i} - 1] \tag{4.30}$$

i.e.

$$t^{l(\sigma)-l(v)} \sum_{i=1}^{\frac{1}{2}(l(\sigma)-l(v)-1)} \alpha_i t^{-2i} = \sum_{i=1}^{\frac{1}{2}(k-1)} \alpha_i t^{k-2i}, \tag{4.31}$$

where $k = l(\sigma) - l(v)$ observe that

$1 \leq i \leq \frac{1}{2}(k-1) \Rightarrow 2 \leq 2i \leq (k-1) \Rightarrow 1 - k \leq -2i \leq -2 \Rightarrow 1 \leq k - 2i \leq (k-2)$

Hence, $f(t)$ is a polynomial with no constant term.

By Deodhar inequality, and Differentiating with respect to t at $t = 1$ we have

$$\frac{d}{dt}[t^{l(\sigma)-l(v)} P_{v,\sigma}(t^{-2})]_{t=1} = \sum_{r \in R | v < rv \leq \sigma} P_{rv,\sigma}(1). \tag{4.32}$$

i.e.

$$\frac{d}{dt}[f(t) + t^{l(\sigma)-l(v)}]_{t=1} = \sum_{r \in R | v < rv \leq \sigma} P_{rv,\sigma}(1). \tag{4.33}$$

$$f^{'}(1) + l(\sigma) - l(v) = \sum_{r \in R | v < rv \leq \sigma} P_{rv,\sigma}(1). \tag{4.34}$$

$$f^{'}(1) = \sum_{r \in R | v < rv \leq \sigma} P_{rv,\sigma}(1) - [l(\sigma) - l(v)]. \tag{4.35}$$

Let $r \in R$ be such that $v < rv \leq \sigma$.

$$\begin{aligned} v < rv &\Rightarrow l(v) < l(rv) \\ &\Rightarrow l(v) \leq l(rv) - 1 \\ &\Rightarrow -l(rv) \leq -l(v) - 1 \\ &\Rightarrow l(\sigma) - l(rv) \leq l(\sigma) - l(v) - 1 = k - 1 \end{aligned}$$

So from the induction hypothesis $P_{rv,\sigma}(1) = 1$.

Thus

$$f^{'}(1) = \sum_{r \in R | v < rv \leq \sigma} 1 - [l(\sigma) - l(v)]. \tag{4.36}$$

i.e.

$$\sharp\{r \in R | v < rv \leq \sigma\} - [l(\sigma) - l(v)] = 0. \tag{4.37}$$

From (4.28) and (4.29) we have

$$f(t) = \sum_{i=1}^{\frac{1}{2}(k-1)} \alpha_i t^{k-2i}. \tag{4.38}$$

where

$$P_{v,\sigma}(t) = \sum_{i=0}^{\frac{1}{2}(k-1)} \alpha_i t^i. \tag{4.39}$$

$$f^{'}(t) = \sum_{i=1}^{\frac{1}{2}(k-1)} \alpha_i (k - 2i) t^{k-2i-1}. \tag{4.40}$$

$$f^{'}(1) = \sum_{i=1}^{\frac{1}{2}(k-1)} \alpha_i (k - 2i) = 0. \tag{4.41}$$

The coefficients $\alpha_i$ of the Kazhdan-Lusztig polynomials are non negatives and

$k - 2i \geq 1, \forall i$ hence $\alpha_i = 0, \forall i$

So, $f(t) = 0, \forall t$

$$t^{l(\sigma)-l(v)}[P_{v,\sigma}(t^{-2}) - 1] = 0, \forall t. \tag{4.42}$$

$$P_{v,\sigma}(t^{-2}) - 1 = 0, \forall t. \tag{4.43}$$

i.e.

$$P_{v,\sigma}(t^{-2}) = 1, \forall t. \tag{4.44}$$

i.e.

$$P_{v,\sigma}(t) = 1, \forall t. \tag{4.45}$$

which shows that the Schubert variety $X_\sigma$ is rationally smooth at every vertex implies smoothness in type A. $\square$

**Example 4.1.6.** *For the $X_\sigma$ where $\sigma$ is the exponent of the monomials of the $X_\sigma$ for the permutation of $S_4$, we have the Bruhat order.*

| *Length* | *Exponents* |
|---|---|
| 6 | $(3,2,1)$ |
| 5 | $(3,2,0),(3,1,1),(2,2,1)$ |
| 4 | $(3,0,1),(3,1,0),(2,2,0),(1,2,1),(2,1,1)$ |
| 3 | $(0,2,1),(3,0,0),(1,2,0),(2,0,1),(2,1,0),(1,1,1)$ |
| 2 | $(0,2,0),(0,1,1),(1,0,1),(2,0,0),(1,1,0)$ |
| 1 | $(0,0,1),(0,1,0),(1,0,0)$ |
| 0 | $(0,0,0)$ |

$$P_\sigma(\mathcal{F}\ell_4(\mathbb{C}), t) = t^6 + 3t^5 + 5t^4 + 6t^3 + 5t^2 + 3t + 1. \tag{4.46}$$

$$(1 \quad 3 \quad 5 \quad 6 \quad 5 \quad 3 \quad 1).$$

*Hence, $\mathcal{F}\ell_4(\mathbb{C})$ is smooth.*

**Example 4.1.7.** *For the $X_\sigma$ where $\sigma$ is the exponent of the monomials of the $X_\sigma$ for the permutation of $S_4$ we have the Bruhat order.*

| *Length* | *Exponents* |
|---|---|
| 4 | $(2,2,0)$ |
| 3 | $(2,0,1),(2,1,0),(1,1,1),(3,0,0)$ |
| 2 | $(0,1,1),(2,0,0),(1,1,0),(0,2,0),(1,0,1)$ |
| 1 | $(0,1,0),(0,0,1),(1,0,0))$ |
| 0 | $(0,0,0)$ |

$$P_\sigma((X_{2,2,0}), t) = t^4 + 4t^3 + 5t^2 + 3t^1 + t^0 = t^4 + 4t^3 + 5t^2 + 3t + 1. \tag{4.47}$$

$$(1 \quad 4 \quad 5 \quad 3 \quad 1).$$

*Hence $(X_{2,2,0})$ is singular since the Poincaré polynomial is not palindromic.*

**Remark 4.1.8.** *The following are the observations when showing smoothness and singularity of Schubert variety using the exponent of its monomials ;.*

- *Smoothness is understood in terms of the exponents of the monomial of the Schubert variety.*
- *The sum of each exponent of a monomial term gives the length of the Schubert variety.*
- *The addition of the exponent term on same row gives the coefficient of the Poincaré polynomial.*
- *The sum of the exponent terms are reducing as we move down the bruhat order.*

**Remark 4.1.9.** *The result of Carrell (1994) shows that for $\sigma \in S_n$, the Schubert variety $X_\sigma$ is smooth iff it is palindromic and thls led to the result of Oh et al. (2008). They showed that $P_\sigma(t) = R_\sigma(t)$ iff the Schubert variety is smooth. The Schubert variety is smooth iff the poincaré polynomial is palindromic otherwise it is singular. but since the rank generating function $R_\sigma(t)$ is always palindromic then $P_\sigma(t) \neq R_\sigma(t)$ in all cases. This lead us to show that given any $\sigma \in \mathbb{Z}_n^+$ to be the exponent of the monomials of the Schubert variety $X_\sigma$ then the following are equivalent;*

1. *The Schubert variety $X_\sigma$ is rationally smooth at every point.(since smoothness in type A is equivalent to rational smoothness);*
2. *The Poincaré polynomial $P_\sigma(t)$ is Palindromic; for $\sigma \in \mathbb{Z}_n^+$*
3. *The Bruhat graph $\Gamma(id, \sigma)$ is regular, that is every vertex has the same number of edges, $l(\sigma)$.;*

*Characterising smoothness and singularity of the Schubert varieties by using the exponents of the monomials of the Schubert varieties has reviewed the results of Carrell (1994) and have extended the underlying group from $S_n$ to $\mathbb{Z}_n^+$ .*

## 4.2 Smoothness of the Equations Defining the Ideal of Schubert Varieties using the Jacobian Criterion

In this session smoothness is determined using the equation defining the ideal of the Schubert varieties.

### 4.2.1 Polynomial Rings and Tangent Spaces

This subsection comprises of some basic definitions that leads to the proof of the results.

**Definition 4.2.1.** *Let $K$ be a ring, A polynomial $f(x)$ with coefficient in $K$ is an infinite formal sum*

$$\sum_{i=0}^{\infty} a_i x^i = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots . \tag{4.48}$$

*where $a_i \in K$ and $a_i \neq 0$ for all but a finite number of values of $i$.*

**Definition 4.2.2.** *An affine $n-$space over $K$ is given by $A^n = \{(a_1, a_2, \cdots, a_n) \ni a_i \in K\}$.*

**Definition 4.2.3.** *Let $X \subset A^n$, a polynomial function $f$ is a map $f : X \to K$ defined by $x \mapsto f(x)$ for some $f \in K[x_1, \cdots, x_n]$.*

**Remark 4.2.4.**

1. *$\forall f, g \in K[x_1, \cdots, x_n], f(x) = g(x)$ iff $f = g \in I(X)$ .*
2. *$A(X) = K[x_1, \cdots, x_n]/I(X)$, coordinate ring of $X$ .*
3. *$A(X) \approx$ ring of all polynomial functions on $X$.*

**Definition 4.2.5.** *Let $X \subset A^n$ be an affine variety the ideal of $X$ denoted by $I(X) = \{f \in K[x_1, \cdots, x_n] : f(p) = 0, \forall, p \in X\}$ This is the set of all polynomials vanishing on $X$.*

**Definition 4.2.6.** *Let $p$ be any point in the Schubert variety i.es $p = (a_1, a_2, \cdots, a_n) \in X_v$ then $\forall f \in K[x_1, x_2, \cdots, x_n]$,*

$$f(x_1, x_2, \cdots, x_n) = f(p) + \sum_i \frac{\partial f}{\partial x_i}(p)(x_i - a_i) + terms\ at\ least\ quadratic\ in(x_i - a_i).$$

**Definition 4.2.7.** *The linear parts of the polynomials*

$$\mathbf{L_p} = span\left\{\sum_i \frac{\partial f}{\partial x_i}(p)(x_i - a_i)\right\} \subset K[x_1, x_2, \cdots, x_n]. \tag{4.49}$$

**Definition 4.2.8.** *The tangent space of the Schubert variety at the point $p$ is*

$$T_p(X_v) = \left\langle \{(x_1, x_2, \cdots, x_n) : \sum_i \frac{\partial f}{\partial x_i}(p)(x_i - a_i) = 0 \forall f \in I(X_v)\} \right\rangle. \tag{4.50}$$

**Definition 4.2.9.** *The dimension of the linear parts of the polynomials is exactly the rank of the jacobian matrix of the ideals of the Schubert variety. i.es $dim\mathbf{L_p} = rankJ(I(X_v)$.*

**Definition 4.2.10.** *The dimension of the flag which is equal to the dimension of the Schubert variety at identity (when the dimension is complete) is given by $N = dim\mathbf{L_p} + dimT_p(X_v)$.*

**Definition 4.2.11.** *The Schubert varieties are smooth at the point $p$ if $dimT_p(X_v) = dimX_v$.*

**Theorem 4.2.12.** *Let $S_n$ be the symmetric group of $n$ letters, with $\sigma, v \in S_n$ such that $\sigma$ is of maximal length. Then the Schubert variety $X_v$ is smooth iff $R(J(I(X_v))) = N - l(v)$ .*

*Proof.* $X_v$ is smooth at $p \Leftrightarrow dimT_p(X_v) = dimX_v$. (by definition)

But the dimension of the flag is $N = dim\mathbf{L_p} + dimT_p(X_v)$

$\Leftrightarrow dim\mathbf{L_p} = N - dimT_p(X_v)$

$\Leftrightarrow dim\mathbf{L_p} = N - dim(X_v)$

$\Leftrightarrow rankJ(I(X_v)) = N - dimX_v = l(\sigma) - l(v)$

Hence the Schubert variety $X_v$ is smooth whenever $rankJ(I(X_v)) = N - dimX_v = l(\sigma) - l(v)$ where ($N = l(\sigma)$ and $dimX_v = l(v)$).

□

**Example 4.2.13.** *To Show that the equation defining the ideal of $X_{321}$ is smooth we must show that the rank of the Jacobian matrix $R(J(I)) = l(\sigma) - l(v) = codim(X_\sigma)$.*

*The equation defining the ideal of the Schubert variety $X_{321}$ is given by*

$$x_{11}(x_{12}x_{23} - x_{13}x_{22}) - x_{12}(x_{11}x_{23} - x_{13}x_{21}) + x_{13}(x_{11}x_{22} - x_{12}x_{21}) = 0. \quad (4.51)$$

$$= p_1p_{23} - p_2p_{13} + p_3p_{12} \quad (4.52)$$

*Where* $p_1 = x_{11}, p_2 = x_{12}, p_3 = x_{13}, p_{12} = (x_{11}x_{22} - x_{12}x_{21}), p_{13} = (x_{11}x_{23} - x_{13}x_{21}), p_{23} = (x_{12}x_{23} - x_{13}x_{22})$ *and*

$f_1 = p_1p_{23} = x_{11}(x_{12}x_{23} - x_{13}x_{22})$.

$f_2 = p_2p_{13} = x_{12}(x_{11}x_{23} - x_{13}x_{21})$.

$f_3 = p_3p_{12} = x_{13}(x_{11}x_{22} - x_{12}x_{21})$.

Therefore, we have

$$J(x_1, \cdots, x_n) = \left(\frac{\partial f_i}{\partial x_j}\right) = \begin{bmatrix} \frac{\partial f_1}{\partial x_{11}} & \frac{\partial f_1}{\partial x_{12}} & \frac{\partial f_1}{\partial x_{13}} & \frac{\partial f_1}{\partial x_{21}} & \frac{\partial f_1}{\partial x_{22}} & \frac{\partial f_1}{\partial x_{23}} \\ \frac{\partial f_2}{\partial x_{11}} & \frac{\partial f_2}{\partial x_{12}} & \frac{\partial f_2}{\partial x_{13}} & \frac{\partial f_2}{\partial x_{21}} & \frac{\partial f_2}{\partial x_{22}} & \frac{\partial f_2}{\partial x_{23}} \\ \frac{\partial f_3}{\partial x_{11}} & \frac{\partial f_3}{\partial x_{12}} & \frac{\partial f_3}{\partial x_{13}} & \frac{\partial f_3}{\partial x_{21}} & \frac{\partial f_3}{\partial x_{22}} & \frac{\partial f_3}{\partial x_{23}} \end{bmatrix}.$$

Differentiating with respect to $\{x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}\}$ we have the matrix

$$J(I) = \begin{pmatrix} x_{12}x_{23} - x_{13}x_{22} & x_{11}x_{23} & -x_{22}x_{11} & 0 & -x_{13}x_{11} & x_{11}x_{12} \\ x_{12}x_{23} & x_{11}x_{23} - x_{13}x_{21} & -x_{21}x_{11} & -x_{13}x_{12} & 0 & x_{12}x_{11} \\ x_{13}x_{22} & -x_{21}x_{13} & x_{11}x_{22} - x_{12}x_{21} & x_{12}x_{13} & x_{13}x_{22} & 0 \end{pmatrix}.$$

Setting the variables to be equal to zero the Jacobian matrix of I denoted

$$J(I) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Rank of $J(I) = 0 =$ number of non zero rows,

$codimX_\sigma = \begin{pmatrix} 3 \\ 2 \end{pmatrix} - l(\sigma) = 3 - 3 = 0$

Hence, the Schubert variety $X_{321}$ is smooth at identity.

**Example 4.2.14.** *To Show that the equation defining the ideal of $X_{3412}$ is singular we must show that the rank of the jacobian matrix $R(J(I)) \neq l(\sigma) - l(v) = codim(X_\sigma)$. The equation defining the ideal of the Schubert variety $X_{3412}$ is given by $\{p_4, p_{234}\} = (x_{41}, x_{21}(x_{43}x_{32} - x_{42}) - (x_{43}x_{31} - x_{41})$ therefore we have*

$$J(x_{21}, x_{31}, x_{32}, x_{41}, x_{42}, x_{43}) = \left(\frac{\partial f_i}{\partial x_{ij}}\right) = \begin{bmatrix} \frac{\partial f_1}{\partial x_{21}} & \frac{\partial f_1}{\partial x_{31}} & \frac{\partial f_1}{\partial x_{32}} & \frac{\partial f_1}{\partial x_{41}} & \frac{\partial f_1}{\partial x_{42}} & \frac{\partial f_1}{\partial x_{43}} \\ \frac{\partial f_2}{\partial x_{21}} & \frac{\partial f_2}{\partial x_{31}} & \frac{\partial f_2}{\partial x_{32}} & \frac{\partial f_2}{\partial x_{41}} & \frac{\partial f_2}{\partial x_{42}} & \frac{\partial f_2}{\partial x_{43}} \end{bmatrix}.$$

*Where $f_1 = x_{41}, f_2 = x_{21}(x_{43}x_{32} - x_{42}) - (x_{43}x_{31} - x_{41})$*

*Differentiating the $f_i$ with respect to $(x_{21}, x_{31}, x_{32}, x_{41}, x_{42}, x_{43})$ we have the matrix,*

$$J(I) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ x_{43}x_{32} - x_{42} & -x_{43} & x_{21}x_{43} & 1 & -x_{21} & x_{21}x_{32} - x_{31} \end{pmatrix}.$$

*Setting the variables to be equal to zero, we obtain*

*Rank of $J(I) = 1$, $codim X_\sigma = \begin{pmatrix} 4 \\ 2 \end{pmatrix} - l(\sigma) = 6 - 4 = 2$,*

*Hence, the Schubert variety $X_{3412}$ is singular since $R(J(I)) \neq l(\sigma) - l(v)$. .*

**Example 4.2.15.** *Show that the equation defining the ideal of $X_{2413}$ is not singular we must show that; $R(J(I)) \neq l(\sigma) - l(v) = codim(X_\sigma)$.*

*The equation defining the ideal of the Schubert variety $X_{2413}$ is given by*

$$\{p_3, p_4, p_{34}, p_{134}, p_{234}\},$$

*where*

$p_3 = x_{31}$.

$p_4 = x_{41}$.

$p_{34} = x_{42}x_{31} - x_{41}x_{32}$.

$p_{134} = x_{43}x_{32} - x_{42}$.

$p_{234} = x_{21}(x_{43}x_{32} - x_{42}) - (x_{43}x_{31} - x_{41})$.

*Therefore, the equation defining $X_{2413}$ is*

$\{x_{31}, x_{41}, x_{42}x_{31} - x_{41}x_{32}, x_{43}x_{32} - x_{42}, x_{21}(x_{43}x_{32} - x_{42}) - (x_{43}x_{31} - x_{41})\}$.

*Therefore, we have,*

$$J(x_{21},x_{31},x_{32},x_{41},x_{42},x_{43}) = \left(\frac{\partial f_i}{\partial x_{ij}}\right) = \begin{bmatrix} \frac{\partial f_1}{\partial x_{21}} & \frac{\partial f_1}{\partial x_{31}} & \frac{\partial f_1}{\partial x_{32}} & \frac{\partial f_1}{\partial x_{41}} & \frac{\partial f_1}{\partial x_{42}} & \frac{\partial f_1}{\partial x_{43}} \\ \frac{\partial f_2}{\partial x_{21}} & \frac{\partial f_2}{\partial x_{31}} & \frac{\partial f_2}{\partial x_{32}} & \frac{\partial f_2}{\partial x_{41}} & \frac{\partial f_2}{\partial x_{42}} & \frac{\partial f_2}{\partial x_{43}} \\ \frac{\partial f_3}{\partial x_{21}} & \frac{\partial f_3}{\partial x_{31}} & \frac{\partial f_3}{\partial x_{32}} & \frac{\partial f_3}{\partial x_{41}} & \frac{\partial f_3}{\partial x_{42}} & \frac{\partial f_3}{\partial x_{43}} \\ \frac{\partial f_4}{\partial x_{21}} & \frac{\partial f_4}{\partial x_{31}} & \frac{\partial f_4}{\partial x_{32}} & \frac{\partial f_4}{\partial x_{41}} & \frac{\partial f_4}{\partial x_{42}} & \frac{\partial f_4}{\partial x_{43}} \\ \frac{\partial f_5}{\partial x_{21}} & \frac{\partial f_5}{\partial x_{31}} & \frac{\partial f_5}{\partial x_{32}} & \frac{\partial f_5}{\partial x_{41}} & \frac{\partial f_5}{\partial x_{42}} & \frac{\partial f_5}{\partial x_{43}} \end{bmatrix}.$$

*Where* $f_1 = x_{31}, f_2 = x_{41}, f_3 = x_{42}x_{31} - x_{41}x_{32}, f_4 = x_{43}x_{32} - x_{42}, f_5 = x_{21}(x_{43}x_{32} - x_{42}) - (x_{43}x_{31} - x_{41})$.

*Differentiating the* $f_i$ *with respect to* $(x_{21}, x_{31}, x_{32}, x_{41}, x_{42}, x_{43})$.

*we have the matrix,*

$$J(I) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & x_{42} & -x_{41} & -x_{32} & x_{31} & 0 \\ 0 & 0 & x_{43} & 0 & -1 & x_{32} \\ x_{43}x_{32} - x_{42} & -x_{43} & x_{21}x_{43} & 1 & -x_{21} & x_{32}x_{21} - x_{31} \end{pmatrix}.$$

*Setting all the variables to zero, we have,*

*Rank of* $J(I) = 3$, $codimX_\sigma = \begin{pmatrix} 4 \\ 2 \end{pmatrix} - l(\sigma) = 6 - 3 = 3.$

*Hence the Schubert variety* $X_{2413}$ *is not singular since* $R(J(I)) = l(\sigma) - l(v)$ .

**Remark 4.2.16.**

- *The equation of the ideal is contained in the kernel of the variety. Hence it is equal to zero.*
- *Differentiating with respect to each of terms of the equation of the ideal gives us a zero matrix.*

**Remark 4.2.17.** *The results of Lakshmibai & Seshadri (1984) showed that* $X_\sigma$ *is smooth at* $v \in S_n$ *if and only if* $dimT_v(X_\sigma) := \sharp\{(i < j) : vt_{ij} \leq \sigma\} = l(\sigma)$ *which is also equivalent to* $\sharp\{(i < j) : v < vt_{ij} \leq \sigma\} = l(\sigma) - l(v)$, *that gave rise to the theorem of Lakshmibai & Seshadri (1984) that for* $v \leq \sigma \in S_n$, *the tangent space of* $X_\sigma$ *at* $v$ *is given by* $dimT_v(X_\sigma) = \sharp\{(i < j) : vt_{ij} \leq \sigma\}$. *Hence we show using the* *equations defining the ideals of the Schubert varieties that, if given any $\sigma, v \in S_n$ where $S_n$ is the symmetric group of n letters, such that $\sigma$ is of maximal length the the Schubert variety $X_v$ is smooth iff $R(J(I(X_v))) = N - l(v)$ .*

*This has established a connection between smoothness in differentials equations and smoothness in algebraic geomertry. Hence the concept of smoothness is successfully generalised.*

**Remark 4.2.18.**

- *The equation defining the ideal of the Schubert variety $X_\sigma$ through the essential set is derived by determining the rank matrix of the Schubert variety while the equation of the ideal through the plücker embedding is derived by determining the Schubert varieties embedded in the Grassmannians which are in turn embedded in the product of higher dimensional projective spaces by means of the Plücker embeddings map.*

- *The equation defining the ideal of the Schubert variety $X_\sigma$ obtained through the essential set is not in the kernel of the varieties while that of the Plücker embedding is in the kernel of the varieties and is equal to zero.*

- *Differentiating with respect to each of the terms in the equation of the ideal obtained through the essential set do not give a zero matrix whereas that of the Plücker embedding gives us a zero matrix.*

- *Using the essential set and the Plücker embedding map the Schubert varieties are always smooth at the origin.*

# Chapter 5

# SUMMARY AND CONCLUSION

## 5.1 Summary of Findings

The smoothness of type A Schubert varieties are reviewed, extended and generalised to other groups with supporting examples. Chapter Two gives the basic definitions and general review of the study.

In chapter three the methods, adopted in showing for smoothness of type A Schubert varieties are presented.

In section 4.1 smoothness is defined in terms of the exponents of the monomials of the Schubert varieties using the method of Palindromic Poincaré polynomials. This has sucessfully extended the underlying group $S_n$ to $\mathbb{Z}_n^+$.

In section 4.2 smoothness is established using the equations defining the ideals of the Schubert varieties and this shows that smoothness in algebra geometry is same as that in differential equations. Some examples that supports the results are included.

The relationship and differences between the essential set method and the Plücker coodinate method are given.

## 5.2 Conclusion

This research work has successfully shown smoothness of type A Schubert varieties using the exponents of the monomials of the Schubert varieties. The thesis has reviewed the result of Carrell (1994) and successfully extends the underlying group from $S_n$ to $Z_n^+$.

Smoothness using the equations defining the ideals of the Schubert varieties is established and this shows that smoothness in the theory of differential equations is same as in algebraic geometry

## 5.3 Limitations

The limitations of this work are mainly in the area of concrete applications of the results to concrete problems.

## 5.4 Contributions to Knowledge

The following contributions are achieved;

1. This study has successfully reviewed the results of Carrell (1994) on smoothness and singularities of Schubert varieties.
2. Smoothness using $S_n$ as the underlying group have been extended to $\mathbb{Z}_n^+$.
3. smoothness using the exponents of the monomials of the Schubert varieties by means of the Palindromic Poincaré polynomial is established.
4. This thesis investigate smoothness of Schubert varieties in terms of the equations defining the ideal of the Schubert varieties.
5. A connection between smoothness in differential equations and smoothness in algebraic geometry is established .

## 5.5 Areas of Further Research

Further research that will be of interest includes :

- Unification of the many conditions of Schubert varieties to obtain a condition that brings them all together.
- Establishing that for $\sigma$ is smooth, then $r_\sigma(t)$ can be factorise out nicely, hence $r_\sigma(t)$ is Palindromic.
- Verifying that for $\sigma$ singular, then $r_\sigma(t)$ can not be factorise out nicely, hence $r_\sigma(t)$ is not Palindromic.
- Establing smoothness of type A Schubert varieties using the exponents of the monomials of the varieties through pattern avoidance method.

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