AN ALTERNATIVE GENERALISED WEIGHTED WEIBULL REGRESSION MODEL

BY

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ABSTRACT

Classical Regression Model (CRM) such as Weibull regression is commonly used for estimating relationship among variables. The problem with CRM is its dependence on the assumptions of normality and homoscedasticity of the residual terms. However, the assumption of normality is not valid for several real life events especially time-to-event phenomenon where the data exhibit a high level of skewness. Previous research on CRM has generally excluded non-normality of the residual terms. Therefore, this study was aimed at developing an Alternative Generalised Weighted Weibull Regression Model (AGWWRM) for improved inference when the residual terms are not normal.

The Weighted Weibull Distribution (WWD)
$f(x) = \frac{(\gamma+1)}{\gamma}\left(\frac{\alpha}{\beta}\right)^{\alpha} x^{\alpha-1}\exp\left(-\frac{x}{\beta}\right)^{\alpha}\left(1-\exp\left(-\gamma\left(\frac{x}{\beta}\right)^{\alpha}\right)\right)$ where γ, α and β are: weighted, scale and shape parameters. The WWD was redefined by the introduction of two shape parameters, a and b to accommodate skewness in the data; based on the beta link function: $g(x) = \frac{1}{B(a,b)}[F(x)]^{a-1}[1-F(x)]^{(b-1)}f(x)$ where, β is the beta function and $F(x)$ is the distribution function of the WWD. To obtain a location-scale regression model that would link the response variable $y_i (= X_i^T\beta^* + \sigma z_i$ and $z_i = \left(\frac{(y_i-\mu)}{\sigma}\right)$ is the error term, where β^* is the regression model, μ and σ are the location and dispersion parameters for $i = 1, 2, \cdots, n$; to a vector X of p explanatory variables. The transformations $Y = \log(T), \alpha = \frac{1}{\sigma}$ and $\mu = \log(\beta)$ were used. T is a random variable having beta Weighted Weibull (BWW) density function and Y is a log-beta WW variable. The statistical properties namely: moments, moment generating functions, skewness and kurtosis were determined for the Alternative Generalised Weighted Weibull (AGWW) distribution. The performance of the AGWWRM was determined using secondary data on time-to-completion of a Ph.D. programme using a sample of 187 Ph.D. graduates from the University of Ibadan. The explanatory variables usedwere supervisor (x_1), employment (x_2), marital status (x_3), age (x_4) and faculty (x_5),whileybeing dependent variable was time-to-completion. The AGWWRM was compared with six existing gener-

alised WW regression models: log-beta Weibull, log-beta normal, log-Weibull, log-normal, log-logistic and log-weighted. The Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC) were used as the assessment criteria for AGWWRM.

The derived AGWW distribution was $g(z; a, b, \gamma, \mu, \sigma) = \frac{(\gamma+1)}{\sigma\gamma B(a,b)} \exp\left(-\exp(z_i)\right) \left(1 - \exp(-\gamma \exp(z_i))\right) [F(z)]^{a-1}[1-F(z)]^{b-1}$ where $F(z) = \frac{\gamma+1}{\gamma} \left\{(1 - \exp(-\gamma \exp(z_i))) - \frac{1}{\gamma+1} (1 - \exp(-\exp(1+\gamma)(z_i)))\right\}$. The developed AGWWRM was $y = \beta_0^* + \beta^* x_1 + \beta_2^* x_2 + \beta_3^* x_3 + \beta_4^* x_4 + \beta_5^* x_5 + \sigma z_i$ where $0 < \sigma, \infty$. The parameters of the regression model $(\beta_0^*,\ \beta_1^*,\ \beta_2^* \beta_3^*,\ \beta_4^*\ and\ \beta_5^*) = (2.550, 3.250, 1.250, 4.150, 1.310, 5.150)$. The AIC and BIC for the AGWWRM were -9112754.000 and -9112751.000 while the AIC for the six generalised WW regression models were -474248.600 for log-beta Weibull, -3076234.000 for log-beta normal, -487430.400 for log-Weibull, -1541.182 for log-normal, -1102.662 for log-logistic and -252807.000 for log-weighted, respectively. Also the corresponding BIC were -474249.500, -3076205.000, -487428.500, -1518.564, -1080.044 and -252804.800, respectively. The assessment criteria for the AGWWRM were consistently lower than those from the existing generalised WW regression models indicating improved inference.

The developed Alternative Generalised Weighted Weibull Regression model exhibited improved inference when the residual terms are not normal.

Keywords: Beta link function, Log-beta distribution, Location-scale regression model, Log-normal distribution.

Word count: 480

DEDICATION

I dedicate this work to the Almighty God, who Himself represents wisdom, knowledge and understanding; to my dear wife (Temitope) whose love, support and endurance has enabled me be where I am today; I cherish you; and lastly to my wonderful children; Iyanuoluwa and Ireoluwa.

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T. K. Idowu, Mr. R. S. Idowu. Indeed, you are wonderful people and I appreciate you all. Thank you.

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CERTIFICATION

I certified that this work was carried out by **Mr. N. I. BADMUS** with Matric Number 134544, in the Department of Statistics, Faculty of Science, University of Ibadan under my supervision.

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Chapter 1

INTRODUCTION

1.1 General Introduction

Classical Regression Model is commonly used in regression analysis. The problem with it is its dependence on the assumptions of normality and homoscedasticity of the residual term. However, the assumption of normality is not valid for several real life events especially time-to-event phenomenon where data naturally exhibit a high level of skewness. Several works have been done on robust regression models in the literature; Steve (1983) described a log-logistic regression model in which the hazard functions for separate samples converge with time and applied the model to lung cancer survival data. Hongshik (1996) discussed a method for fitting log-normal regression model to censored survival data through binary decision trees. Silva *et al.* (2009) studied on the traditional lifetime distributions i.e Weibull, log-logistics and log-normal and explained that these distributions could not capture the behaviour of a lifetime data that exhibited skewness among others. There is still a need to develop some robust estimation strategy for flexible parametric models to overcome the problem of non-normality.

In addition, most of the survival data are usually non-normal in nature and invariably call for robust, flexible and versatile model to accommodate non-normality of the residual terms; one of the important concerns in Statistics is the development of flexible and versatile models that would be applicable to a wide spectrum of data in medicine, biology, engineering, economics, quality control among others.

However, in these fields, various methods/approaches have been developed using non-parametric, semi-parametric and parametric model; for example, James (1994) worked on semi-parametric modelling as a hybrid of parametric and non-parametric models. Kristin (2003) introduced some survival analysis based on non-parametric method. Guolin (2008) introduced nonparametric and parametric survival analysis of censored data with possible violation of method assumptions. Abadie (2003) investigated semi-parametric on instrumental variable estimation of response models. Paul (2011) introduced flexible parametric survival model. Vincenzo (2013) based his research on estimation of semi-parametric models among others; but these methods appear in-appropriate and inadequate for modelling survival data in some critical areas.

Furthermore, flexible distributions (parametric family) were studied by many authors in literature for modelling and handling survival data using generator approach in the case of logit of beta function as introduced by Eugene *et al.* (2002) and Jones (2004); this involved convoluting two or more distributions to obtain a composite distribution that was better in scope of applicability than individual alternatives. Famoye *et al.* (2005) introduced and defined the beta Weibull distribution from the logit of beta random variable using logit function; Lee *et al.* (2007) investigated beta-Weibull distribution and applied it to censored data; Akinsete *et al.* (2008) studied the beta pareto distribution; Razzaghi (2009) worked on beta-normal distribution in dose-response modelling as risk assessment of quantitative response. The beta-Hyperbolic secant distribution defined and studied by Fisher and Vaughan (2010) author provided a thorough treatment of statistical properties of the beta hyperbolic secant distribution. The beta log-logistic distribution was a new distribution investigated by Lemonte (2012) that extended the log-logistic distribution; some other distributions were studied in the work. According to Shittu and Adepoju (2013), Nakaganni distribution was used to measure alternation of wireless signal traversing multiple paths. All the parametric models mentioned above could only estimate univariate survival function.

Fortunately, different forms of regression models have been proposed in survival analysis, among which location-scale regression model was studied by several authors/researchers in literature such as Ortega *et al.* (2009) who investigated the log-generalised modified Weibull regression model. Hashimoto *et al.* (2010) worked on the log-exponentiated Weibull regression model for interval-censored data. Pescim *et al.* (2010) studied what could be referred to as the log-beta generalised half normal regression model, which was a feasible alternative for modelling the four exiting types of failure rate functions. Ortega *et al.* (2013) based their research on the log-beta Weibull regression model with application to predict recurrence of prostate cancer. Lawless (2003) used clinical trial data in his studies, Ortega *et al.* (2013)

fitted their model with clinical trial data and applied censored data. Pescim *et al.* (2013) also made use of censored data.

Therefore, this study focused on developing a parametric model that would be adequate for a wider spectrum of survival data, accommodate a greater scope of risk functions, improve modelling capabilities and mitigate empiricism of the model.

1.2 Statement of the Problem

Survival events could be very complex and intractable events that require flexible, versatile and robust model to capture its essence and accommodate a wide range of conditions and possibilities that commonly prevail in survival events. Previous attempts have not been able to adequately meet these requirements particularly about the shape. For instance, nonparametric model does not rely on any particular distribution and the reason being that it is a distribution free method. In nonparametric methods, disadvantage of biasness and precision of the estimators and a problem called curse of dimensionality have been discovered. The Kaplan-Meier which is a common nonparametric estimator has the following limitations: it is mainly descriptive, has no control for covariates, requires categorical predictors, and cannot accommodate time-dependent variables; it made no mathematical assumptions either about the underlying hazard function or about proportional hazard and nonparametric plots cannot be relied upon because, virtually all of them are unconditional. i.e. there are no covariates and cannot deal adequately with censored data and other issues. Semi-parametric modelling is hybrid of the parametric and nonparametric to construction, fitting, and validation of statistical models; and an intermediate between parametric and nonparametric model. Therefore, semi-parametric approach was generally used to fit parametric models in which the distribution made no assumption about the error term; attempts to provide semi-parametric approaches to the Accelerated Failure Time (AFT) model met with limited success. However, in the field of survival analysis, there is still a need to construct robust estimation for flexible parametric models. This study is aimed at meeting this need by developing some appropriate generalised parametric models that are adequate for a wider spectrum of survival data and accommodates a greater scope of risk functions.

1.3 Motivation

In the past, very little attention has been paid to the convolution/mixed distribution because of the tedious and rigorous mathematics involved. And recently, logit of beta function/generator approach by Eugene *et al.* (2002) and Jones (2004) is now being applied to convolute or merge two distributions to develop some mixed model to simultaneously meet the different requirements of the each of the component models. Some success has been recorded in this area. However, this work would explore and upgrade weighted Weibull distribution to Beta weighted Weibull distribution using logit of beta function to construct a regression model based on the new AGWW distribution to improve on the descriptive and predictive status of a more versatile generalised model for survival data.

1.4 Justification

The justification of this work can be highlighted as follows:

- In the field of survival analysis, there is still a need to construct robust estimation for flexible parametric models (Catherine, 2010)

- Nonparametric plots cannot be relied upon because; virtually all of them are not conditioned on covariates (Cleves *et al.*, 2008)

- Nonparametric methods cannot deal adequately with censored data and other related issues (Cleves *et al.*, 2008)

- Semi-parametric models are often used when the researcher wants to use a parametric model but the functional form with respect to a subset of the regressors or the density of the errors is not known.

- The CRM/traditional lifetime distributions i.e Weibull, log-logistics and log-normal cannot capture the behaviour of a lifetime data that exhibits skewness (Silva *et al.*, 2009).

- The problem about whether or not certain independent variables are correlated with the survival or failure times has to be addressed.

1.5 Significance of the Study

The work is of high significance, as it attempts to provide better alternative skew model for modelling non-normal, appreciably skewed survival data that would meet the needs of data from a wider scope of disciplines or fields.

1.6 Aim and Objectives of the Study

The main focus of the work is to develop generalised distributions that are capable of handling and accommodating non-normal data e.g survival data, and constructing a robust parametric model that can be used more effectively in survival analysis. The objectives of the proposed work are as follows:

(i) To obtain a distribution that is better in scope of applicability than the individual alternatives and construct a parametric model that can be used more effectively in survival analysis.

(ii) To obtain the statistical properties of the proposed distributions such as moment, moment generating, survival and hazard function, skewness, kurtosis and the entropy.

(iii) To investigate the parameters by the method of MLEs.

(iv) To measure the effect of covariate on survival.

(v) To assess the performance of the model on a set of performance criteria as a basis for determining the best modelling option, i.e. MLEs of the model parameters, fitting the models for each explanatory variable etc; and assess robustness of the MLE method to model failure.

Chapter 2

LITERATURE REVIEW

Different discussions by various authors/researchers regarding needs/reasons for flexible parametric models in survival analysis and some different developed distributions in the literature have been established. The essence of this section is to establish some of the reasons for parametric models, why flexible models, why not nonparametric models, semi-parametric models and why generalised beta weighted Weibull (GBWW) distribution.

Survival analysis consists of parametric, semiparametric, and nonparametric methods. These methods could be used to estimate the most commonly used measures in survival studies, including survivor and hazard functions and compare them across different groups; survival analysis also assesses the relationship of predictor variables and their relevance to survival time. Some statistical probability distributions describe survival times well. Simple and commonly used distributions are exponential, Weibull, lognormal, Gompertz, log-logistic, Burr, and Birnbaum-Saunders among others. One important concept in survival analysis is censoring and independence of censoring (or non-informative censoring) is the most important assumption in all methods (McGilchrist and Aisbett, 1991). Censoring is said to be present when information on time to an event is not available for all study participants. Participant is said to be censored when information on time to event is not available due to failure to follow-up or non-occurrence of outcome event before the trial ends (Elashoff and Afifi, 1997). Censoring affects the shape of survival curve in a situation when a large number of individuals are censored at a single point of time leading to sudden spurious large jumps or large flat section in survival curve (Machin *et al.* 2006).

2.1 Essence of Flexible Parametric Model

Parametric methods of survival analysis assume distribution of hazard rate as a function of time and derive estimates of failure time statistics while accounting for the presence of censoring in the data (Machin *et al.*, 2006). The parametric models specify a specific form for the hazard function over time. These models fully characterize the distribution of failure time as a function of time (and covariates). Parametric methods overcome the disadvantage and are used for extrapolation of study results beyond the study period. Reasons for Parametric models consist in the following provisions and advantages

- Highly Predictive power
- Modelling time-dependent effects
- Understanding
- Complex models in large data sets (time-dependent effects /multiple time-scales) and
- Quantification (Paul, 2011)

2.1.1 Reasons for Flexible Model

The arguments in favour of flexible models include the following:

- It is good for modelling non-normal data e.g. (common in survival data)
- More complex models such as generalised gamma are only available for accelerated failure form (Nelson *et al.* and Paul, 2011)
- Basic models including Exponential, Weibull, Gompertz are available for proportional hazard
- The estimates obtained from flexible parametric survival models are incredibly similar to those obtained from a Cox model. (Lambert *et al.* 2010 and Paul, 2011)

2.2 Prospects of Nonparametric and Semi-parametric Models in Survival Analysis

We examine nonparametric and semi-parametric models respectively and identify their prospects or otherwise in survival analysis.

2.2.1 Nonparametric Models in Survival Analysis

Nonparametric or distribution free inference does not rely on assumptions about the shape or parameters of the underlying population distribution; however, the procedures generally have less power for the same sample size than the corresponding parametric procedure when the data are truly normal. Interpretation of nonparametric procedures can also be more difficult than for parametric procedures. Kaplan Meier method is suitable when no assumption about the functional distribution of hazard rate with time is made. However, since no assumption is made about hazard rate, no extrapolation of study results beyond the study period is possible (Kaplan and Meier, 1958 and Machin *et al.*, 2006). The advantages of the nonparametric approach include the following:

- it makes none or liberal mathematical assumptions about the underlying hazard function or about proportional hazard (Kristin, 2003),
- it functions on lack of independence within the sample,
- it is insensitive to censoring and
- it assumes uniform behaviour within a time interval,
- it is not affected by problems that could arise from presence of large number of censored values,
- violations of assumptions may be graphically isolated

The disadvantages of nonparametric approach relate to the precision and statistical tractability of the estimators and problem called curse of dimensionality (Abadie, 2003). The disadvantages include the following

- The Kaplan-Meier as a common nonparametric estimator has the following limitations:
  - Mainly descriptive
  - Does not control for covariates
  - Require categorical predictors
  - Cannot accommodate time-dependent variables (Kristin, 2003).
- The log-rank test cannot deal with continuous variables (Stated in Machin *et al.* (2006).
- Other inefficiencies as noted by Miller (1983).

Some violations may have little practical effect on the analysis; but other violations may render the Kaplan-Meier results useless and of poor interpretability. Therefore, small sample sizes may increase the adverse effect of violations of assumption. Heavy censoring may also affect the reliability of the Kaplan-Meier estimates. Meanwhile, small data size may exacerbate problems arising from potential assumption violations including lack of independence within the sample, independence of censoring, uniformity within a time interval, presence of large number of censored values, and limitations arising from non-availability of analytical alternative to graphical detection of assumption failures (*www.basic.northwestern; edu/statguidefiles/survival-tests-ass-viol.html*).

2.2.2 Semi-parametric Models in Survival Analysis

Semi-parametric models are intermediate between parametric and nonparametric model (Peter *et al.*, 2005). Semi-parametric models combine elements of the parametric and non-parametric models respectively in its construction, fitting, and validation (James, 1994). Cox-proportional hazard model is a most widely semi-parametric model; it is not a pure parametric method since it does not assume any functional form of distribution of hazard rate. However, it assumes that hazard function of any two individuals is proportional with the ratio being determined by the covariates that is constant over time. (Cox, 1977 and Breslow, 1972). It is generally used to fit a parametric model in which the distribution of the error term cannot be assumed to be of a specific type (reported in Verardi, 2013). Attempts to adopt semi-parametric approach to the Accelerated Failure Time (AFT) model met with limited success; Cox Proportional hazard model is flexible but has its problem. Cox modelled the effect of predictors and covariates on the hazard rate but had the baseline hazard poorly specified. Also, attempts to provide semi-parametric approaches to the AFT model had limited success (Millar, 1983; Beckley and James, 1979; Koul *et al.*, 1981).

2.2.3 The Generalised Beta Weighted Weibull (GBWW) Distribution

Reasons for GBWW Distribution include the following:

(1.) Due to its flexibility the distribution

(i) has wider scope of applicability.

(ii) could be more real to life situation; there is mitigated empiricism.

(iii) has bathtub and a unimodal shaped hazard function.

(2.) It is a generalised model and many models including some cited in literature are special cases defined by specifications of the parameters

(3.) It defines a parametric model that could provide a good description of lifetime data. e.g survival data yielding more precise estimate of the parameters and predictions of interest.

(4.) The model can be applied and used more effectively in the analysis of any survival data since it represents a parametric family of models being sub-models of the main model

(5.) It is flexible and useful in modelling a broader spectrum of survival data

2.2.4 Gap in Literature

(1.) The existing literature on Generalised forms of the Weighted Weibull distribution is very few and scanty. This work develops a generalised form of Weighted Weibull distribution by means of logit of Beta as the link function; the proposed distribution is be used to develop a location-scale regression model linking the response and the explanatory variables.

(2.) Azzalini (1985) proposed a method of obtaining weighted distributions from independently identically distributed random variables; there is a need to explore his proposal to come up with a more realistic model.

(3.) Gupta and Kundu (2009) used Azzalinis method to propose a weighted exponential distribution. The proposed model could be used as an alternative to Gamma and Weibull distribution.

(4.) Then, Shahbaz *et al.* (2010) slightly modifying the method of Azzalini proposed the weighted Weibull distribution which could be used to model life time data as a much more flexible concept

(5.) Ramadan (2013) also introduced a skewness parameter to a Weibull distribution using an idea of Azzalini that gives a new class of weighted Weibull distribution.

(6.) Again, Aleem *et al.* (2013) modified Ramadans work by introducing additional two shape parameters which creates a class of modified weighted Weibull distribution.

(7.) Ortega *et al.* (2013) proposed the log-beta Weibull distribution that led to the log-beta Weibull regression model using the beta Weibull density function (Famoye et al, 2005; Lee et al 2007).

(8.) Ortega *et al.* (2009) also applied and extended the modified Weibull density function by Lai et al (2003) to produce the log-generalised modified Weibull regression model

(9.) Pescim *et al.* (2013) introduced a log-linear regression model based on the beta generalised half normal distribution; they investigated Pescim *et al.* (2010) to formulate and develop a log-linear model using a new distribution called the log-beta generalized half normal distribution.

(10.) Cancho *et al.* (2009), modified log-exponentiated-Weibull regression model to allow the possibility that long term survivors are present in the data; and their modification led to a log-exponentiated-Weibull regression model with cure rate.

None of the aforementioned had considered weighted Weibull in conjunction with beta distribution or a distribution that could give/generate log-beta weighted Weibull distribution and log-beta weighted Weibull regression formulation.

2.3 Essence of The Study

The essence of this study is to develop generalised models that would fill identified gaps in survival analysis and meet urgent needs concerning model specification, estimation process and specified study focus

2.3.1 Model Specification Needs

In the field of survival analysis, there is still a need for a model with the following characteristics among others

(a) Composite and all-embracing generalised model

(b) Versatile and flexible, model with enhanced scope of application

(c) Parameters that explain underlying survival phenomenon

(d) Distribution structure that is real to life

2.3.2 Estimation Process Needs

There is also a need for an estimation process that is:

(a) Robust and insensitive to failures of underlying assumptions

(a) Conducive to estimation of the point and interval of parameters of the model

(a) Optimal in terms of precision and interpretation of parameter estimates

2.3.3 Study Focus

In the light of the needs specified above, this study is focused on

(a) Developing generalised distributions that are capable of handling and accommodating survival data including non-normal data types

(a) Constructing a robust parametric model that can be used more effectively in survival analysis.

However, different distributions have been studied and developed by researchers by means of combining two or more existing distributions together. The beta distribution has been widely identified and applied as a powerful and well probability distribution to address several kinds of problems in reliability analysis. In recent years, development has focused on new techniques for building meaningful distributions, including the use of the logit of beta with a link function associated with the beta generalised distribution pioneered by Jones (2004). The beta link function introduced by Jones has probability density function (pdf).

$$g(x) = \frac{1}{B(a,b)}[F(x)]^{(a-1)}[1 - F(x)]^{(b-1)}f(x) \tag{2.1}$$

where $f(x)$ and $F(x)$ are the pdf and cdf of X.
Furthermore, we have some of the distributions combining with the beta distribution previously considered in literature.

Chapter 3

METHODOLOGY

In this section, we explore and upgrade two classes of well-known and useful distributions namely the Weighted Weibull and skewed distribution (beta distribution). These distributions are mixed under certain classical rule to obtain generalised weighted Weibull distributions that are capable of handling any non-normal data e.g. skewed data and accommodating different forms of the risk functions. The performance of the proposed distributions is compared with existing distributions with a view to determine their ability and capability that will yield a wider scope of applicability than the individual alternatives.

The study focused on comprehensive treatment of statistical properties of the distributions. Attempts is also made to derive the functional form of the distribution and determine moment generating function, asymptotic behaviour, the first four moments, variance, skewness, kurtosis, Renyi Entropy and order statistics.

We also propose the method of maximum likelihood estimation (MLEs) for estimating the parameters of the distributions. Interval estimation of the set of parameters that constitute the respective distributions and Fishers information matrix are also derived.

Furthermore, the study develops a log model using the newly proposed distribution called the generalised weighted Weibull distribution. The following statistical properties are derived; density function and cumulative distribution, survival and hazard function, moments and moment generating function, quantile function and mean deviation.

Finally, a regression model is constructed using the generalised weighted Weibull distribution and the maximum likelihood estimation (MLEs) are obtained.

The generalised beta distribution of the first kind or beta type I was introduced by McDonald (1984) and used by Alexander *et al.* (2011). Its density function is given as

$$f_{GB}(k, a, b, m) = \frac{m}{B(a, b)} k^{am-1} (1 - k^m)^{b-1} \qquad 0 < k < 1$$

where, $a > 0, b > 0$ and $m > 0, \quad k = F(x; \tau)$

The class of distributions such as Generalised Beta compounded distributions, Kumaraswamy -generated distributions as well as classical Beta distributions were developed respectively by McDonald (1984) and Jones (2004). Jones (2004) stated that the shape of a probability distribution is often determined by the distribution's skewness and kurtosis. Starting from a symmetric baseline density $f(x)$, he modified the shape by introducing skewness and some modification of kurtosis; if f(x) was weighted, he chose the Beta distribution as underlying weighting function W. In particular every density W on the interval is a specific weighting function. With this, Eugene *et al.* (2002) investigated the beta-normal distribution generated from the logit of a beta random variable and used the result to model dose-response. Nadarajah and Kotz (2004) introduced the beta-Gumbel distribution from logit of a beta random variable. In literature, there are other logit transforms of beta family which were derived from the beta-Weibull distribution which is believed to be very useful in many areas of studies (Famoye *et al.*, 2005). Nadarajah and Kotz applied their distribution to censored data set on head-and-neck cancer in clinical trials among other applications. Nadarajah and Kotz (2005) again studied and defined the beta-exponential distribution as alternative options in real life problems. The beta-Rayleigh distribution was developed for estimating the reliability functions in medicine, engineering, hydrology and so on. Also, Akinsete *et al.* (2008) worked on the beta-pareto distribution in a flood data set.

The new distribution would be used to construct location-scale regression models; these would be appropriate extensions of some extant parametric family of distributions and would involve some standardization of parent distributions that have been proposed in literature for survival analysis. Lawless (2003) studied generalised log-gamma regression models with censored data. A location-scale regression model based on BurXII distribution was investigated by Silva *et al.* (2008); it was called log-BurXII distribution. Hashimoto *et al.* (2010) introduced the log-exponentiated Weibull regression model for interval-censored data. The log-beta Weibull regression model was developed by Ortega *et al.* (2013) to predict recurrence prostate cancer for patients with clinically localized prostate cancer treated by open radical prostatectomy. Another study of the log-beta generalised half-normal (LBGHN) regression model provided some estimators appropriate for developing Asymptotic Inference.

This work intends to extend convolution of conventional weighted Weibull distributions in a way to increase its modelling capabilities, improve inference and provide better fitting to real life data.

3.1 Beta Generalised Link Function

Jones (2004) developed families of univariate continuous distributions from a novel perspective. The distributions of order statistics of i.i.d samples from various underlying distributions are central to the immense literature in order statistics, Arnold *et al.* (1992). They succeeded in extracting order statistic distributions from literature on the order statistics by generating them and suggested the use of the collection of generalised distributions of order statistics as empirically useful families of distributions in their own right.

The intention was to provide future families of distributions for use in the empirical modelling of data, replacing the normal and the like, where necessary and appropriate, by wider families of distributions that automatically account for skewness and/or heavy tail weights in the data. The beta generated link function, a new family of continuous distributions, is an extant continuous distribution F with density f together with the parameters $a > 0$ and $b > 0$. By letting $B(.,.)$ be the beta function, Jones (2004) introduced

$$g_f(x; a, b) = \frac{1}{B(a, b)}[F(x)]^{a-1}[1 - F(x)]^{b-1}f(x) \tag{3.1}$$

where $f(x)$ and $F(x)$ are the pdf and cdf of the parent distribution. In a situation where F is symmetric about zero with no free parameters other than location and scale support, the distribution has tails that are at least as heavy as those of the normal distribution.

Let $\{x_1, x_2, \cdots, x_n\}$ be a random sample of size n on X such that $X \sim BG$. The family G_f then affords a class of distributions generated from symmetric F with a and b governing skewness and tail weight; when $a = b$, g_F remains symmetric; but if $a \neq b$, g_F is invariably skewed where skewness and excess kurtosis could be either positive or negative; the tails get higher as sample size n increases and heavier as a decreases.

3.2 Estimation and Statistical Inference of Likelihood Function

We derive the maximum likelihood estimates (MLEs) of the parameters of the BG family of distributions. Suppose $x_1, x_2, \cdots, , x_n$ be a random sample of size n on X i.e $X \sim BG(\tau, a, b)$, where τ is a $K \times 1$ vector of unknown parameters in the baseline distribution $F(x;\tau)$ then, given a beta generated family, the likelihood function for $\theta = (\tau, a, b)$ can be written as:

$$l(\theta) = [B(a,b)]^{-n} \prod_{i=1}^{n} f(x)[F(x)]^{a-1}[1-F(x)]^{b-1} \tag{3.2}$$

Taking log of both sides, we obtain

$$\log l(\theta) = -n \log B(a,b) + \sum_{i=1}^{n} \log f(x_i, \tau) + (a-1)\sum_{i=1}^{n} \log F(x_i, \tau) + \\ (b-1)\sum_{i=1}^{n} \log[1 - \log F(x_i, \tau)] \tag{3.3}$$

take $l(\theta) = \log(\theta)$.

The above equation can be maximized either directly using Newton Raphson or SAS (Proc NLMIXED) or OX (subroutine Max BFGS) or by solving the nonlinear likelihood equations by differentiating (3.3). This leads to the components of the score vector obtained as follows:

$$\frac{\partial l(\theta)}{\partial a} = -n[\psi(a) - \psi(a+b)] + \sum_{i=1}^{n} \log F(x_i;\tau) \tag{3.4}$$

$$\frac{\partial l(\theta)}{\partial b} = -n[\psi(a) - \psi(a+b)] + \sum_{i=1}^{n} \log[1 - F(x_i;\tau)] \tag{3.5}$$

$$\frac{\partial l(\theta)}{\partial \tau} = \sum_{i=1}^{n} \frac{\dot{f}(x_i)}{F(x_i;\tau)} + (a-1)\sum_{i=1}^{n} \frac{\dot{F}(x_i)}{F(x_i;\tau)} + (b-1)\sum_{i=1}^{n} \frac{f(x_i)}{[1 - F(x_i;\tau)]} \tag{3.6}$$

where $\psi(.)$ is the digamma function,

$$\dot{f}(x_i) = \frac{\partial f(x_i;\tau)}{\tau} \; and \; \dot{F}(x_i) = \frac{\partial F(x_i;\tau)}{\partial \tau} \;\; are \;\; px' \;\; vector$$

3.3 Information Matrix

Information matrix is required for the estimation of the model parameters. The $(p+2)\times(p+2)$ information matrix is derived as follows:

$$\frac{\partial^2 l}{\partial a^2} = -n[\psi^1(a) - \psi^1(a+b)] \tag{3.7}$$

$$\frac{\partial^2 l}{\partial a \partial b} = -n\psi^1(a+b) \tag{3.8}$$

$$\frac{\partial^2 l}{\partial a \partial b} = \sum_{i=1}^{n} \frac{F(x_i)}{F(x_i;\tau)} \tag{3.9}$$

$$\frac{\partial^2 l}{\partial b^2} = -n[\psi^1(a) - \psi^1(a+b)] \tag{3.10}$$

$$\frac{\partial^2 l}{\partial b \partial \tau} = \sum_{i=1}^{n} \frac{F(x_i)}{[1 - F(x_i;\tau)]} \tag{3.11}$$

$$\frac{\partial^2 l}{\partial \tau^2} = \sum_{i=1}^{n} \frac{\ddot{f}(x_i) f(x_i,\tau) - (\dot{f}(x_i))^2}{f^2(x_i)} + (a-1) \sum_{i=1}^{n} \frac{F(x_i)F(x_i;\tau) - (F(x_i))^2}{F^2(x_i)} + (b-1) \sum_{i=1}^{n} \frac{\left[\ddot{F}(x_i)\left[(1 - F(x_i,\tau))\right] - \dot{F}^2(x_i)\right]}{[1 - F(x_i;\tau)]^2} \tag{3.12}$$

where $\Psi(.)$ is the derivative of the digamma function, $\dot{f}(x_i;\tau) = \frac{\partial F(x_i;\tau)}{\partial \tau}$

$$\dot{F}(x_i) = \frac{\partial F(x_i;\tau)}{\partial \tau}$$

$$\ddot{f}(x_i) = \frac{\partial^2 f(x_i;\tau)}{\partial \tau \tau}$$

$$\ddot{F}(x_i) = \frac{\partial^2 F(x_i;\tau)}{\partial \tau \tau}$$

Fisher information matrix can be used to construct approximate confidence intervals and confidence regions for the parameters based on the multivariate normal $N_{(P+2)}(0, J(\hat{\theta})^{-1})$ distribution.

3.4 Logarithm of the Distribution

Here, we also define the extended form of the density function of the new distribution (for $t > 0$) as

$$g_f(t;a,b) = \frac{1}{B(a,b)}[F(t)]^{a-1}[1 - F(t)]^{b-1} f(t) \tag{3.13}$$

A random variable T where $T \sim B(a,b)$ can be re-defined by random variable $Y = \log(T)$. Variable Y can be normalized as standardized random variate $z = \frac{y-\mu}{\sigma}$ with density function given as

$$g_f(z;a,b) = \frac{1}{B(a,b)}[F(z)]^{a-1}[1-F(z)]^{b-1}f(z) \tag{3.14}$$

We write $Z \sim LogB(a,b)$.

3.4.1 The Log of the Distribution with Regression Model

A location-scale regression model linking the response variable y_i and the explanatory variable vector $x_i^T = x_{i1}, \cdots, x_{ip}$ is represented as

$$y_i = X_i^T\beta + \sigma z_i, \qquad i = 1, 2, \cdots, n \tag{3.15}$$

where $X_i^T\beta = \mu_i$ is the location of y_i, z_i is the random error, $\beta = (\beta_1, \cdots, \beta_p)^T, \sigma, a,$ and b are unknown parameters.

3.4.2 Maximum Likelihood Estimation (MLEs)

Again, we derive the maximum likelihood estimates (MLEs) as follows: Let (y_i, x_i), $i = 1, 2, \cdots, n$ be a random sample of size n where $y_i = \min(\log(t_i), \log(c_i)$ is a random response; t_i and c_i are failure times and censoring indicator. The likelihood function to (a, b, z_i) has the form $l(a,b,z_i) = \sum_{i\in F}\log[f(y_i)] + \sum_{i\in C}\log[s(y_i)]$, is where $f(y_i)$ is the density function and $S(y_i)$ is the survival functionof Y_i.

The log-likelihood function $l(a, b, z_i)$ reduces to

$$\log l(a,b,z_i) = -m\log B(a,b) + (a-1)\sum_{i\in F}\log F(z_i) + \tag{3.16}$$
$$(b-1)\sum_{i\in F}\log[1-F(z_i)] + \sum_{i\in C}[1 - l_{F(z_i)}(a,b)]$$

where m is the number of uncensored observations and $z_i = \frac{(y_i - X_i^T\beta)}{\sigma}$.

3.5 Review of Conventional Weighted Weibull Distribution

3.5.1 The Weighted Distribution

Azzalini (1985) initiated a method of obtaining weighted distributions from independently identically distributed (i.i.d.) random variables Y_1 and Y_2 based on the expression

$$f_Y(y) = \frac{1}{P(\alpha X_1 > X_2)} f_Y(y)F_Y(\alpha y), \quad \alpha > 0 \tag{3.17}$$

where $f(y)$ and $F(y)$ were the pdf and cdf of Y respectively and α was an unknown parameter.

3.5.2 The Weighted Exponential (WE) Distribution

Gupta and Kundu (2009) slightly modified Azzalini's approach to obtain the Weighted Exponential (WE) distribution which is defined here as

$$f_Y(y) = \frac{1}{P\left(\alpha^{\frac{1}{\beta}} Y_1 > Y_2\right)} f_Y(y) F_Y\left(\alpha^{\frac{1}{\beta}}\right), \quad \alpha, \beta > 0 \tag{3.18}$$

where, α and β were unknown parameters.
According to Gupta and Kundu (2009) the random variable X is said to have weighted exponential (WE) distribution, and the pdf of X given as

$$f_X(x, \alpha, \lambda) = \frac{\alpha+1}{\alpha} \lambda e^{-\lambda x} \left(1 - e^(- \alpha\lambda x)\right); \quad x > 0 \tag{3.19}$$

where $\alpha > 0$ and $\lambda > 0$ were the shape and scale parameters, respectively.

The cumulative distribution function (cdf) of the Weighted Exponential (WE) variable X, derived from the pdf defined in equation (3.19) is given as

$$\begin{aligned} F_X(x; \alpha, 1) = P(X \leq x) &= \int_0^x f_X(x) dx \\ &= \frac{\alpha+1}{\alpha} \int_0^x \lambda e^{-\lambda x} \left(1 - e^{-\alpha\lambda x}\right) dx \\ &= \frac{\alpha+1}{\alpha} \left\{1 - e^{-\lambda x} - \frac{1}{\alpha+1} \left(1 - e^{-(1+\alpha\lambda)x}\right)\right\} \end{aligned}$$

$$\int_0^x \lambda e^{-\lambda x} \left(1 - e^{-\alpha\lambda x}\right) dx = 1 - e^{-\lambda x} - \frac{1}{\alpha+1} \left(1 - e^{-(1+\alpha\lambda)x}\right) \tag{3.20}$$

It was assumed, without loss of generality, that $\lambda = 1$. The distribution function (CDF) of WE (α) then becomes

$$F_X(x; \alpha, 1) = \frac{\alpha+1}{\alpha} \left\{1 - e^{-x} - \frac{1}{\alpha+1} \left(1 - e^{-(1+\alpha)x}\right)\right\} \tag{3.21}$$

Hence, the survival rate function of a random weighted exponential variable X with cumulative distribution function $F_X(x)$ is given by

$$S_X(x; \alpha, 1) = 1 - F_X(x; \alpha, 1) \tag{3.22}$$

where $F_X(x)$ is in equation (3.21)

$$\begin{aligned} S_X(x; \alpha, 1) &= 1 - \left(\frac{\alpha+1}{\alpha} \left\{1 - e^{-x} - \frac{1}{\alpha+1}(1 - e^{(1+\alpha)x}\right\}\right) \\ &= 1 + \alpha - e^{-\alpha x} \end{aligned} \tag{3.23}$$

Meanwhile, the hazard rate function of a random weighted exponential variable X is given as

$$H_X(x;\alpha,1) = \frac{f_X(x;\alpha,1)}{1 - F_X(x;\alpha,1)} \tag{3.24}$$

where, $1 - F_X(x;\alpha,1) = S_X(x;\alpha,1)$ in (3.23)

$$H_X(x;\alpha,1) = \frac{(\alpha+1)\left(1 - e^{-\alpha x}\right))}{1 + \alpha - e^{-\alpha x}} \tag{3.25}$$

3.5.3 The Weighted Weibull Distribution (WWD)

The Weighted Weibull (WW) distribution was studied extensively by Shahbaz *et al.* (2010). They extended the work of Gupta and Kundu (2009) in (3.19), when X_1 and X_2 are i.i.d Weibull random variable with and as shape and scale parameters respectively; then substituting for $f_y(x) = \lambda\beta x^{\beta-1} e^{-\lambda x^\beta}$ and $F_Y(x) = 1 - e^{-\lambda x^\beta}$ in to equation 3.18, they obtain the following weighted Weibull distribution:

$$f_y(x) = \frac{\alpha+1}{\alpha}\lambda\beta x^{\beta-1} e^{-\lambda x^\beta}\left[1 - e^{-\alpha\lambda x^\beta}\right], \qquad \alpha, \beta, \lambda, x > 0 \tag{3.26}$$

The weighted exponential distribution of Gupta and Kundu (2009) can be immediately obtained from (3.18) by using $\beta = 1$.

The density (3.26) can be used for modelling life time data with greater flexibility. They presented some common distributional properties; and the parameter estimation of distribution (3.26) could be achieved as follows.

The cumulative distribution of a random variable from (3.26) is given as

$$f_x(x) = \int_0^x +(w)dw$$

the distribution function becomes

$$F_X(x) = \frac{\alpha+1}{\alpha}\left[1 - e^{-x^\beta} - \frac{1}{\alpha+1}\left(1 - e^{-(1+\alpha)x^\beta}\right)\right] \tag{3.27}$$

The survival rate function of a random weighted Weibull variable X with cumulative distribution function $F_X(x)$ is given as:

$$S_X(x;\alpha,\beta) = 1 - F_X(x;\alpha,\beta) \tag{3.28}$$

With $F_X(x)$ as given in equation (3.21), we have

$$S_X(x;\alpha,\beta) = 1 + \alpha - e^{-\alpha x^\beta} \tag{3.29}$$

Using (3.26) and (3.27), the hazard rate function of weighted Weibull distribution is given as:

$$h(x) = \frac{f_X(x)}{1 - F_X(x)} \tag{3.30}$$

where, $f_X(x)$ and $F_X(x)$ are the pdf and cdf of the weighted Weibull distribution.
With the survival function given as $1 - F_X(x) = S_X(x)$ (see equation (3.29)), we have hazard rate function $h(x)$ given as

$$h_X(x; \alpha, \beta) = \frac{\beta(1+\alpha)x^{\beta-1}\left(1 - e^{-\alpha x^{\beta}}\right)}{1 + \alpha - e^{-\alpha x^{\beta}}} \tag{3.31}$$

The hazard rate function given in (3.31) may be decreasing when $\beta < 1$ and so the distribution (3.26) is applied to life time of components data. The moment generating function of density (3.26) is also given as:

$$\begin{aligned} M_X(t) &= \int_0^{\infty} e^{tx} f_X(x) dx \\ &= \int_0^{\infty} e^{tx} \frac{\alpha+1}{\alpha} \beta x^{\beta-1} e^{-x\beta}\left(1 - e^{-\alpha x^{\beta}}\right) dx \end{aligned}$$

and the moment generating function is given as

$$M_X(t) = \sum \frac{t!}{j!_{\alpha}} \left[1 - \alpha - (1+\alpha)^{\frac{j}{\beta}}\right] r\left(1 + \frac{j}{\beta}\right) \tag{3.32}$$

While r-th moment is written from (3.32) as

$$\mu_r' = E(X^r) = \frac{1}{\alpha}\left[1 + \alpha - (1+\alpha)^{-\frac{r}{\beta}}\right] \Gamma\left(1 + \frac{r}{\beta}\right) \tag{3.33}$$

3.5.4 Parameter Estimation

The density function of weighted Weibull estimation is given in (3.26) by Shahbaz et al. (2010). The likelihood function for a sample of size n can be immediately written as

$$L(x_1, x_2, \cdots, x_n/\alpha, \beta) = \left(\frac{\alpha+1}{\alpha}\right)^n \beta^n \prod_{i=1}^{n} x_i^{\beta} \exp\left[-\sum_{i=1}^{n} x_i^{\beta} \prod_{i=1}^{n} \left(1 - e^{-\alpha x_i \beta}\right)\right] \tag{3.34}$$

The logarithm of (3.34) is:

$$\ln L = n \ln\left(\frac{\alpha+1}{\alpha}\right) + n\ln\beta + \sum_{i=1}^{n}(\beta-1)\ln x_i - \sum_{i=1}^{n} x_i^{\beta} + \sum_{i=1}^{n} \ln\left(1 - e^{-\alpha x_i^{\beta}}\right) \tag{3.35}$$

Taking the partial derivatives of (3.35) with respect to the unknown parameters α and β, for the estimating equations unknown parameters are:

$$\frac{\partial \ln Ln}{\partial \alpha} = -\frac{n}{\alpha} + \frac{n}{\alpha+1} + \sum_{i=1}^{n} \frac{x_i^{\beta} e^{-\alpha x_i^{\beta}}}{1-e^{-\alpha x_i^{\beta}}} = 0 \tag{3.36}$$

$$\frac{\partial \ln Ln}{\partial \beta} = -\frac{n}{\beta} + \sum_{i=1}^{n} \ln x_i - \sum_{i=1}^{n} y_i^{\beta} \ln x_i + \sum_{i=1}^{n} \frac{\alpha y_i^{\beta} \ln x_i e^{-\alpha x_i^{\beta}}}{1-e^{-\alpha x_i^{\beta}}} = 0 \tag{3.37}$$

The unknown parameters can be obtained by solving (3.36) and (3.37) iteratively.

3.6 The Developed Beta-Weighted Weibull (BWW) Distribution

The probability density function (pdf) and the cumulative distribution function (cdf) are given in equations (3.26) and (3.27) respectively. Then, the new developed Beta-Weighted Weibull distribution is derived as follows.

Let X be a random variable with the probability distribution based on parametric form from the logit of Beta distribution as defined by Jones (2004) (3.1) and given as:

$$g(x) = \frac{m}{B(a,b)}[F(x)]^{am-1}[1-F(x)^m]^{b-1}f(x) \tag{3.38}$$

Then by substituting the expressions in (3.26) and (3.27) above, the probability density function (pdf) of the Beta-Weighted Weibull (BWW) distribution is derived as follows:

$$g_{BWWD}(x) = \frac{1}{B(a,b)}\left[\frac{\alpha+1}{\alpha}\left\{1-e^{-x_i^{\beta}} - \frac{1}{\alpha+1}\left(1-e^{-(1+\alpha)x^{\beta}}\right)\right\}\right]^{a-1} \times \tag{3.39}$$
$$\left[1-\frac{\alpha+1}{\alpha}\left\{1-e^{-x^{\beta}} - \frac{1}{\alpha+1}\left(1-e^{-(1+\alpha)x^{\beta}}\right)\right\}\right]^{b-1}$$
$$\frac{\alpha+1}{\alpha}\lambda\beta x^{\beta-1}e^{-\lambda x^{\beta}}\left(1-e^{-\alpha\lambda x^{\beta}}\right)$$

where $\alpha, \beta, \lambda, a, b$ and $x > 0$, $X \sim BWWD(\alpha, \beta, \lambda, a, b)$. In equation (3.39), when $b = 1$, it becomes exponentiated Weighted Weibull, and when $a = 1$, it becomes Lehmann type II weighted Weibull and when $\beta = 1$ the distribution also leads to beta weighted distribution (all are new special sub-models). Then, set $a = b = 1$, it becomes weighted Weibull distribution (baseline distribution). Again, assume we set $\lambda = 1$ in (3.39), as we can see in (3.21) and

(3.26) (without loss of generality), we have

$$g_{BWWD}(x) = \frac{1}{B(a,b)}\left[\frac{\alpha+1}{\alpha}\left\{1-e^{-x_i^{\beta}}-\frac{1}{\alpha+1}\left(1-e^{-(1+\alpha)x^{\beta}}\right)\right\}\right]^{a-1}\times \left[1-\frac{\alpha+1}{\alpha}\left\{1-e^{-x^{\beta}}-\frac{1}{\alpha+1}\left(1-e^{-(1+\alpha)x^{\beta}}\right)\right\}\right]^{b-1} \frac{\alpha+1}{\alpha}\beta x^{\beta-1}e^{-x^{\beta}}\left(1-e^{-\alpha x^{\beta}}\right) \tag{3.40}$$

Then, $X \sim BWWD(a, b, \alpha, \beta, 1)$ Some new distributions can be obtained from the BWW distribution in (3.40).

3.6.1 Some Reductions of the Generalised Beta Weighted Weibull Distribution $g_{BWW}(x)$ and Probability Density Function of BWW Distribution

Some distributions emanate from the BWW distribution depending on values of the parameters;

1. When $\beta = 1$ in expression (3.40), the distribution becomes beta weighted exponential distribution and the density function is given as

$$f_{BWE}(x) = \frac{1}{B(a,b)}\left[\frac{(1+\alpha)\left(1-e^{-\lambda x}\right)+e^{-\lambda x(1+\alpha)}-1}{\alpha}\right]^{a-1} \left[1-\frac{(1+\alpha)\left(1-e^{-\lambda x}\right)+e^{-\lambda x(1+\alpha)}-1}{\alpha}\right]^{b-1} \frac{\lambda(1+\alpha)e^{-\lambda x}\left(1-e^{-\lambda\alpha x}\right)}{\alpha} \tag{3.41}$$

   where, $\alpha > 0, \lambda > 0, a > 0, b > 0$ and $x > 0$. $x \sim BWE(\alpha, \lambda, a, b)$.

2. When $a = 1$, the distribution reduces to the Lehmann Type II weighted Weibull (LWW) distribution, and the pdf is written as

$$f_{LWW}(x) = b\left[1-\frac{\alpha+1}{\alpha}\left\{\left(1-e^{-x^{\beta}}\right)-\frac{1}{\alpha+1}\left(1-e^{-(1+\alpha)x^{\beta}}\right)\right\}\right]^{b-1} \left[\frac{\alpha+1}{\alpha}\beta x^{\beta-1}e^{-x^{\beta}}\left(1-e^{-x^{\beta}}\right)\right] \tag{3.42}$$

   where $a = 1, b, \alpha, \beta, x > 0$

3. when $b = 1$, expression (3.40) reflects the Exponentiated Weighted Weibull (EWW) distribution with probability density function

$$f_{EWW}(x) = \alpha \left[\frac{\alpha+1}{\alpha} \left\{ \left(1 - e^{-x^{\beta}}\right) - \frac{1}{\alpha+1} \left(1 - e^{-(1+\alpha)x^{\beta}}\right) \right\} \right]^{a-1} \left[\frac{\alpha+1}{\alpha} \beta x^{\beta-1} e^{-x^{\beta}} \left(1 - e^{-\alpha b x^{\beta}}\right) \right] \tag{3.43}$$

where $a, b = 1, \alpha, \beta, x > 0$

4. For $\alpha = \alpha^{\beta}$ in (3.40), we have a version of beta weighted Weibull distribution, with pdf

$$f_{BWE}(x) = \frac{1}{B(a,b)} \left[\frac{(1+\alpha^{\beta})(1 - e^{\lambda x^{\beta}}) + e^{-\lambda x^{\beta}(1+\alpha^{\beta})-1}}{\alpha^{\beta}} \right]^{a-1} \left[1 - \frac{(1+\alpha^{\beta})(1 - e^{\lambda x^{\beta}}) + e^{-\lambda x^{\beta}(1+\alpha^{\beta})-1}}{\alpha} \right]^{b-1} \frac{\lambda\beta\left(1+\alpha^{\beta}\right) x^{\beta-1} e^{-\lambda x^{\beta}} \left(1 - e^{-\lambda(x\alpha)^{\beta}}\right)}{\alpha^{\beta}} \tag{3.44}$$

where $\alpha^{\beta} > 0, a > 0, b > 0$ and $x > 0$. $X \sim BWE(\alpha, \beta, \lambda, a, b)$

From (3.40), we set

$$K = K(x) = \frac{\alpha+1}{\alpha} \left[\left(1 - e^{-x^{\beta}}\right) - \frac{1}{\alpha+1} \left(1 - e^{-(1+\alpha)x^{\beta}}\right) \right] \tag{3.45}$$

Then,

$$\frac{dK}{dx} = \frac{\alpha+1}{\alpha} \left[\beta x^{\beta-1} e^{-x^{\beta}} - \frac{1}{\alpha+1}(1+\alpha)\beta x^{\beta-1} e^{-(1+\alpha)x^{\beta}} \right] \times \frac{\alpha+1}{\alpha} \left(1 - e^{-x^{\beta}}\right) - \frac{1}{\alpha+1} \left(1 - e^{-(1+\alpha)x^{\beta}}\right)$$

$$\frac{dK}{dx} = \frac{\alpha+1}{\alpha} \beta x^{\beta-1} e^{-x^{\beta}} \left(1 - e^{\alpha x^{\beta}}\right) \left[\frac{\alpha+1}{\alpha} \left\{ \left(1 - e^{-x^{\beta}}\right) - \frac{1}{\alpha+1} \left(1 - e^{-(1+\alpha)x^{\beta}}\right) \right\} \right]$$

Then,

$$dx = \frac{dK}{\frac{\alpha+1}{\alpha} \beta x^{\beta-1} e^{-x^{\beta}} \left(1 - e^{\alpha x^{\beta}}\right) \left[\frac{\alpha+1}{\alpha} \left\{ \left(1 - e^{-x^{\beta}}\right) - \frac{1}{\alpha+1} \left(1 - e^{-(1+\alpha)x^{\beta}}\right) \right\} \right]}$$

Substituting dx into (3.40), we get

$$g_{BWWD}(x) = \frac{1}{B(a,b)} \int_0^{\infty} \left[\frac{\alpha+1}{\alpha} \left\{ \left(1 - e^{-x^{\beta}}\right) - \frac{1}{\alpha+1} \left(1 - e^{-(1+\alpha)x^{\beta}}\right) \right\} \right]^{a-1} \left[1 - \frac{\alpha+1}{\alpha} \left\{ \left(1 - e^{-x^{\beta}}\right) - \frac{1}{\alpha+1} \left(1 - e^{-(1+\alpha)x^{\beta}}\right) \right\} \right]^{b-1} dK \tag{3.46}$$

and K in expression (3.45) can be written as

$$k(x) = \frac{dK(x)}{dx} = \frac{\alpha+1}{\alpha}\beta x^{\beta-1}e^{-x^{\beta}}\left(1-e^{-\alpha x^{\beta}}\right)$$

Equation (3.46) is the density function of the Beta-Weighted Weibull (BWW) distribution and can be re-written as

$$g_{BWWD}(x) = \frac{1}{B(a,b)}K^{a-1}(1-K)^{b-1}\frac{dK}{dx} \tag{3.47}$$

If X has BWW distribution with parameters a, b, α and β, we denote $X \sim BWW(a, b, \alpha, \beta)$. In figure 3.1-3.6, we plot the density function of different distributions i.e. BWW and some new distributions that was obtained from the BWW for different values of parameters are displayed in the graph below. Therefore, as the values of the parameters (a, b) decrease, the skewness of the distribution decreases and they have bell shape.

$$g_{BWWD}(x) = \frac{1}{B(a,b}\left[\frac{\alpha+1}{\alpha}\left\{\left(1-e^{-x^{\beta}}\right)-\frac{1}{\alpha+1}\left(1-e^{-(1+\alpha)x^{\beta}}\right)\right\}\right]^{a-1}\times$$
$$\left[1-\frac{\alpha+1}{\alpha}\left\{\left(1-e^{-x^{\beta}}\right)-\frac{1}{\alpha+1}\left(1-e^{-(1+\alpha)x^{\beta}}\right)\right\}\right]^{b-1}\times$$
$$\frac{\alpha+1}{\alpha}\beta x^{\beta-1}e^{-x^{\beta}}\left(1-e^{-\alpha x^{\beta}}\right)$$

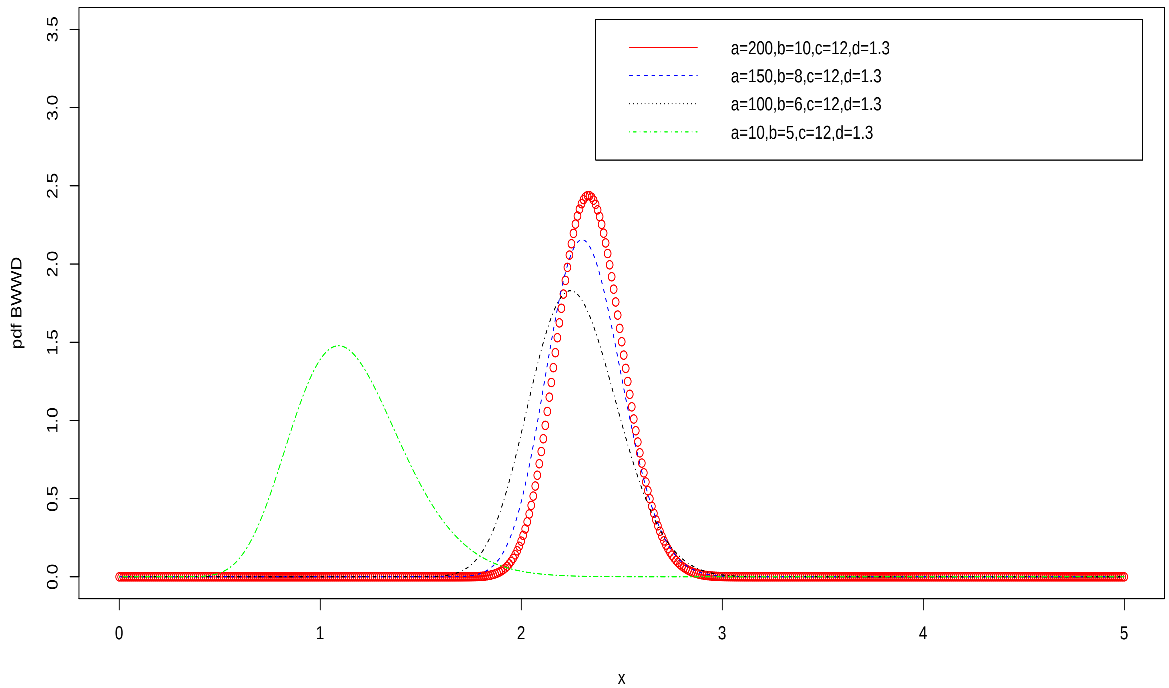

Figure 3.1: The graph of the density function of BWWD (a, b, α, β) for selected parameter values; i.e. fixed: $c = \alpha = 12$ and $d = \beta = 1.3$, some combinations of (a, b) from $a = 200, 150, 100, 10$ while $b = 10, 8, 6, 5$ respectively.

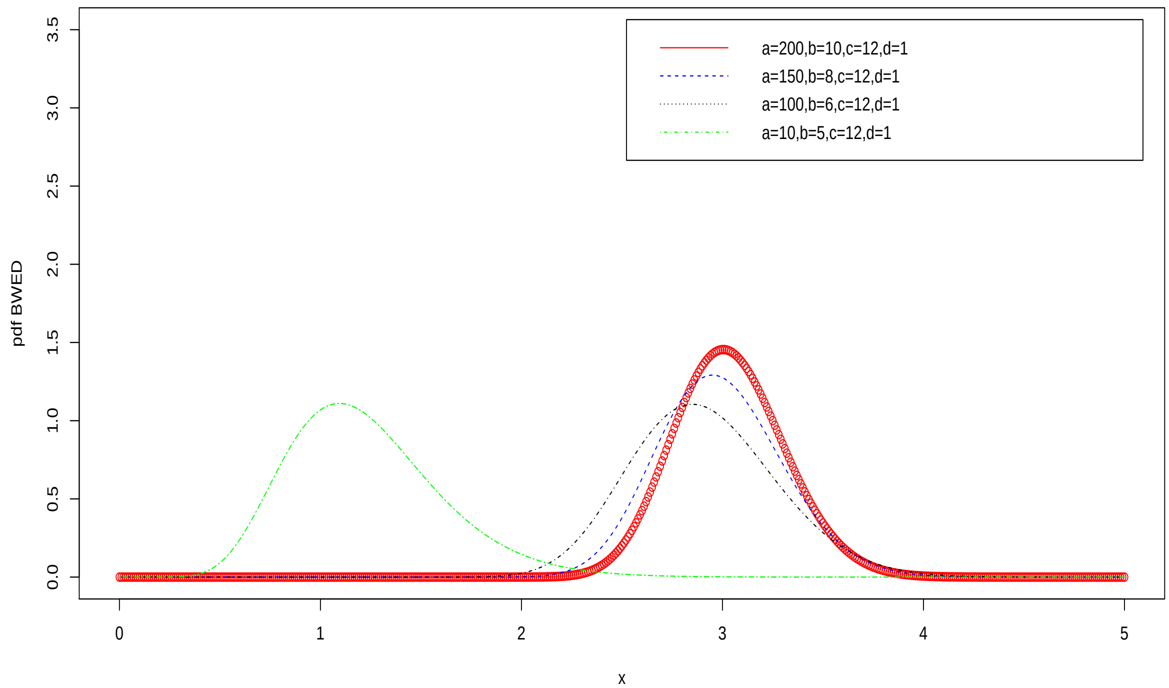

Figure 3.2: The graph of the density function of BWWD (a, b, α, β) for selected parameter values: i.e. fixed $c = \alpha = 12$ and $d = \beta = 1$ and some combinations of (a, b) from, $a = 200, 150, 100, 10$ and $b = 10, 8, 6, 5$ respectively.

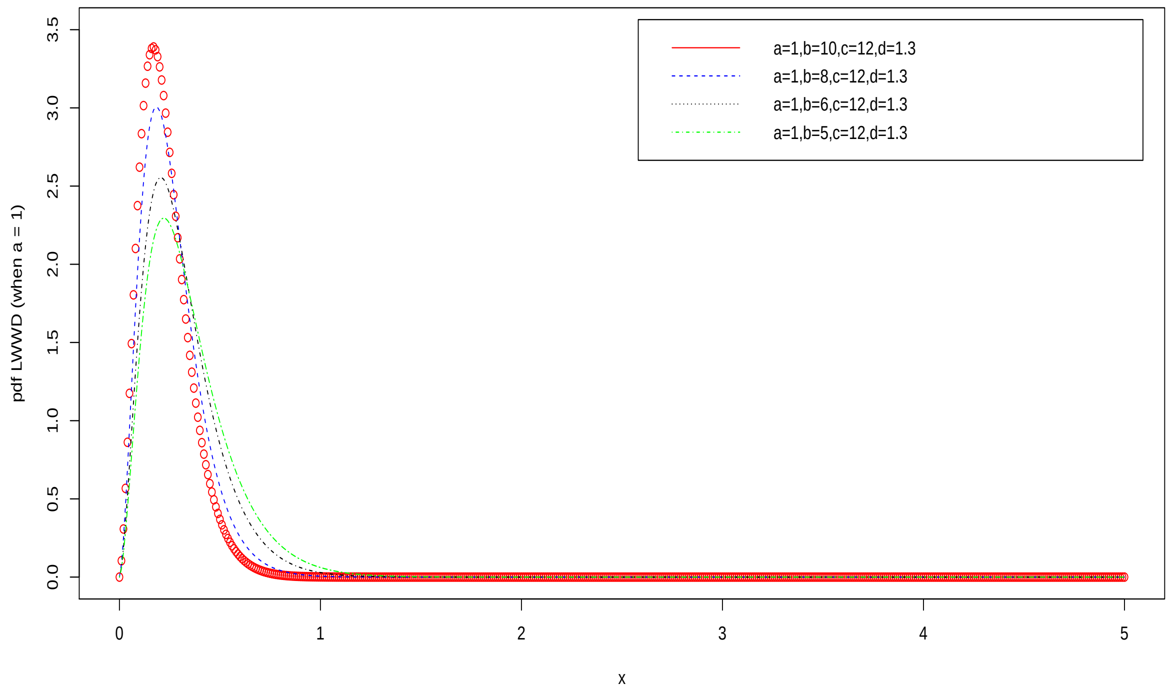

Figure 3.3: The graph of the density function of LWWD= BWWD $(1, b, \alpha, \beta)$ for selected parameter values, e.g. $c = \alpha = 12$ and $d = \beta = 1.3$. Then, $a = 1$, $b = 10, 8, 6, 5$ respectively.

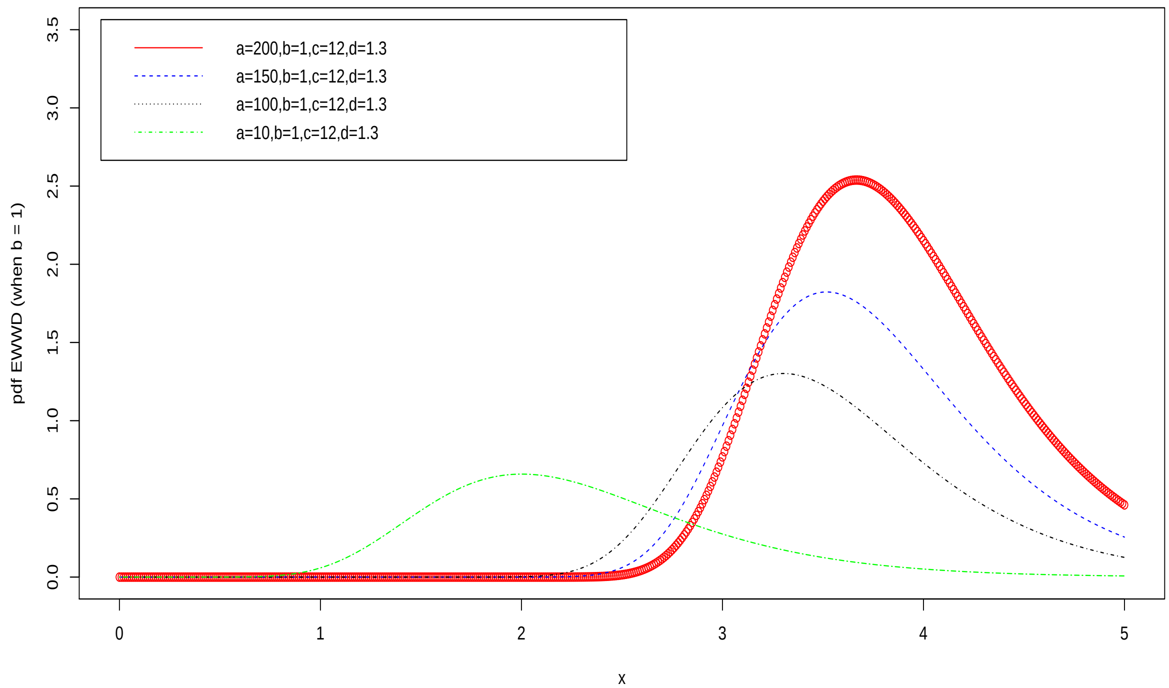

Figure 3.4: The graph of the density function of EWWD = BWWD $(a, 1, \alpha, \beta)$ for selected parameter values, e.g. $c = \alpha = 12$ and $d = \beta = 1.3$. Then, $a = 200, 150, 100\&10$ while b=1 respectively.

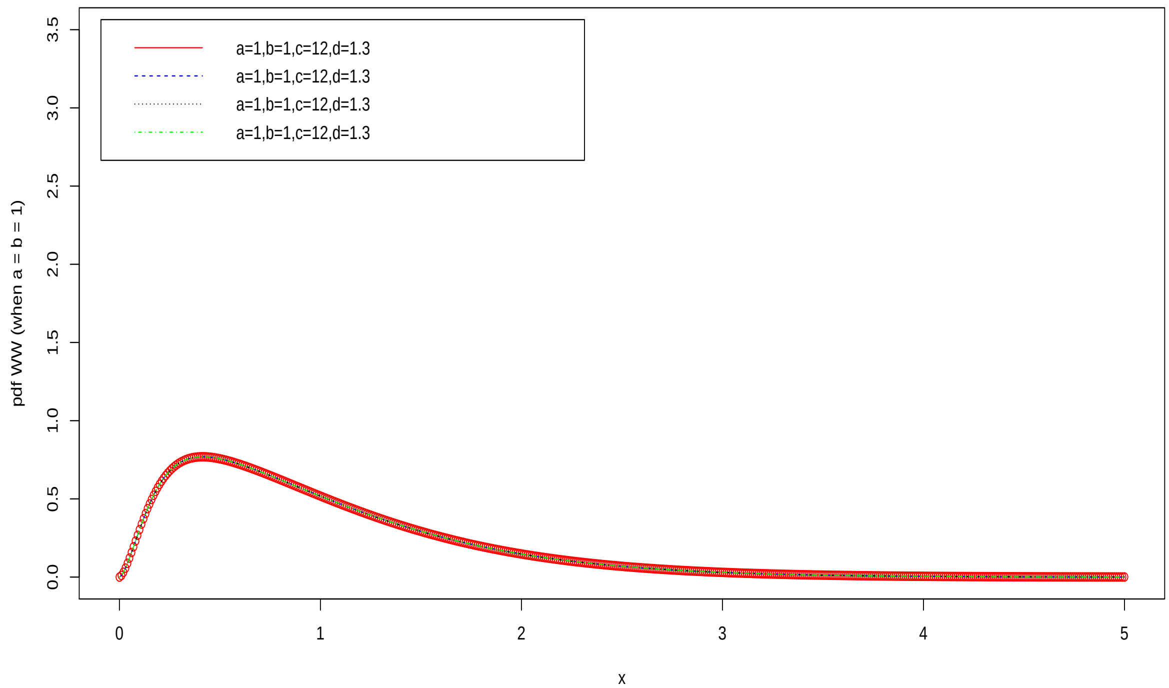

Figure 3.5: The graph of the density function of $WWD = BWWD(1, 1, \alpha, \beta)$ for selected parameter values: e.g. $c = \alpha = 12$, $d = \beta = 1.3$ and $a = b = 1$ respectively.

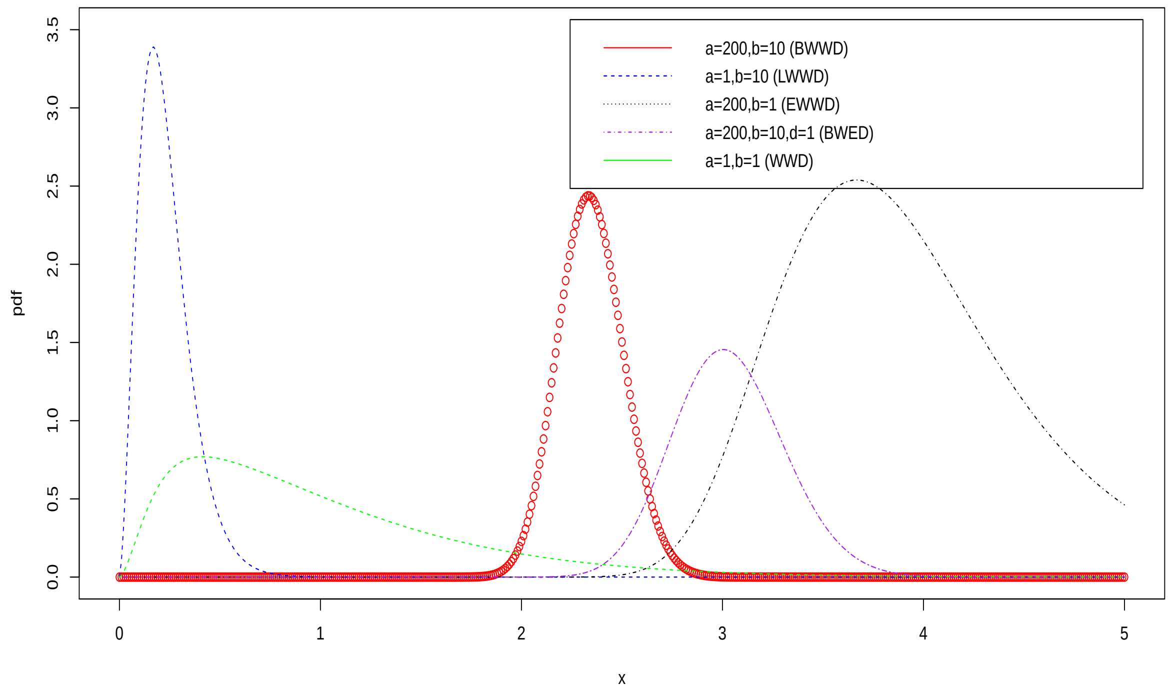

Figure 3.6: The graph of the density function of BWWD ($a = 200, b = 10$), LWWD blue colour ($a = 1\&b = 10$), EWWD black colour ($a = 200, b = 1$) and WWD green colour ($a = 1, b = 1$) for selected parameter values, e.g. $c = \alpha = 12, d = \beta = 1.3$.

We also show that the generalised BWW has a true probability density function by showing that

$$g_{BWW}(x) \geq 0,\ 0 < x < \infty\ \ and\ \ \int_0^{\infty} g_{BWW}(x)dx = 1 \tag{3.48}$$

Following McDonald (1984) and Jones (2004) generalised beta distribution of first kind given as:

$$g_x(x, a, b, m) = \frac{m}{B(a, b)}[F(x)]^{am-1}[1 - F(x)^m]^{b-1}f(x) \quad 0 < x < 1$$

where, $a > 0, b > 0$ and $m > 0$ and using differentiation of expression (3.45) above. then

$$g_{BWWD}(x) = \frac{m}{B(a, b)}K^{am-1}(1 - K^m)^{b-1}\frac{dK}{dx}$$

$$\int_{-\infty}^{\infty} g_{BWWD}(x) = \int_{-\infty}^{\infty} \frac{m}{B(a, b)}K^{am-1}(1 - K^m)^{b-1}dK$$

also, let $W = K^m$ and differentiate W with respect to K

$$\frac{dW}{dK} = mK^{m-1}$$

$$dK = \frac{dW}{mK^{m-1}}$$

$$K = W^{\frac{1}{m}}$$

$$\int_{-\infty}^{\infty} g_{BWWD}(x)d(x) = \int_0^1 \frac{m}{B(a,b)} \left(W^{\frac{1}{m}}\right)^{am-1} (1-W)^{b-1} \frac{dW}{mK^{m-1}}$$

$$g_{BWWD}(x) = \int_0^1 \frac{W^{a-\frac{1}{m}}(1-W)^{b-1}}{W^{1-\frac{1}{m}}} dW$$

$$g_{BWWD}(x) = \frac{1}{B(a,b)} \int_0^1 W^{a-1}(1-W)^{b-1} dW$$

$$g_{BWWD}(x) = \frac{B(a,b)}{B(a,b)} = 1 \tag{3.49}$$

Therefore, this shows that $g_B WW(x)$ is a probability density function of a continuous distribution.

3.6.2 Cumulative Distribution Function (CDF)

We have defined the beta-weighted Weibull distribution in (3.46). The expression above may be given as

$$G_{BWWD}(x) = P(X \leq x) = \int_0^x f(k)dk \tag{3.50}$$

$$= \int_0^x \frac{1}{B(a,b)} \left[\frac{\alpha+1}{\alpha}\left\{\left(1-e^{-x^\beta}\right) - \frac{1}{\alpha+1}\left(1-e^{(1+\alpha)x^\beta}\right)\right\}\right]^{a-1}$$

$$\left[1 - \frac{\alpha+1}{\alpha}\left\{\left(1-e^{-x^\beta}\right) - \frac{1}{\alpha+1}\left(1-e^{(1+\alpha)x^\beta}\right)\right\}\right]^{b-1}$$

$$\frac{\alpha+1}{\alpha}\beta x^{\beta-1}e^{-x^\beta}\left(1-e^{-\alpha x^\beta}\right) dK$$

using (3.40) in (3.47), yields

$$G_{BWWWD}(x) = P(X \leq x) = \int_0^x \frac{1}{B(a,b)} K^{a-1}(1-K)^{b-1} dK$$

$$= \frac{1}{B(a,b)} \int_0^x K^{a-1}(1-K)^{b-1} dK$$

where, $\int\limits_0^x K^{a-1}(1-K)^{b-1}dk = B(k;a,b)$

Now,

$$G_{BWWD}(x) = \frac{1}{B(a,b)} \int_0^x K^{a-1}(1-K)^{b-1} dK = \frac{B(k;a,b)}{B(a,b)} \tag{3.51}$$

where $B(k;a,b)$ is called an incomplete beta function.

The plots below show the graph of the CDF of the Beta-weighted Weibull distribution (BWWD)

$$G_{BWWD}(x) = \frac{1}{B(a,b)} \int_0^x K^{a-1}(1-K)^{b-1} dK = \frac{B(k;a,b)}{B(a,b)}$$

where K as in (3.45) above.

For some values of the parameters coinciding with specific sub-models and baseline variants of distribution i.e. $c = \alpha$ and $d = \beta$. In figure 3.7 as the values of the parameters (a, b) increases, the cdf increases. Then, in figure 3.8, 3.9 and 3.10, as parameter a is constant and b increases, the cdf increases, a increases and b is constant, the cdf increases and as a and b are constant, the cdf also increases. Lastly, in figure 3.11 the higher the values of a and b, the higher the cdf of BWW distribution.

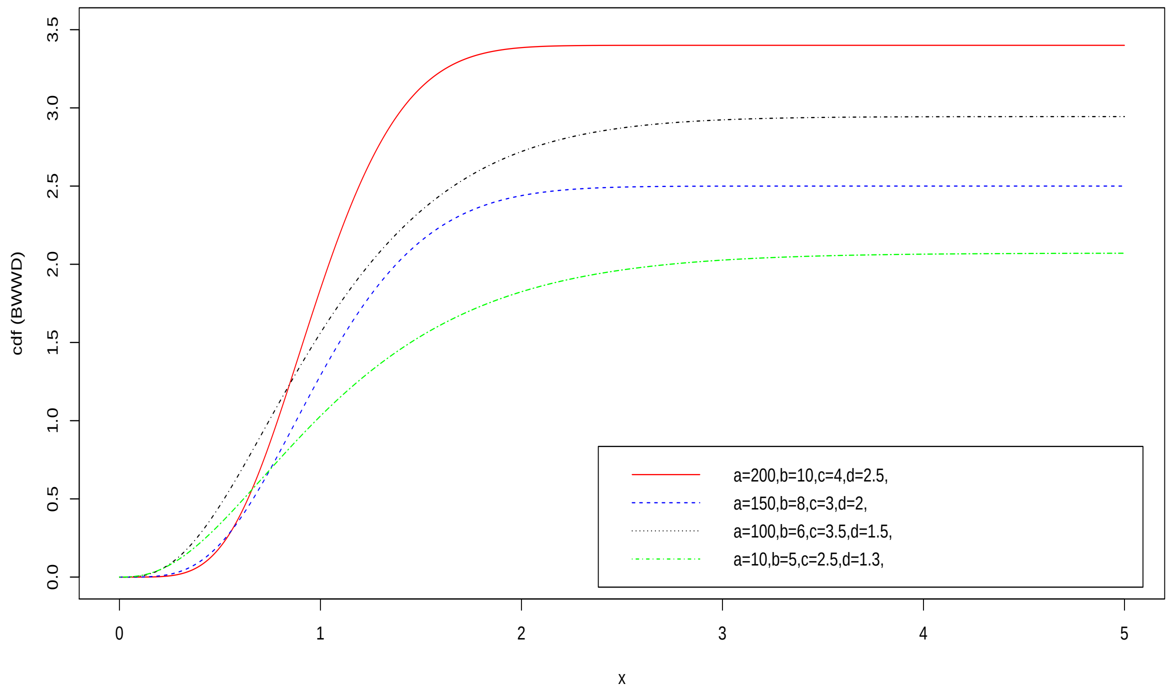

Figure 3.7: The plots of the distribution function of BWWD at (a, b, α, β)= (200, 10, 4, 2.5), (150,8,3,2), (100,6,1.5,3.5), (10,5,1.5,1.3) respectively.

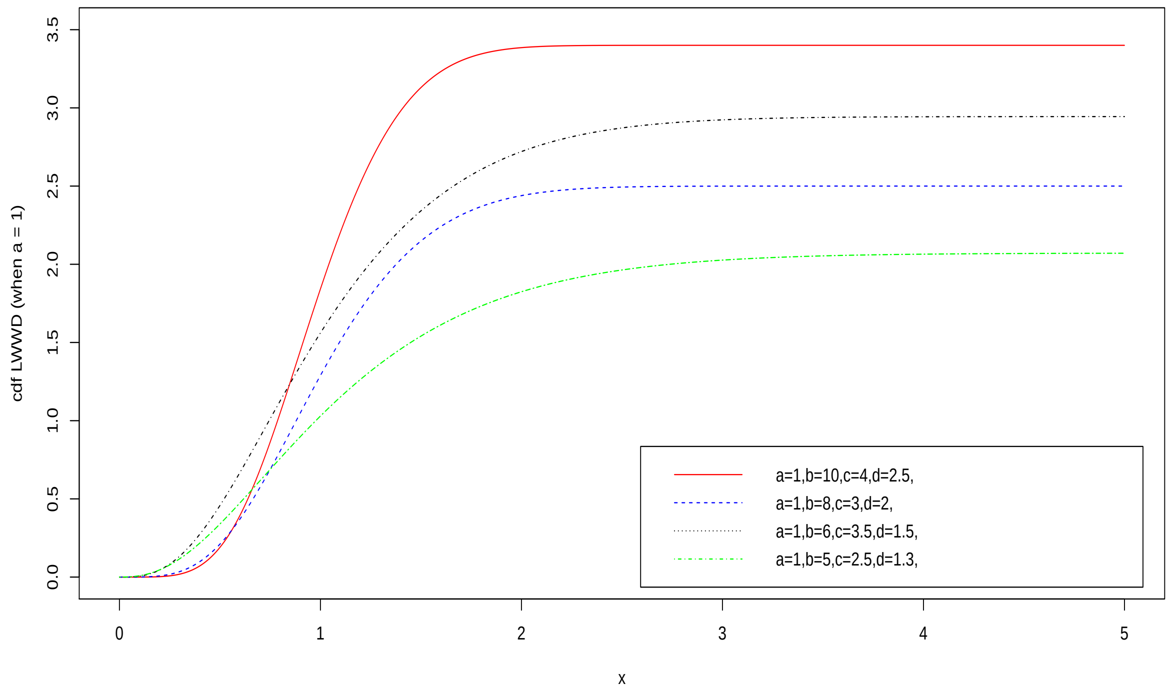

Figure 3.8: The plot of the distribution function of LWWD i.e. BWWD $(a = 1, b, \alpha, \beta)$ at 1 and other parameters at specified values of $(b, \alpha, \beta) = (10, 4, 2.5),\ (8, 3, 2)$; and $(b, \alpha, \beta) = (10, 4, 2.5),\ (8, 3, 2),\ (6, 3.5, 1.5)$ and (5,2.5,1.3)

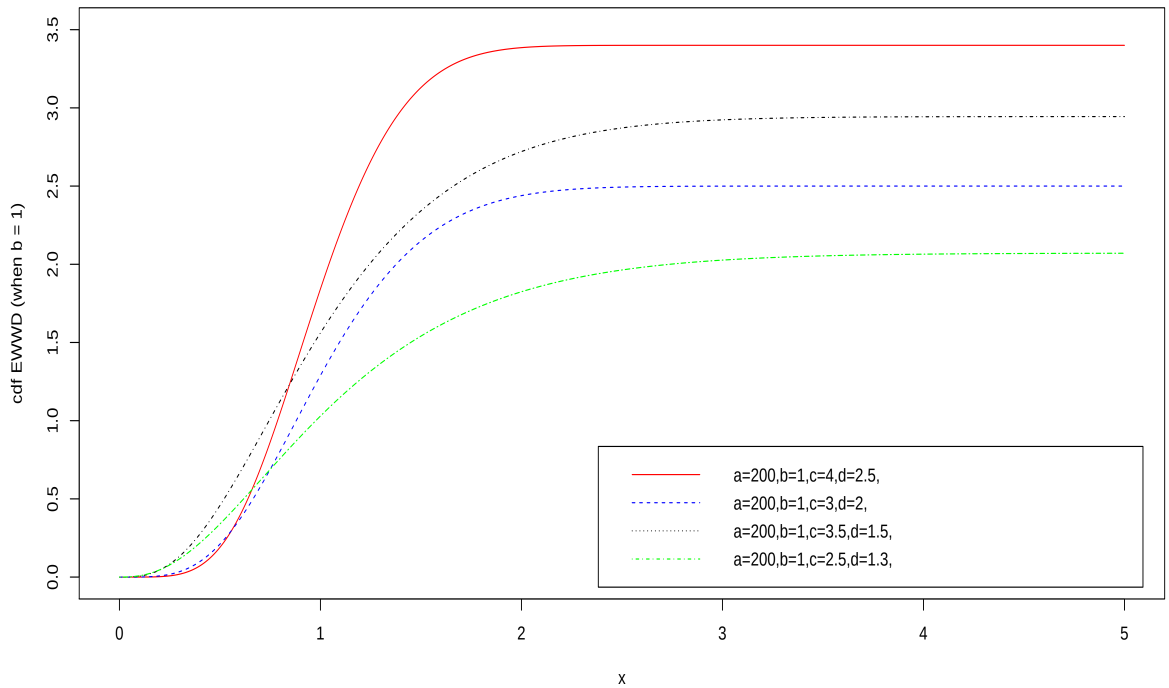

Figure 3.9: The plot of the distribution function of EWWD i.e. $BWWD(a, 1, \alpha, \beta)$ at $(a, \alpha, \beta) = (200, 4, 2.5)$, $(150, 3, 2)$, $(100, 3.5, 1.5)$, $(10, 2.5.1.3)$ respectively.

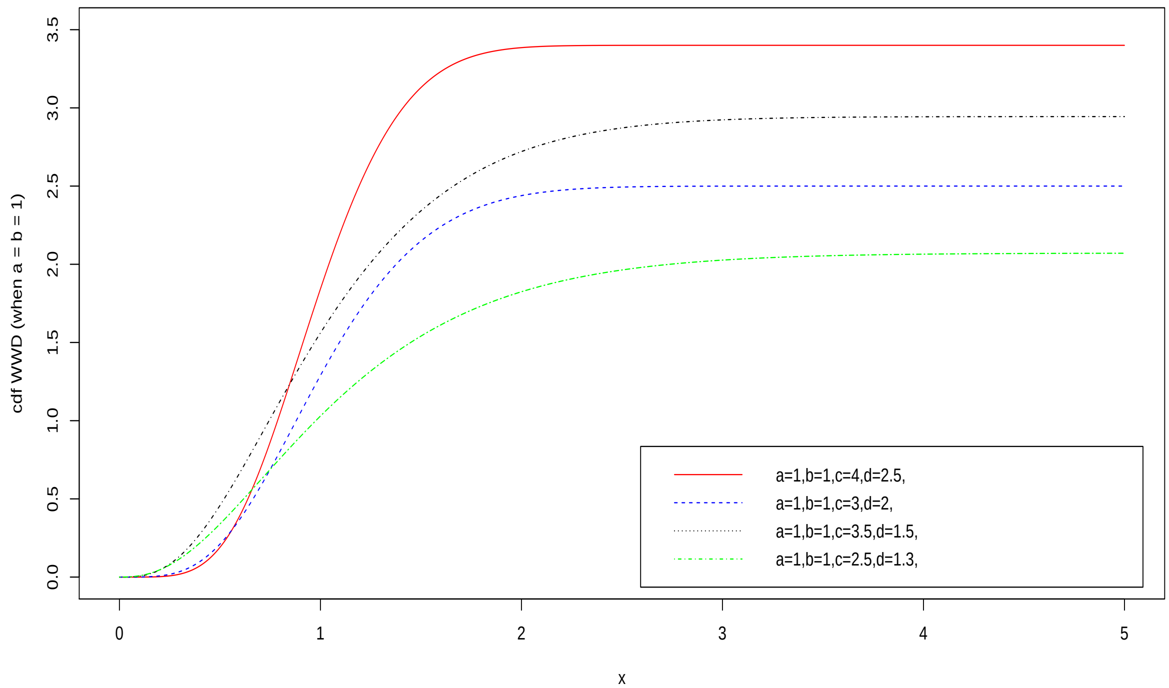

Figure 3.10: The plot of the distribution function of WWD i.e. BWWD $(1, 1, \alpha, \beta)$ at $(\alpha, \beta) = (4, 2.5)$, $(3, 2)$, $(3.5, 1.5)$, $(2.5, 1.3)$ respectively.

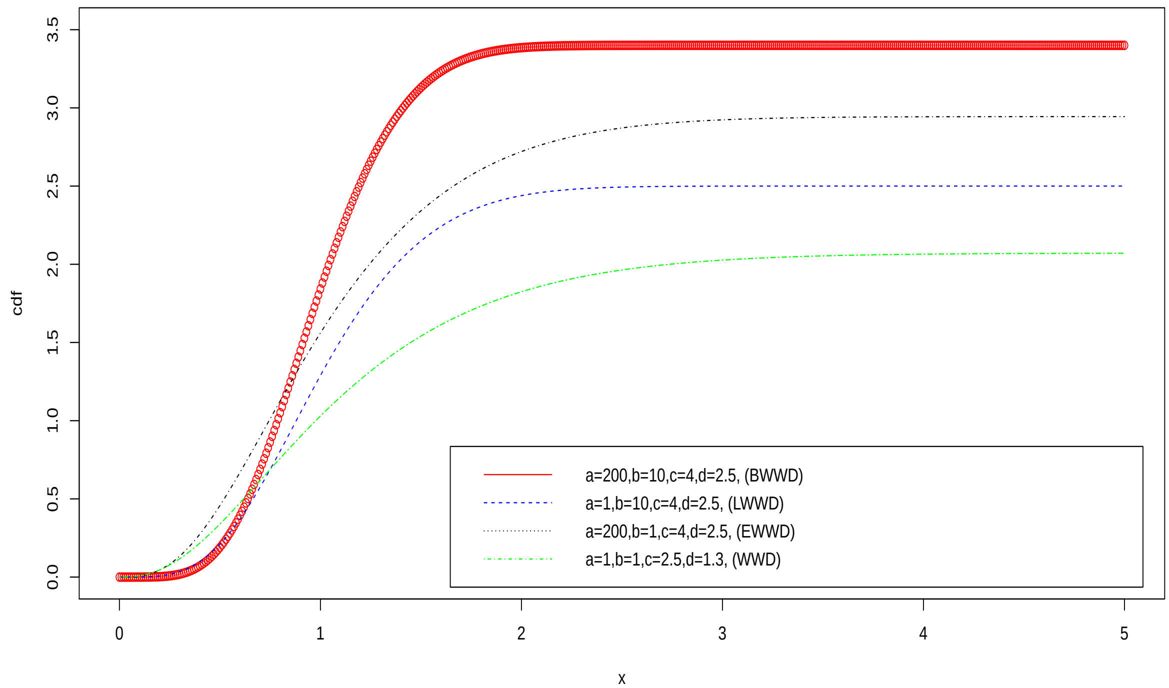

Figure 3.11: The plot of the distribution function of BWWD, LWWD, EWWD and WWD at $(a, b, \alpha, \beta) = (200, , 4, 2.5),\ (1, 8, 3, 2),\ (200, 1, 3.5, 1.5)$ and $(1, 1, 2.5, 1.3)$ respectively

3.6.3 The Survival Rate Function

The survival rate function of the beta-Weighted Weibull distribution (BWWD) is defined as:

$$S_{BWWD}(x) = 1 - G_{BWWD}(x) = 1 - \int_0^x f(k)dk \tag{3.52}$$

$$S_{BWWD}(x) = 1 - G_{BWWD}(x) = 1 - \int_0^x \frac{1}{B(a,b)} K^{a-1}(1-K)^{b-1}dK$$

where,

$$\int_0^x \frac{1}{B(a,b)} K^{a-1}(1-K)^{b-1}dK = \frac{B(k;a,b)}{B(a,b)}$$

$$S_{BWWD}(x) = 1 - \frac{B(k;a,b)}{B(a,b)} = \frac{B(a,b) - B(k;a,b)}{B(a,b)} \tag{3.53}$$

Below are graphs of the survival rate of the BWWD (a, b, α, β) including the general model and some sub-models as specified by fixed values of some of the parameters. In figure 3.12 as the values of the parameters (a, b) change, the survival rate decrease negatively and tends to zero. Parameter a is contact in figure 2.13, the survival rate tends to zero and parameter b is constant in figure 3.14, the survival rate also tends to zero. In figure 3.15 parameters (a, b) are constant, the survival rate tends to zero and in figure 3.16 the survival rate of the BWW distribution decreases while other distributions tend to zero.

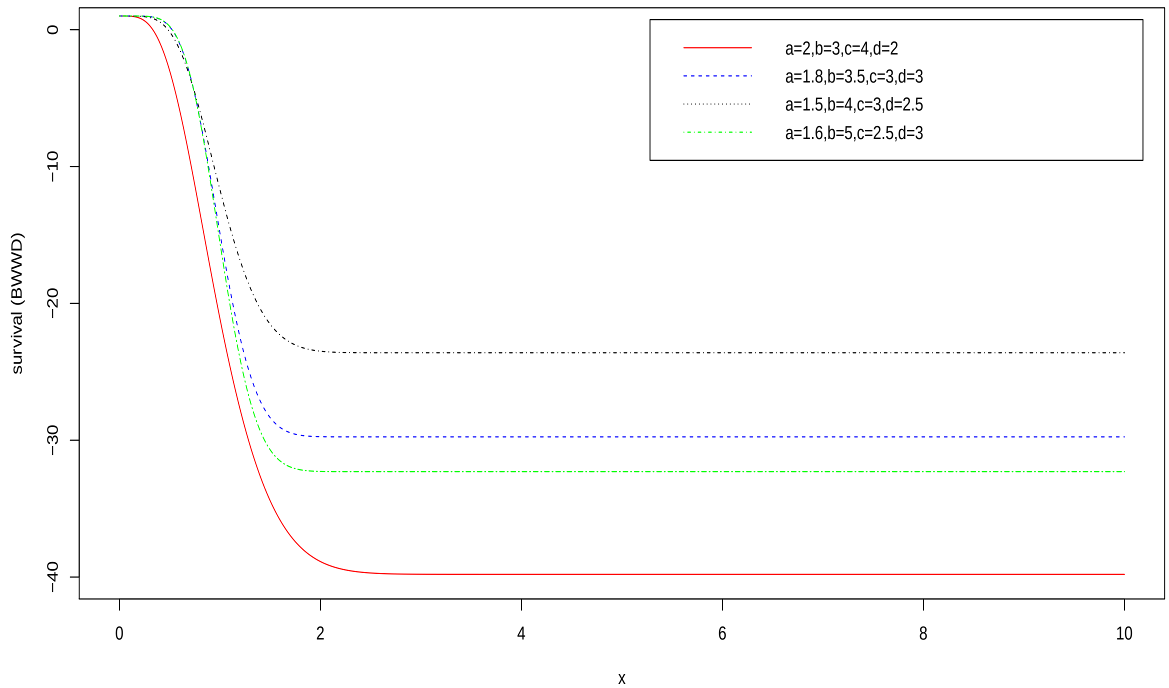

Figure 3.12: The graph of survival rate function of the BWWD(a, b, α, β) at parameter values:$(a, b, \alpha, \beta) = (2, 3, 4, 2), (1.8, 3.5, 3, 3), (1.5, 4, 3, 2.5)$ respectively.

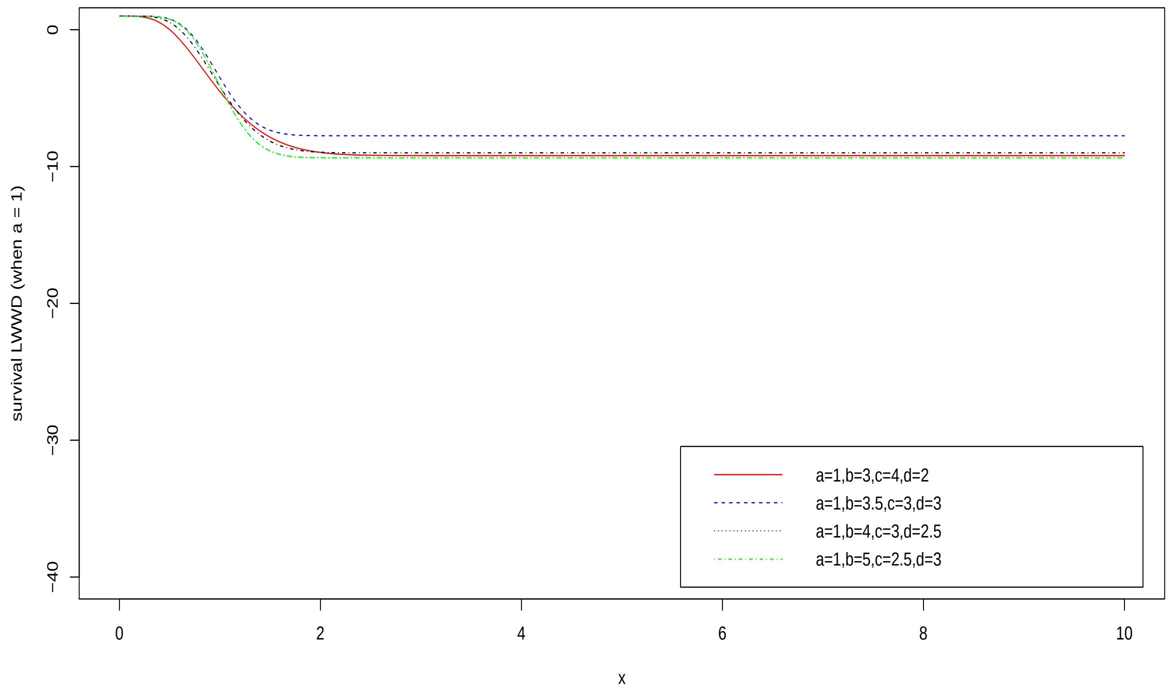

Figure 3.13: The graph of survival rate function of the $LWWD = BWWD(1, b, \alpha, \beta)$ at $(b, \alpha, \beta) = (1, 3, 4, 2)$, $(1, 3.5, 3, 3)$, $(1, 4, 3, 2.5)$, respectively.

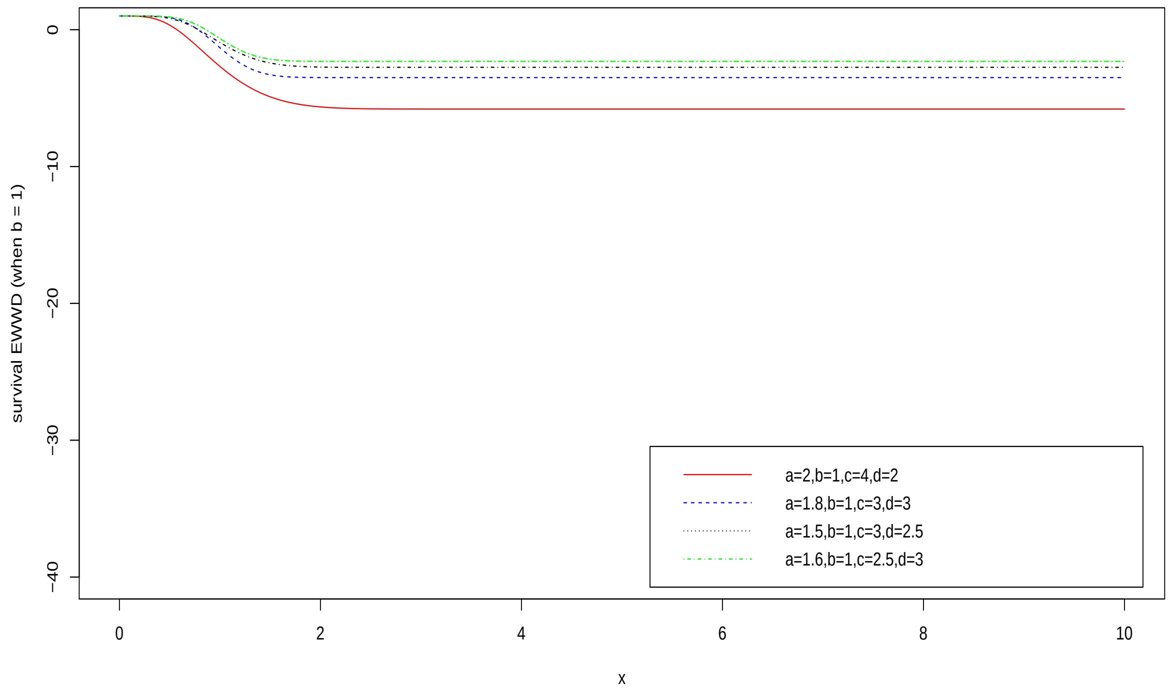

Figure 3.14: The graph of survival rate function of the EWWD= BWWD $(a, 1, \alpha, \beta)$ at $(a, b, \alpha, \beta) = (2, 1, 4, 2)$, $(1.8, 1, 3, 3)$, $(1.5, 1, 3, 2.5)$, $(1.6, 1, 2.5, 3)$, respectively.

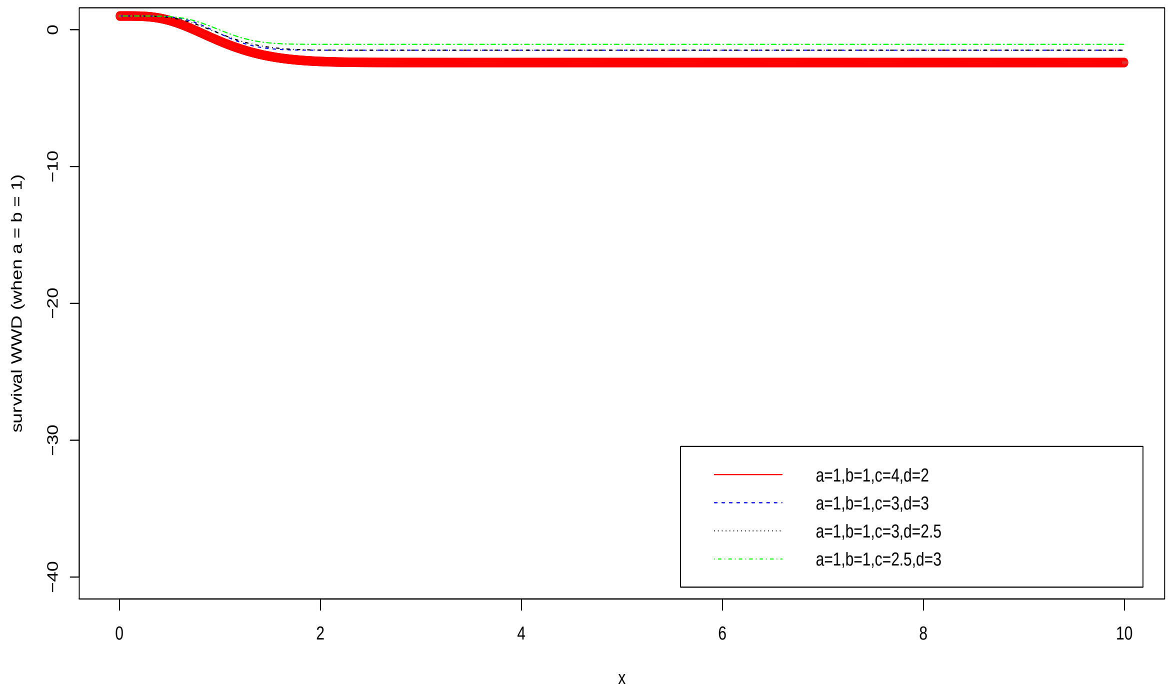

Figure 3.15: The graph of survival rate function of the $WWD = BWWD(1, 1, \alpha, \beta)$ at $(a, b, \alpha, \beta) = (1, 1, 4, 2),\ (1, 1, 3, 3),\ (1, 1, 3, 2.5),\ (1, 1, 2.5, 3)$, respectively.

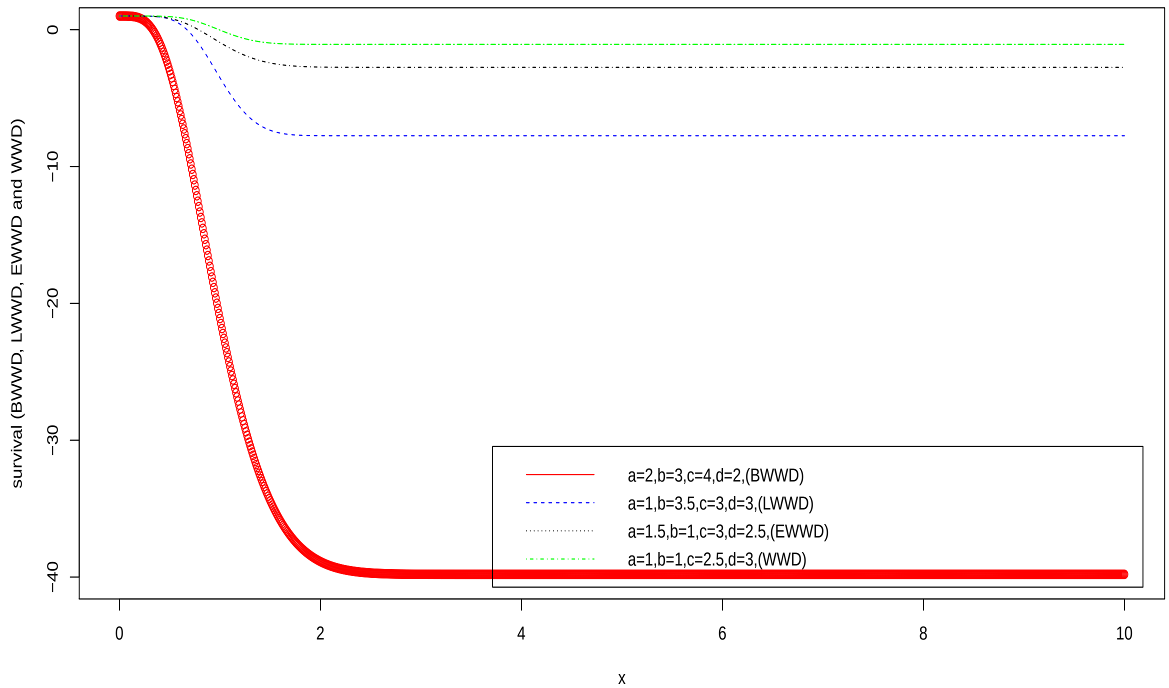

Figure 3.16: The graph of survival rate function of the $BWWD = (a, b, \alpha, \beta)$, $LWWD = BWWD(1, b, \alpha, \beta)$, $EWWD = BWWD(a, 1, \alpha, \beta)$, and $WWD = BWWD(1, 1, \alpha, \beta)$ at $(a, b, \alpha, \beta) = (2, 3, 4, 2)$; $(1, 3.5, 3, 3)$, $(1.5, 1, 3, 2.5)$, $(1, 1, 2.5, 3)$, respectively.

3.6.4 The Hazard Rate Function

The hazard rate function of a random variable X with the probability density function $g_{BWWD}(x)$ and cumulative density function $G_{BWWD}(x)$ is given by

$$h_{BWWD}(x) = \frac{g_{BWWD}(x)}{1 - G_{BWWD}(x)} \tag{3.54}$$

When, we have

$$\begin{aligned} h_{BWWD}(x) &= \frac{\frac{1}{B(a,b)} K^{a-1}(1-K)^{b-1}K'}{1 - \frac{B(k;a,b)}{B(a,b)}} \\ &= \frac{K^{a-1}(1-K)^{b-1}K'}{B(a,b)} \div \frac{B(a,b) - B(k;a,b)}{B(a,b)} \end{aligned}$$

Hence, the beta-weighted Weibull distribution with $g_{BWWD}(x)$ and $G_{BWWD}(x)$ respectively, defined in (3.40) and (3.53) constitutes an hazard rate function which can be expressed as:

$$h_{BWWD}(x) = \frac{K^{a-1}(1-K)^{b-1}K'}{B(a,b) - B(k;a,b)} \tag{3.55}$$

where K is the expression in (3.45) and K' is the pdf of the parent distribution (3.26).

Therefore, since hazard rate function is a measure of the tendency of a component to fail. The higher the value of the hazard function is, the higher the probability of the impending failure is. Technically, the hazard function is a probability of failure in a very small time interval.

The Plots of the hazard rate function of the BWWD and its sub-models shown below in figure 3.17 and 3.18 corroborates the above statements about the hazard rate function.

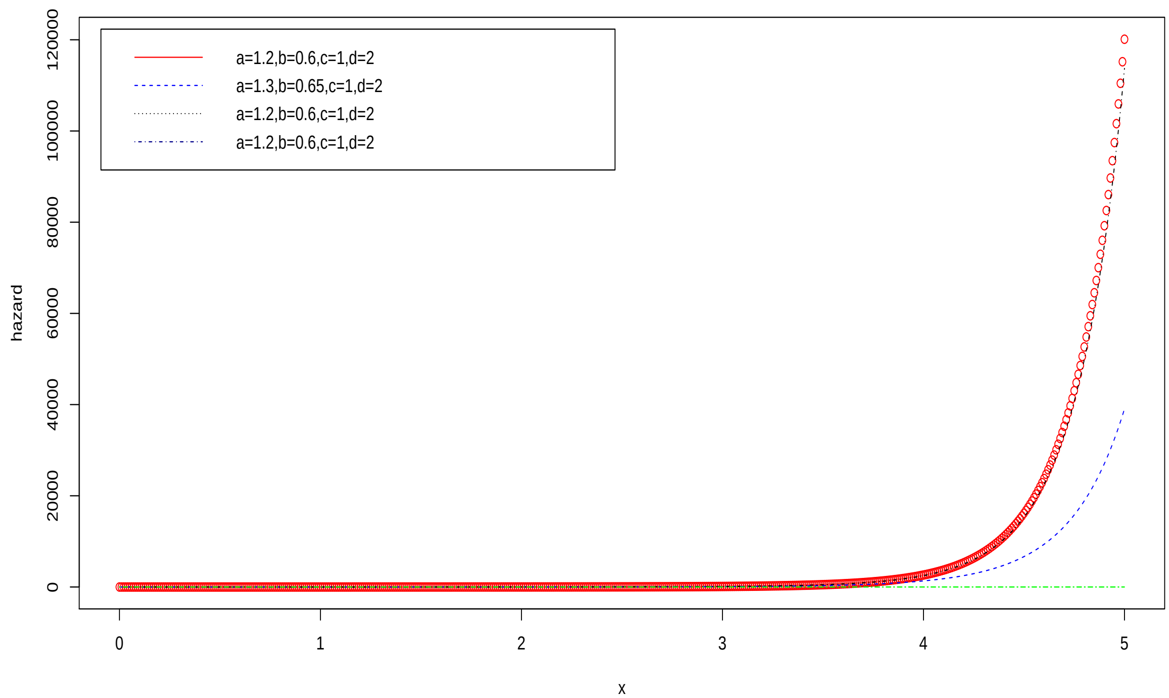

Figure 3.17: The graph of hazard rate of the BWWD $(a, b, 1, 2)$ at $(a, b) = (1.2, 0.6)$, $(1.2, 0.65)$, $(1.3, 0.6)$, $(1.3, 0.65)$, respectively.

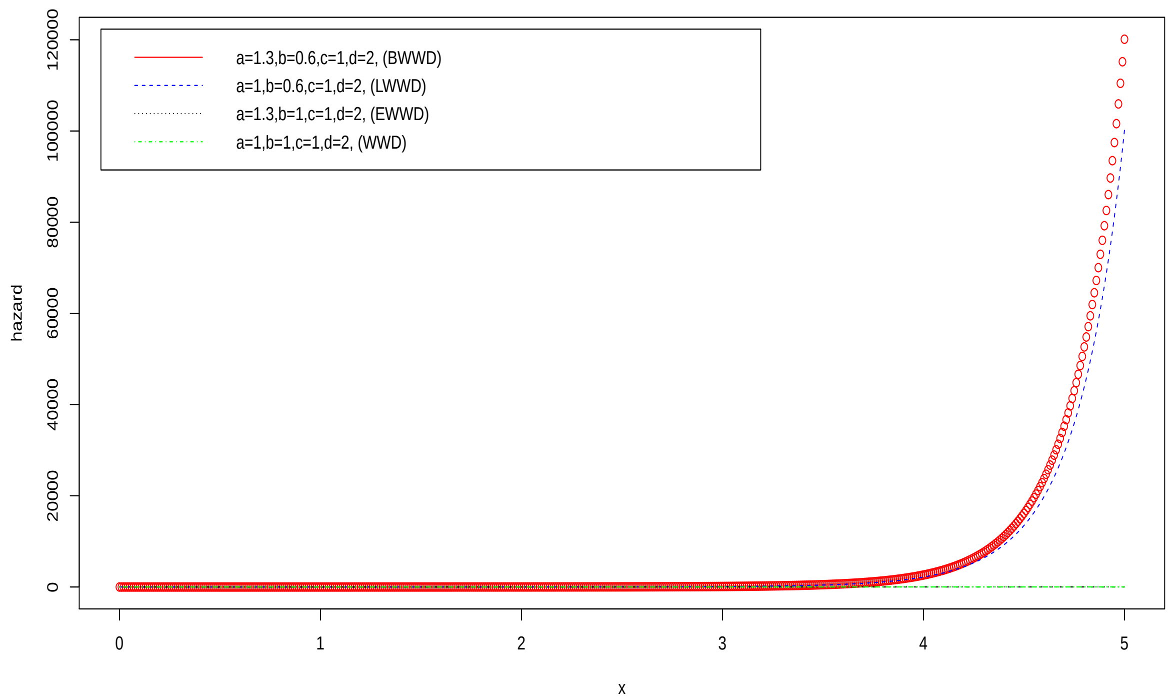

Figure 3.18: The graph of hazard rates of the $BWWD(a, b, \alpha, \beta)$, $LWWD = BWWD(1, b, \alpha, \beta)$, $EWWD = BWWD(a, 1, \alpha, \beta)$ and WWD= BWWD $(1, 1, \alpha, \beta)$ at (a, b, α, β)=(1,3,0.6,1,2), (1,0.6,1,2), (1.3,1,1,2), (1,1,1,2), respectively.

Furthermore, to show that $\lim_{x\to\infty} h_{BWWD}(x) = 0$ and $\lim_{x\to 0} h_{BWWD}(x) = 0$, we obtain the following

$$\lim_{x\to\infty} h_{BWWD}(x) = \lim_{x\to\infty} \frac{K^{a-1}(1-K)^{b-1}K'}{B(a,b) - B(k;a,b)}$$

where, $K' = \frac{\alpha+1}{\alpha}\beta x^{\beta-1}e^{-x^\beta}\left(1-e^{-\alpha x^\beta}\right)$ and K as in (3.45)

$$= \lim_{x\to\infty} \frac{K^{a-1}(1-K)^{b-1}K'}{B(a,b) - B\left[\frac{\alpha+1}{\alpha}\left\{\left(1-e^{-x^\beta}\right) - \frac{1}{\alpha+1}\left(1-e^{-(1+\alpha)x^\beta}\right)\right\};a,b\right]}$$

For easy simplification for the above expression, we take the limit of the base line distribution in asymptotic behaviour below when $x \longrightarrow \infty$ and $x \longrightarrow 0$

3.6.5 The Asymptotic Behaviours

The asymptotic properties of the beta-weighted Weibull distribution are investigated by considering the behaviour of $\lim_{x\to 0} g_{BWWD}(x)$ and $\lim_{x\to\infty} g_{BWWD}(x)$ as follows: Considering the situation when $x \longrightarrow 0$ and $x \longrightarrow \infty$ respectively in equation (3.40), we get

$$\begin{aligned}\lim_{x\to 0} g_{BWWD}(x) = \lim_{x\to 0} \frac{1}{B(a,b)} &\left[\frac{\alpha+1}{\alpha}\left\{\left(1-e^{-x^\beta}\right) - \frac{1}{\alpha+1}\left(1-e^{-(1+\alpha)x^\beta}\right)\right\}\right]^{a-1} \\ &\left[\frac{\alpha+1}{\alpha}\left\{\left(1-e^{-x^\beta}\right) - \frac{1}{\alpha+1}\left(1-e^{-(1+\alpha)x^\beta}\right)\right\}\right]^{b-1} \\ &\frac{\alpha+1}{\alpha}\beta x^{\beta-1}e^{-x^\beta}\left(1-e^{-\alpha x^\beta}\right)\end{aligned} \tag{3.56}$$

To make it simpler then, we take the limits of the density function of the baseline distribution in (3.56).

$$\lim_{x\to 0} g_{WW}(x;\alpha,\beta) = \lim_{x\to 0} \frac{\alpha+1}{\alpha}\beta x^{\beta-1}e^{-x^\beta}\left(1-e^{-\alpha x^\beta}\right) = 0 \tag{3.57}$$

Furthermore, for $x \longrightarrow \infty$

$$\lim_{x\to\infty} g_{WW}(x;\alpha,\beta) = \lim_{x\to\infty} \frac{\alpha+1}{\alpha}\beta x^{\beta-1}e^{-x^\beta}\left(1-e^{-\alpha x^\beta}\right) = 0 \tag{3.58}$$

since $\lim_{x\to\infty} e^{-x^\beta} = 0$ and $\lim_{x\to\infty}\left(1-e^{-\alpha x^\beta}\right) = 1$

In a nut shell, the asymptotic behaviour when $x \longrightarrow 0$ and $x \longrightarrow \infty$ are equal to 0. Then, from the limits, we conclude that the pdf of BWW distribution is unimodal i.e. the distribution has one mode.

3.6.6 Moments and Generating Function (MGF)

Following Hosking (1990), when a random variable follows a generalized beta generated distribution, then $X \sim GBG(f, a, b, c)$, then $\mu'_m = E\left[F^{-1}\mu^{\frac{1}{c}}\right]^m$, where $\mu \sim \beta(a, b)$, c is a constant and $F^{-1}(x)$ is the inverse of CDF of the Weighted Weibull distribution.

Since Beta-Weighted Weibull is a special form where $C = 1$; we derive the moment generating function (mgf) of beta-Weighted Weibull distribution $M_{(t)} = E\left[e^{tx}\right]$ as the general rth moment of a beta-generated distribution which is given by

$$\mu'_m = \frac{1}{B(a, b)} \int_0^1 [F^{-1}(x)]^m x^{a-1}(1-x)^{b-1}dx \tag{3.59}$$

Then, we use the Taylor series expansion around the point $E(x_f) = \mu_f$ to obtain

$$\mu'_m = \sum_{\mu=0}^{m} \binom{m}{u} \left[F^{-1}(x_f)\right]^{m-\mu} \left[F^{-1(1)}\mu_f\right]^{\mu} \sum_{\mu=0}^{m}(-1)^j \binom{m}{i} \tag{3.60}$$

Cordeiro *et al.* (2011) gave an alternative series expansion for μ'_m in terms of $r(r, k) = E\left[Y^r F(Y)^k\right]$ where Y follows the parent distribution; then for $m = 0, 1, \cdots$ we have

$$\mu'_m = \frac{1}{B(a, b)} \sum_{c=0}^{\infty}(-1)^i \binom{b-1}{i} r(r, a, i-1)$$

Again, according to Cordeiro *et al.* (2011), using another moment generating function of X for generated beta-distribution, we have

$$M_{(t)} = \frac{1}{B(a, b)} \sum_{i=0}^{\mathbb{N}}(-1)^i \binom{b-1}{i} p(t, a, i-1) \tag{3.61}$$

where

$$p(t, \alpha, r) = \int_{-\infty}^{\infty} e^{tx}[F(x)]^r f(x)dx$$

Then

$$M_{BWWD(x)}(t) = \frac{1}{B(a, b)} \sum_{i=0}^{\infty}(-1)^i \binom{b-1}{i} \int_{-\infty}^{\infty} e^{tx}[F(x)]^{a(i+1)-1} f(x)dx \tag{3.62}$$

Substituting density function $f(x)$ and distribution function $F(x)$ defined in (3.26) and (3.27) into the MGF $M(t)$ in equation (3.62) gave

$$M_{BWWD(x)}(t) = \frac{1}{B(a, b)} \sum_{i=0}^{\infty}(-1)^i \binom{b-1}{i} \int_{-\infty}^{\infty} e^{tx} \tag{3.63}$$
$$\left[\frac{\alpha+1}{\alpha}\left\{\left(1-e^{-x^{\beta}}\right) - \frac{1}{\alpha+1}\left(1-e^{-(1+\alpha)x^{\beta}}\right)\right\}\right]^{a(i+1)-1}$$
$$\frac{\alpha+1}{\alpha}\beta x^{\beta-1}e^{-x^{\beta}}\left(1-e^{-\alpha x^{\beta}}\right)dx$$

Equation (3.63) becomes the moment generating function of the Beta-Weighted Weibull distribution. When $a = b = 1$, (3.63) reduces to the moment generating function of the Weighted Weibull distribution. Again, to obtain the rth moment of the beta-weighted Weibull distribution, we have the following:

The moment generating function and rth moment of weighted Weibull distribution by Shahbaz *et al.* (2010) follows from (3.32) and (3.33) as

$$M_x(t) = \int_0^\infty \frac{\alpha+1}{\alpha}\beta x^{\beta-1}e^{-x^\beta}\left(1-e^{-\alpha x^\beta}\right)dx = \sum_{j=0}^{\infty}\frac{t^j}{j!\alpha}\left[1+\alpha-(1+\alpha)^{-\frac{j}{\beta}}\right]\Gamma\left(1+\frac{j}{\beta}\right) \tag{3.64}$$

and

$$\mu_r = E(X^r) = \frac{1}{\alpha}\left[1+\alpha-(1+\alpha)^{-\frac{r}{\beta}}\right]\Gamma\left(1+\frac{r}{\beta}\right)$$

Equation (3.63) can be re-written as

$$M_{WWD(x)}(t) = \frac{1}{B(a,b)}\sum_{i=0}^{\infty}(-1)^i\binom{b-1}{i}\sum_{j=0}^{\infty}\frac{t^j}{jk}\left[1+\alpha-(1+\alpha)^{-\frac{r}{\beta}}\right]\Gamma\left(1+\frac{r}{\beta}\right)\times$$
$$\int_0^\infty\left[\frac{\alpha+1}{\alpha}\left\{\left(1-e^{-x^\beta}\right)-\frac{1}{\alpha+1}\left(1-e^{-(1+\alpha)x^\beta}\right)\right\}\right]^{a(i+1)-1}$$

Then,

$$M_{WWD(x)}(t) = \frac{1}{B(a,b)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^i\binom{b-1}{i}\frac{t^j}{jk}\left[1+\alpha-(1+\alpha)^{-\frac{r}{\beta}}\right]\Gamma\left(1+\frac{r}{\beta}\right) \tag{3.65}$$
$$\int_0^\infty e^{tx}\left[\frac{\alpha+1}{\alpha}\left\{\left(1-e^{-x^\beta}\right)-\frac{1}{\alpha+1}\left(1-e^{-(1+\alpha)x^\beta}\right)\right\}\right]^{a(i+1)-1}$$

Then, the rth moment of the beta-weighted Weibull distribution (BWWD) can be written as

$$\mu'_{(BWWD(r))} = \int_0^\infty x^r g_{BWWD}(x)dx$$

i.e

$$\mu'_{BWWD(r)} = \int_0^\infty x^r\left[\frac{1}{B(a,b)}[K(x)]^{a-1}[1-K(x)]^{b-1}dK(x)\right]$$

where

$$K(x) = \frac{[\eta(1-c(x))-\eta^{-1}(1-c(x)^\eta)]}{\alpha}$$

That is $c(x) = e^{-x^{\beta}}$, $\eta = (\alpha + 1)$ and $(1 + \alpha)$

Then,

$$\mu'_{BWWD(r)} = \left[\frac{\left(\eta - \eta^{-\frac{r}{\beta}}\right) \Gamma\left(1 + \frac{r}{\beta}\right)}{\alpha B(a, b)} \right] \sum_{i=0}^{\infty} (-1)^i \binom{b-1}{i} \times$$
$$\left[\int_0^{\infty} \left\{ \frac{\eta(1 - c(x)) - \eta^{-1}(1 - c(x)^{\eta})}{\alpha} \right\}^{\alpha(i+1)-1} \right] dx$$

$$\mu'_{BWWD(r)} = W \left[\left(\eta - \eta^{-\frac{r}{\beta}}\right) \Gamma\left(1 + \frac{r}{\beta}\right) \right] \tag{3.66}$$

where

$$W = \frac{\sum_{i=0}^{\infty} (-1)^i \binom{b-1}{i} \left[\int_0^{\infty} \left\{ \frac{\eta(1-c(x)) - \eta^{-1}(1-c(x)^{\eta})}{\alpha} \right\}^{\alpha(i+1)-1} \right] dx}{\alpha B(a, b)}$$

Now, the first four non-central moments μ'_r, $r = 1, 2, 3, 4$ are obtained from equation (3.66) by putting $r = 1, 2, 3$ and 4 respectively in the equation, e.g. μ'_1 is given as

$$\mu'_{BWWD(1)} = E(X') = \left[\frac{\left(\eta - \eta^{-\frac{r}{\beta}}\right) \Gamma\left(1 + \frac{r}{\beta}\right)}{\alpha B(a, b)} \right] \sum_{i=0}^{\infty} (-1)^i \binom{b-1}{i} \tag{3.67}$$

Then, central moments μ_r, $r = 1, 2, 3, 4, \cdots$ are related to noncentral moments μ'_r as

$$\mu_r = \sum_{j=0}^{r} \binom{r}{j} \mu'_{r-j} \mu'_j \tag{3.68}$$

where $\mu'_1 = \mu$ and $\mu'_0 = 1$

Consequently, the mean and 2nd, 3rd, and 4th moments of the BWW distribution are given as

$$\mu = \mu'_1$$
$$\mu_2 = \mu'_2 - \mu^2$$
$$\mu_3 = \mu'_3 - 3\mu\mu'_2 + 2\mu^3 \text{ and}$$
$$\mu_4 = \mu'_4 - 4\mu\mu'_3 + 6\mu^2\mu'_2 - 3\mu^4$$

where

$$\mu'_1 = W \left[\left\{ \eta - \eta^{-\frac{1}{\beta}} \right\} \Gamma\left(1 + \frac{1}{\beta}\right) \right] \tag{3.69}$$

$$\mu'_2 = W \left[\left\{ \eta - \eta^{-\frac{2}{\beta}} \right\} \Gamma\left(1 + \frac{2}{\beta}\right) \right] \tag{3.70}$$

$$\mu'_3 = W \left[\left\{ \eta - \eta^{-\frac{3}{\beta}} \right\} \Gamma\left(1 + \frac{3}{\beta}\right) \right] \tag{3.71}$$

$$\mu'_4 = W \left[\left\{ \eta - \eta^{-\frac{4}{\beta}} \right\} \Gamma\left(1 + \frac{4}{\beta}\right) \right] \tag{3.72}$$

Moments measures of Skewness, φ_1, excess kurtosis, φ_2 and coefficient of variance (CV).

The skewness statistic is the standardized third-order moment applied to the residuals, and is

$$\varphi_1 = \frac{\mu_3}{\mu_2^{\frac{3}{2}}} \tag{3.73}$$

The skewness of the BWW distribution decreases as the values of the parameters (a, b) increases in figure 3.19 (i and ii) and figure 3.20 (i and ii) below:

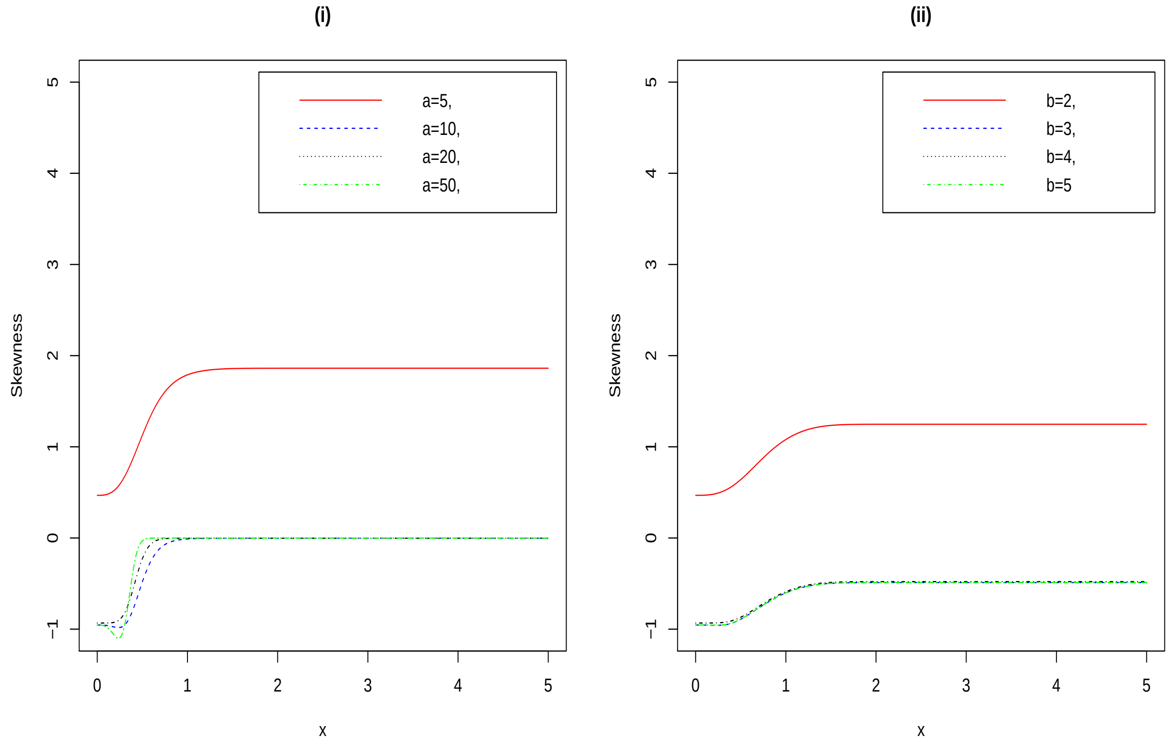

Figure 3.19: Skewness of the beta-weighted Weibull distribution BWWD (a, b, α, β) for a and $b : a = b = 5, 10, 20, 50$ and (ii) values of a and $b : a = b = 2, 3, 4, 5$.

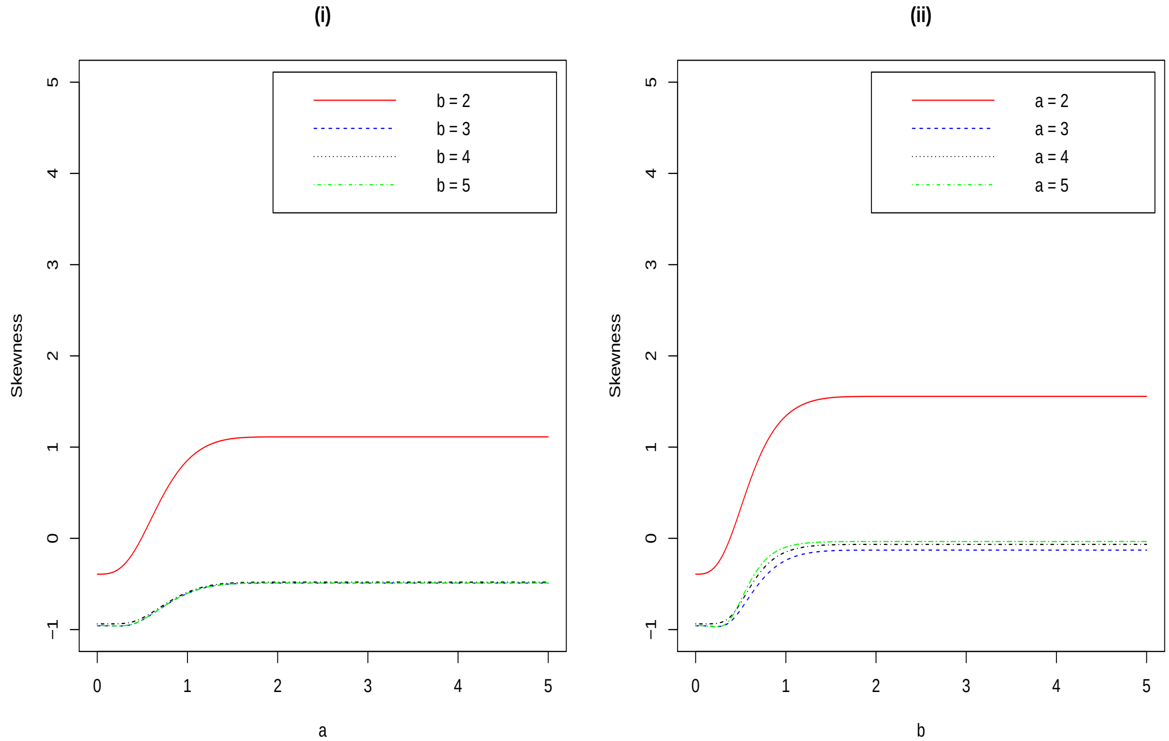

Figure 3.20: Skewness $\varphi_1 = \frac{\mu_3}{\mu_2^{\frac{3}{2}}}$ of the distribution BWWD (see equation (3.73)) (i) at $a = 1$ for values of $b : b = 2, 3, 4, 5$ and (ii) at $b = 1$ for some values of $a : a = 2, 3, 4, 5$

.

The kurtosis statistic is the standardized forth-order moment applied to the residuals, and its expression is given by

$$\varphi_2 = \frac{\mu_4}{\mu_2^2} - 3 \tag{3.74}$$

The level of the kurtosis of the BWW distribution increases as the values of the parameters (a, b) increases in figure 3.21 (i and ii) and figure 3.22 (i and ii).

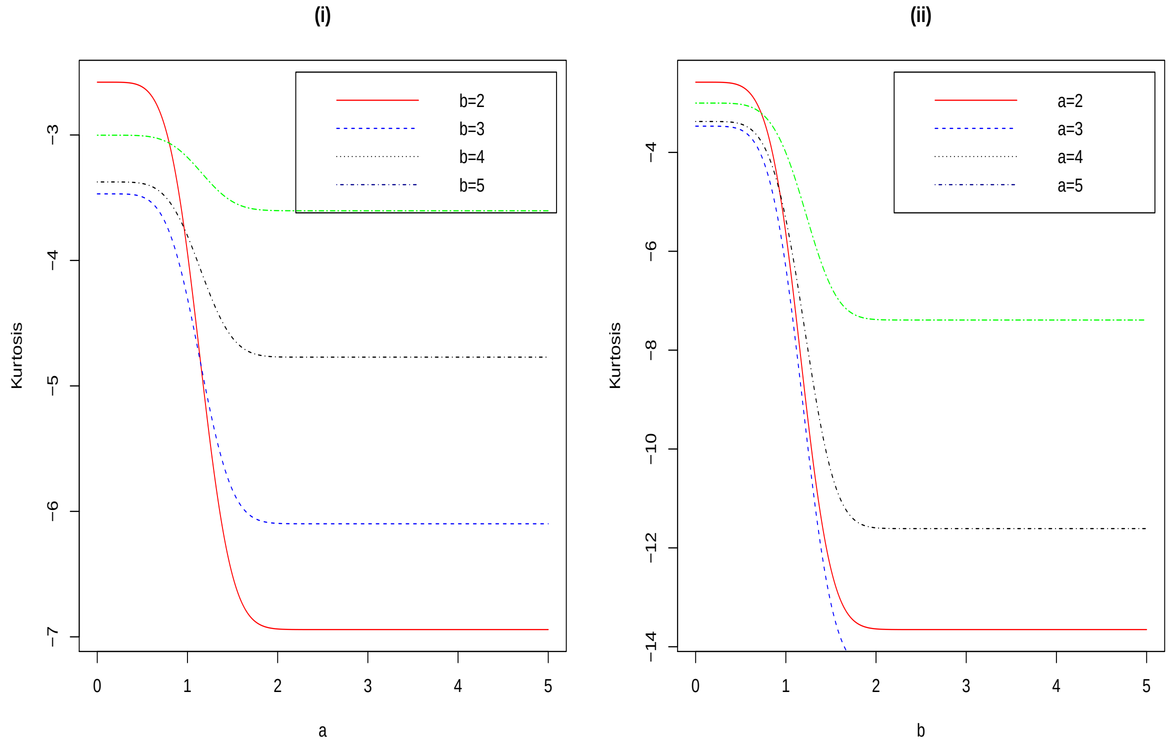

Figure 3.21: Kurtosis of the BWW distribution: (i) $(c = \alpha, d = \beta)$ fixed at $(2, 10)$ and (a, b) varied as $(a, b) = (1, 1),\ (0.5, 50),\ (0.8, 1),\ (0.6, 5)$, respectively and (ii) $(c = \alpha, d = \beta)$ fixed at $(2, 10)$ and $(a, b) = (1, 1),\ (0.5, 50),\ (0.6, 5),\ (0.8, 1)$ respectively

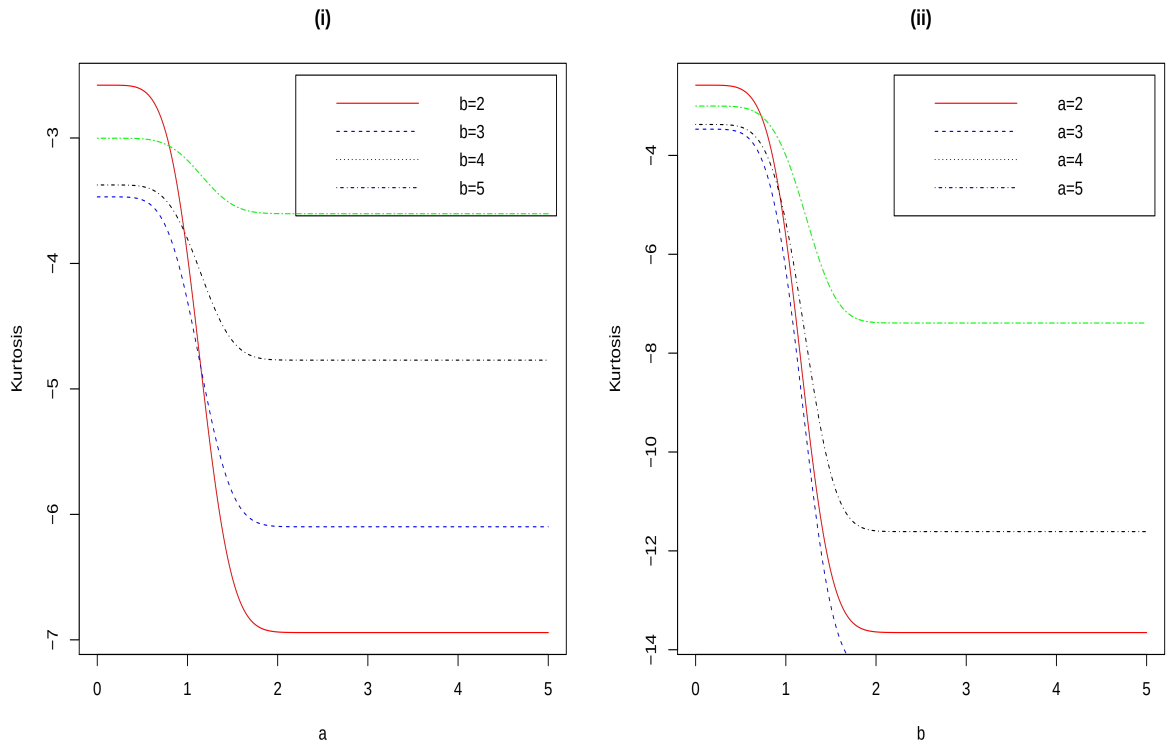

Figure 3.22: Kurtosis of the Beta-weighted Weibull distribution (BWWD) in (3.74) (i) function of $a(=1)$ for some values of $b : b = 2, 3, 4$ and 5 (ii) function of $b(=1)$ for some values of $a : a = 2, 3, 4$ and 5.

and

$$CV = \frac{\sqrt{\mu_2}}{\mu} \tag{3.75}$$

3.6.7 Estimation and Information Matrix

An attempt is made to derive the maximum likelihood estimates (MLEs) of the parameters of the Beta-Weighted Distribution following Cordeiro, *et al.* (2011); the log-likelihood function was given as

$$L(\theta) = n \log c - n \log[B(a,b)] + \sum_{i=1}^{n} \log[f(X_i, \tau)] + \\ (a-1)\sum_{i=1}^{n} \log F(X_i, \tau) + (b-1)\sum_{i=1}^{n} \log[1 - F^c(X_i, \tau)] \tag{3.76}$$

where $\theta = (a, b, c, \tau)$, and $\tau = (\alpha, \beta)$ are vectors Parameter $c = 1$ reduces (3.76) from the class of Generalised Beta distribution to the class of Beta generated distribution; then $\theta = (a, b, 1, \tau)$ and equation (3.76) reduces to

$$L(\theta) = Const - n \log[B(a,b)] + \sum_{i=1}^{n} \log[f(X_i, \tau)] + \\ (a-1)\sum_{i=1}^{n} \log F(X_i, \tau) + (b-1)\sum_{i=1}^{n} \log[1 - F^c(X_i, \tau)] \tag{3.77}$$

where $f(x, \tau)$ is as given in equation (3.26) and is as defined in equation (3.27). The log-likelihood function beta-weighted Weibull distribution is given as

$$L_{BWWD}(\theta) = -nlog[B(a,b)] + \sum_{i=1}^{n} \log\left[\frac{\alpha+1}{\alpha}\beta x^{\beta-1} e^{-x^{\beta}}\left(1 - e^{-\alpha x^{\beta}}\right)\right] + \\ (a-1)\left[\sum_{i=1}^{n} \log \frac{\alpha+1}{\alpha}\left(1 - e^{-x^{\beta}}\right) - \frac{1}{\alpha+1}\left(1 - e^{-(1+\alpha)x^{\beta}}\right)\right] + \\ (b-1)\sum_{i=1}^{n} \log\left[1 - \frac{\alpha+1}{\alpha}\left\{\left(1 - e^{-x^{\beta}}\right) - \frac{1}{\alpha+1}\left(1 - e^{-(1+\alpha)x^{\beta}}\right)\right\}\right] \tag{3.78}$$

By differentiating (3.78) with respect to parameters a, b, α and β gives

$$\frac{\partial L(\theta)}{\partial a} = -n\frac{\Gamma'(a)}{\Gamma(a)} + n\frac{\Gamma'(a+b)}{\Gamma(a+b)} + \sum_{x=1}^{n} \log \frac{\alpha+1}{\alpha}\left[\left(1 - e^{-x^{\beta}}\right) - \frac{1}{\alpha+1}\left(1 - e^{-(1+\alpha)x^{\beta}}\right)\right] \tag{3.79}$$

$$\frac{\partial L(\theta)}{\partial b} = -n\frac{\Gamma'(b)}{\Gamma(b)} + n\frac{\Gamma'(a+b)}{\Gamma(a+b)} + \sum_{i=1}^{n} \log\left[1 - \frac{1}{\alpha+1}\left(1 - e^{-x^{\beta}}\right) - \frac{\alpha+1}{\alpha}\left(1 - e^{-(1+\alpha)x^{\beta}}\right)\right] \tag{3.80}$$

$$\frac{\partial L(\theta)}{\partial \alpha} = \sum_{x=1}^{n}\left[\frac{\frac{\partial}{\partial\alpha}\left(\frac{\alpha+1}{\alpha}\beta x^{\beta-1}e^{-x^{\beta}}\right)}{\frac{\alpha+1}{\alpha}\beta x^{\beta-1}e^{-x^{\beta}}\left(1-e^{-\alpha x^{\beta}}\right)}\right]+ \tag{3.81}$$

$$(a-1)\sum_{x=1}^{n}\frac{\frac{\partial}{\partial\alpha}\left[\frac{\alpha+1}{\alpha}\left(1-e^{-x^{\beta}}\right)-\frac{1}{\alpha+1}\left(1-e^{-(1+\alpha)x^{\beta}}\right)\right]}{\frac{\alpha+1}{\alpha}\left[\left(1-e^{-x^{\beta}}\right)-\frac{1}{\alpha+1}\left(1-e^{-(1+\alpha)x^{\beta}}\right)\right]}+$$

$$(b-1)\sum_{x=1}^{n}\frac{\frac{\partial}{\partial\alpha}\left[1-\frac{\alpha+1}{\alpha}\left(1-e^{-x^{\beta}}\right)-\frac{1}{\alpha+1}\left(1-e^{-(1+\alpha)x^{\beta}}\right)\right]}{1-\frac{\alpha+1}{\alpha}\left[\left(1-e^{-x^{\beta}}\right)-\frac{1}{\alpha+1}\left(1-e^{-(1+\alpha)x^{\beta}}\right)\right]}$$

$$\frac{\partial L(\theta)}{\partial \beta} = \sum_{x=1}^{n}\left[\frac{\frac{\partial}{\partial\beta}\left(\frac{1}{\alpha+1}\beta x^{\beta-1}e^{-x^{\beta}}\right)}{\frac{\alpha+1}{\alpha}\beta x^{\beta-1}e^{-x^{\beta}}\left(1-e^{-\alpha x^{\beta}}\right)}\right]+ \tag{3.82}$$

$$(a-1)\sum_{x=1}^{n}\frac{\frac{\partial}{\partial\beta}\left[\frac{\alpha+1}{\alpha}\left(1-e^{-x^{\beta}}\right)-\frac{1}{\alpha+1}\left(1-e^{-(1+\alpha)x^{\beta}}\right)\right]}{\frac{\alpha+1}{\alpha}\left[\left(1-e^{-x^{\beta}}\right)-\frac{1}{\alpha+1}\left(1-e^{-(1+\alpha)x^{\beta}}\right)\right]}+$$

$$(b-1)\sum_{x=1}^{n}\frac{\frac{\partial}{\partial\alpha}\left[1-\frac{\alpha+1}{\alpha}\left(1-e^{-x^{\beta}}\right)-\frac{1}{\alpha+1}\left(1-e^{-(1+\alpha)x^{\beta}}\right)\right]}{1-\frac{\alpha+1}{\alpha}\left[\left(1-e^{-x^{\beta}}\right)-\frac{1}{\alpha+1}\left(1-e^{-(1+\alpha)x^{\beta}}\right)\right]}$$

The unknown parameters can be obtained by solving (3.79), (3.80), (3.81) and (3.82) using iterative techniques such as Newton Raphson type algorithm.

The Fisher Information Matrix is obtained by finding the second derivatives of the equations (3.79), (3.80), (3.81) and (3.82) respectively; the result is the diagonal elements of the Fishers information matrix obtained as follows

$$\psi'(a) = \frac{\Gamma'(a)}{\Gamma(a)},\ \psi'(a) = \psi(a)$$

$$\frac{\partial^2 L(\theta)}{\partial a^2} = n[\psi(a) = \psi(a+b)] \tag{3.83}$$

$$\frac{\partial^2 L(\theta)}{\partial b^2} = n[\psi(b) = \psi(a+b)] \tag{3.84}$$

$$\frac{\partial^2 L(\theta)}{\partial \alpha^2} = \sum_{x=1}^{n}\frac{P_1-Q_1}{\left(\frac{\alpha+1}{\alpha}\right)^2\beta x^{\beta-1}e^{-x^{\beta}}\left(1-e^{-\alpha x^{\beta}}\right)}+(a-1)\sum_{x=1}^{n}\frac{[R_1-S_1]}{\left[\frac{\alpha+1}{\alpha\left(1-e^{-x^{\beta}}\right)-\frac{1}{\alpha+1}\left(1-e^{-(1+\alpha)x^{\beta}}\right)}\right]^2}+ \tag{3.85}$$

$$(b-1)\sum_{x=1}^{n}\frac{[T_1-U_1]}{\left[1-\frac{\alpha+1}{\alpha}\left(1-e^{-x^{\beta}}\right)-\frac{1}{\alpha+1}\left(1-e^{-(1+\alpha)x^{\beta}}\right)\right]^2}$$

where

$$P_1 = \left[\left(\frac{\alpha+1}{\alpha}\beta x^{\beta-1}e^{-x^\beta}\left(1-e^{-\alpha x^\beta}\right)\right)\right]\frac{\partial^2}{\partial\alpha^2}\left(\frac{\alpha+1}{\alpha}\beta x^{\beta-1}e^{-x^\beta}\left(1-e^{-\alpha x^\beta}\right)\right)$$

$$Q_1 = \frac{\partial}{\partial\alpha}\left(\frac{\alpha+1}{\alpha}\beta x^{\beta-1}e^{-x^\beta}\left(1-e^{-\alpha x^\beta}\right)\right)\frac{\partial}{\partial\alpha}\left(\frac{\alpha+1}{\alpha}\beta x^{\beta-1}e^{-x^\beta}\left(1-e^{-\alpha x^\beta}\right)\right)$$

$$R_1 = \left[\frac{\alpha+1}{\alpha}\left(1-e^{-\alpha x^\beta}\right)-\frac{1}{\alpha+1}\left(1-e^{-(1+\alpha)x^\beta}\right)\frac{\partial^2}{\partial\alpha^2}\left\{\left(1-e^{-x^\beta}\right)-\frac{1}{\alpha+1}\left(1-e^{-(1+\alpha)x^\beta}\right)\right\}\right]$$

$$S_1 = \left[\frac{\partial}{\partial\alpha}\left(1-e^{-x^\beta}\right)-\frac{1}{\alpha+1}\left(1-e^{-(1+\alpha)x^\beta}\right)\frac{\partial}{\partial\alpha}\left\{\frac{\alpha+1}{\alpha}\left(1-e^{-x^\beta}\right)-\frac{1}{\alpha+1}\left(1-e^{-(1+\alpha)x^\beta}\right)\right\}\right]$$

$$T_1 = 1-\frac{\alpha+1}{\alpha}\left(1-e^{-\alpha x^\beta}\right)-\frac{1}{\alpha+1}\left(1-e^{-(1+\alpha)x^\beta}\right)\times \frac{\partial^2}{\partial\alpha^2}\left\{\left(\frac{\alpha+1}{\alpha}\right)\left(1-e^{-x^\beta}\right)-\frac{1}{\alpha+1}\left(1-e^{-(1+\alpha)x^\beta}\right)\right\}$$

$$U_1 = \frac{\partial}{\partial\alpha}\left\{1-\left(\frac{\alpha+1}{\alpha}\right)\left(1-e^{-x^\beta}\right)-\frac{1}{\alpha+1}\left(1-e^{-(1+\alpha)x^\beta}\right)\right\}\times \frac{\partial}{\partial\alpha}\left\{\frac{\alpha+1}{\alpha}\left(1-e^{-x^\beta}\right)-\frac{1}{\alpha+1}\left(1-e^{-(1+\alpha)x^\beta}\right)\right\}$$

$$\frac{\partial^2 L(\theta)}{\partial\beta^2} = \sum_{x=1}^{n}\frac{P_2-Q_2}{\left(\frac{\alpha+1}{\alpha}\right)^2\beta x^{\beta-1}e^{-x^\beta}\left(1-e^{-\alpha x^\beta}\right)}+ (a-1)\sum_{x=1}^{n}\frac{[R_2-S_2]}{\left[\frac{\alpha+1}{\alpha\left(1-e^{-x^\beta}\right)-\frac{1}{\alpha+1}\left(1-e^{-(1+\alpha)x^\beta}\right)}\right]^2}+ (b-1)\sum_{x=1}^{n}\frac{[T_2-U_2]}{\left[1-\frac{\alpha+1}{\alpha}\left(1-e^{-x^\beta}\right)-\frac{1}{\alpha+1}\left(1-e^{-(1+\alpha)x^\beta}\right)\right]^2} \tag{3.86}$$

where

$$P_2 = \left[\left(\frac{\alpha+1}{\alpha}\beta x^{\beta-1}e^{-x^\beta}\left(1-e^{-\alpha x^\beta}\right)\right)\right]\frac{\partial^2}{\partial\beta^2}\left(\frac{\alpha+1}{\alpha}\beta x^{\beta-1}e^{-x^\beta}\left(1-e^{-\alpha x^\beta}\right)\right)$$

$$Q_2 = \frac{\partial}{\partial\beta}\left(\frac{\alpha+1}{\alpha}\beta x^{\beta-1}e^{-x^\beta}\left(1-e^{-\alpha x^\beta}\right)\right)\frac{\partial}{\partial\beta}\left(\frac{\alpha+1}{\alpha}\beta x^{\beta-1}e^{-x^\beta}\left(1-e^{-\alpha x^\beta}\right)\right)$$

$$R_2 = \left[\frac{\alpha+1}{\alpha}\left(1-e^{-\alpha x^\beta}\right)-\frac{1}{\alpha+1}\left(1-e^{-(1+\alpha)x^\beta}\right)\frac{\partial^2}{\partial\beta^2}\left\{\left(1-e^{-x^\beta}\right)-\frac{1}{\alpha+1}\left(1-e^{-(1+\alpha)x^\beta}\right)\right\}\right]$$

$$S_2 = \left[\frac{\partial}{\partial\beta}\left(1-e^{-x^\beta}\right)-\frac{1}{\alpha+1}\left(1-e^{-(1+\alpha)x^\beta}\right)\frac{\partial}{\partial\beta}\left\{\frac{\alpha+1}{\alpha}\left(1-e^{-x^\beta}\right)-\frac{1}{\alpha+1}\left(1-e^{-(1+\alpha)x^\beta}\right)\right\}\right]$$

$$T_2 = 1 - \frac{\alpha+1}{\alpha}\left(1 - e^{-\alpha x^\beta}\right) - \frac{1}{\alpha+1}\left(1 - e^{-(1+\alpha)x^\beta}\right) \times$$
$$\frac{\partial^2}{\partial \beta^2}\left\{\left(\frac{\alpha+1}{\alpha}\right)\left(1 - e^{-x^\beta}\right) - \frac{1}{\alpha+1}\left(1 - e^{-(1+\alpha)x^\beta}\right)\right\}$$
$$U_2 = \frac{\partial}{\partial \beta}\left\{1 - \left(\frac{\alpha+1}{\alpha}\right)\left(1 - e^{-x^\beta}\right) - \frac{1}{\alpha+1}\left(1 - e^{-(1+\alpha)x^\beta}\right)\right\} \times$$
$$\frac{\partial}{\partial \beta}\left\{\frac{\alpha+1}{\alpha}\left(1 - e^{-x^\beta}\right) - \frac{1}{\alpha+1}\left(1 - e^{-(1+\alpha)x^\beta}\right)\right\}$$

$$\frac{\partial L(\theta)}{\partial a \partial b} = \psi(a+b) \tag{3.87}$$

$$\frac{\partial L(\theta)}{\partial a \partial \alpha} = \sum_{i=1}^{n} \frac{\frac{\partial}{\partial \alpha}\left[\frac{\alpha+1}{\alpha}\left(1 - e^{-x^\beta}\right) - \frac{1}{\alpha+1}\left(1 - e^{-(1+\alpha)x^\beta}\right)\right]}{\frac{\alpha+1}{\alpha}\left(1 - e^{-x^\beta}\right) - \frac{1}{\alpha+1}\left(1 - e^{-(1+\alpha)x^\beta}\right)}$$

$$\frac{\partial L(\theta)}{\partial a \partial \alpha} = \frac{\sum_{i=1}^{n}[A_1 - B_1]}{\left[1 - \frac{\alpha+1}{\alpha}\left(1 - e^{-x^\beta}\right) - \frac{1}{\alpha+1}\left(1 - e^{-(1+\alpha)x^\beta}\right)\right]^2} \tag{3.88}$$

where

$$A_1 = \left[\frac{\alpha+1}{\alpha}\left(1 - e^{-\alpha x^\beta}\right) - \frac{1}{\alpha+1}\left(1 - e^{-(1+\alpha)x^\beta}\right)\frac{\partial^2}{\partial \alpha^2}\left\{\left(1 - e^{-x^\beta}\right) - \frac{1}{\alpha+1}\left(1 - e^{-(1+\alpha)x^\beta}\right)\right\}\right]$$
$$B_1 = \left[\frac{\partial}{\partial \alpha}\left(1 - e^{-x^\beta}\right) - \frac{1}{\alpha+1}\left(1 - e^{-(1+\alpha)x^\beta}\right)\frac{\partial}{\partial \alpha}\left\{\frac{\alpha+1}{\alpha}\left(1 - e^{-x^\beta}\right) - \frac{1}{\alpha+1}\left(1 - e^{-(1+\alpha)x^\beta}\right)\right\}\right]$$

$$\frac{\partial L(\theta)}{\partial a \partial \beta} = \sum_{x=1}^{n} \frac{\frac{\partial}{\partial \beta}\left[\frac{\alpha+1}{\alpha}\left(1 - e^{-x^\beta}\right) - \frac{1}{\alpha+1}\left(1 - e^{-(1+\alpha)x^\beta}\right)\right]}{\frac{\alpha+1}{\alpha}\left(1 - e^{-x^\beta}\right) - \frac{1}{\alpha+1}\left(1 - e^{-(1+\alpha)x^\beta}\right)}$$

$$\frac{\partial L(\theta)}{\partial a \partial \beta} = \frac{\sum_{x=1}^{n}[A_2 - B_2]}{\left[1 - \frac{\alpha+1}{\alpha}\left(1 - e^{-x^\beta}\right) - \frac{1}{\alpha+1}\left(1 - e^{-(1+\alpha)x^\beta}\right)\right]^2} \tag{3.89}$$

where

$$A_2 = \left[\frac{\alpha+1}{\alpha}\left(1 - e^{-\alpha x^\beta}\right) - \frac{1}{\alpha+1}\left(1 - e^{-(1+\alpha)x^\beta}\right)\frac{\partial^2}{\partial \beta^2}\left\{\left(1 - e^{-x^\beta}\right) - \frac{1}{\alpha+1}\left(1 - e^{-(1+\alpha)x^\beta}\right)\right\}\right]$$
$$B_2 = \left[\frac{\partial}{\partial \beta}\left(1 - e^{-x^\beta}\right) - \frac{1}{\alpha+1}\left(1 - e^{-(1+\alpha)x^\beta}\right)\frac{\partial}{\partial \beta}\left\{\frac{\alpha+1}{\alpha}\left(1 - e^{-x^\beta}\right) - \frac{1}{\alpha+1}\left(1 - e^{-(1+\alpha)x^\beta}\right)\right\}\right]$$

$$\frac{\partial L(\theta)}{\partial b \partial \alpha} = \sum_{i=1}^{n} \frac{\frac{\partial}{\partial \beta}\left[\frac{\alpha+1}{\alpha}\left(1 - e^{-x^\beta}\right) - \frac{1}{\alpha+1}\left(1 - e^{-(1+\alpha)x^\beta}\right)\right]}{\frac{\alpha+1}{\alpha}\left(1 - e^{-x^\beta}\right) - \frac{1}{\alpha+1}\left(1 - e^{-(1+\alpha)x^\beta}\right)}$$

$$\frac{\partial L(\theta)}{\partial b \partial \beta} = \frac{\sum_{x=1}^{n} [C_1 - D_1]}{\left[1 - \frac{\alpha+1}{\alpha}\left(1 - e^{-x^\beta}\right) - \frac{1}{\alpha+1}\left(1 - e^{-(1+\alpha)x^\beta}\right)\right]^2} \tag{3.90}$$

where

$$C_1 = 1 - \frac{\alpha+1}{\alpha}\left(1 - e^{-\alpha x^\beta}\right) - \frac{1}{\alpha+1}\left(1 - e^{-(1+\alpha)x^\beta}\right) \times$$
$$\frac{\partial^2}{\partial \alpha^2}\left\{\left(\frac{\alpha+1}{\alpha}\right)\left(1 - e^{-x^\beta}\right) - \frac{1}{\alpha+1}\left(1 - e^{-(1+\alpha)x^\beta}\right)\right\}$$
$$D_1 = \frac{\partial}{\partial \alpha}\left\{1 - \left(1 - e^{-x^\beta}\right) - \frac{1}{\alpha+1}\left(1 - e^{-(1+\alpha)x^\beta}\right)\right\} \times$$
$$\frac{\partial}{\partial \alpha}\left\{\frac{\alpha+1}{\alpha}\left(1 - e^{-x^\beta}\right) - \frac{1}{\alpha+1}\left(1 - e^{-(1+\alpha)x^\beta}\right)\right\}$$

$$\frac{\partial L(\theta)}{\partial b \partial \beta} = \sum_{x=1}^{n} \frac{\frac{\partial}{\partial \beta}\left[\frac{\alpha+1}{\alpha}\left(1 - e^{-x^\beta}\right) - \frac{1}{\alpha+1}\left(1 - e^{-(1+\alpha)x^\beta}\right)\right]}{\frac{\alpha+1}{\alpha}\left(1 - e^{-x^\beta}\right) - \frac{1}{\alpha+1}\left(1 - e^{-(1+\alpha)x^\beta}\right)}$$

$$\frac{\partial L(\theta)}{\partial b \partial \beta} = \frac{\sum_{x=1}^{n} [C_2 - D_2]}{\left[1 - \frac{\alpha+1}{\alpha}\left(1 - e^{-x^\beta}\right) - \frac{1}{\alpha+1}\left(1 - e^{-(1+\alpha)x^\beta}\right)\right]^2} \tag{3.91}$$

where

$$C_2 = 1 - \frac{\alpha+1}{\alpha}\left(1 - e^{-\alpha x^\beta}\right) - \frac{1}{\alpha+1}\left(1 - e^{-(1+\alpha)x^\beta}\right) \times$$
$$\frac{\partial^2}{\partial \beta^2}\left\{\left(\frac{\alpha+1}{\alpha}\right)\left(1 - e^{-x^\beta}\right) - \frac{1}{\alpha+1}\left(1 - e^{-(1+\alpha)x^\beta}\right)\right\}$$
$$D_2 = \frac{\partial}{\partial \beta}\left\{1 - \left(1 - e^{-x^\beta}\right) - \frac{1}{\alpha+1}\left(1 - e^{-(1+\alpha)x^\beta}\right)\right\} \times$$
$$\frac{\partial}{\partial \beta}\left\{\frac{\alpha+1}{\alpha}\left(1 - e^{-x^\beta}\right) - \frac{1}{\alpha+1}\left(1 - e^{-(1+\alpha)x^\beta}\right)\right\}$$

$$num1 = \sum_{x=1}^{n}\left[\frac{\frac{\partial}{\partial \alpha}\left\{\frac{\alpha+1}{\alpha}\beta x^{\beta-1} e^{-x^\beta}\left(1 - e^{-\alpha x^\beta}\right)\right\}}{\frac{\alpha+1}{\alpha}\beta x^{\beta-1} e^{-x^\beta}\left(1 - e^{-\alpha x^\beta}\right)}\right] +$$
$$(a-1)\sum_{x=1}^{n} \frac{\frac{\partial}{\partial \alpha}\left\{\frac{\alpha+1}{\alpha}\left(1 - e^{-\alpha x^\beta}\right) - \frac{\alpha+1}{\alpha}\left(1 - e^{-(1+\alpha)x^\beta}\right)\right\}}{\frac{\alpha+1}{\alpha}\left(1 - e^{-\alpha x^\beta}\right) - \frac{\alpha+1}{\alpha}\left(1 - e^{-(1+\alpha)x^\beta}\right)} +$$
$$(b-1)\sum_{i=1}^{n} \frac{\partial}{\partial \alpha}\left[1 - \frac{\alpha+1}{\alpha}\left(1 - e^{-\alpha x^\beta}\right) - \frac{1}{\alpha+1}\left(1 - e^{-(1+\alpha)x^\beta}\right)\right]$$

$$\frac{\partial L(\theta)}{\partial \alpha \partial \beta} = \frac{num1}{1 - \frac{\alpha+1}{\alpha}\left[\left(1 - e^{-\alpha x^\beta}\right) - \frac{1}{\alpha+1}\left(1 - e^{-(1+\alpha)x^\beta}\right)\right]} \tag{3.92}$$

$$\frac{\partial L(\theta)}{\partial\alpha\partial\beta} = \frac{\frac{\partial}{\partial\alpha}g_1(x) \times \frac{\partial}{\partial\alpha}g_1(x)}{[g_1]^2} + (a-1)\sum_{x=1}^{n}\frac{\left[g_2(x) \times \frac{\partial^2}{\partial\alpha^2}g_2(x) - \frac{\partial}{\partial\alpha}g_2 \times \frac{\partial}{\partial\alpha}g_2(x)\right]}{[g_2(x)]^2} + \tag{3.93}$$

$$(b-1)\sum_{x=1}^{n}\frac{\left[(1-g_2(x)) \times \frac{\partial^2}{\partial\alpha^2}(1-g_2(x)) - \frac{\partial}{\partial\alpha}(1-g_2(x)) \times \frac{\partial}{\partial\alpha}(1-g_2(x))\right]}{[g_2(x)]^2}$$

where

$$g_1 = \frac{\alpha+1}{\alpha}\beta x^{\beta-1}e^{-x^\beta}\left(1-e^{-\alpha x^\beta}\right)$$
$$g_2 = \frac{\alpha+1}{\alpha}\left(1-e^{-x^\beta}\right) - \frac{1}{\alpha+1}\left(1-e^{-(1+\alpha)x^\beta}\right)$$

The information matrix is obtained as

$$I(a,b,\alpha,\beta) = \begin{bmatrix} E\left(\frac{\partial^2 L(\theta)}{\partial a^2}\right) & E\left(\frac{\partial^2 L(\theta)}{\partial a\partial b}\right) & E\left(\frac{\partial^2 L(\theta)}{\partial a\partial\alpha}\right) & E\left(\frac{\partial^2 L(\theta)}{\partial a\partial\beta}\right) \\ E\left(\frac{\partial^2 L(\theta)}{\partial a\partial b}\right) & E\left(\frac{\partial^2 L(\theta)}{\partial b^2}\right) & E\left(\frac{\partial^2 L(\theta)}{\partial b\partial\alpha}\right) & E\left(\frac{\partial^2 L(\theta)}{\partial b\partial\beta}\right) \\ E\left(\frac{\partial^2 L(\theta)}{\partial a\partial\alpha}\right) & E\left(\frac{\partial^2 L(\theta)}{\partial b\partial\alpha}\right) & E\left(\frac{\partial^2 L(\theta)}{\partial\alpha^2}\right) & E\left(\frac{\partial^2 L(\theta)}{\partial\alpha\partial\beta}\right) \\ E\left(\frac{\partial^2 L(\theta)}{\partial a\partial\beta}\right) & E\left(\frac{\partial^2 L(\theta)}{\partial b\partial\beta}\right) & E\left(\frac{\partial^2 L(\theta)}{\partial\alpha\partial\beta}\right) & E\left(\frac{\partial^2 L(\theta)}{\partial\beta^2}\right) \end{bmatrix} \tag{3.94}$$

The interval estimations of (a, b, α, β) and the Fisher information matrix $I(.)$ is required. The elements of this matrix are the expected values of the second partial derivatives of the positive log-likelihoods shown above in (3.93).

3.6.8 Essence of the Beta Weighted Weibull (BWW) Distribution

The main essence of the Beta Weighted Weibull Distribution (BWWD) is contained in the following

- It has wider scope of applicability
- The distribution of the error term follows the distribution of the response variable
- The model captures most information about the data through its parameters
- Data are approximately normally distributed.

3.6.9 Limitation

The main problem with the BWWD is any substantial deviation of the data from the underlying model assumptions invalidates the conclusions arrived at.

3.6.10 Entropy

In this section, we present the Renyi Entropy; which is a measure of variation of the uncertainty. Renyi Entropy has been used for characterization of several probability distributions is defined by

$$I_R(S) = \frac{1}{1-S}\log[l(S)]$$

where $l(S) = \int_0^\infty f^s(x)dx, \quad S > 0 \; \& S \neq 1.$

Here we followed beta generated distribution defined in (3.1). Using binomial expansion in (3.40), we then write

$$\begin{aligned} g_{BWWD}(x;\alpha,\beta,a,b)^s = & \frac{1}{B(a,b)}\frac{\alpha+1}{\alpha}\beta x^{\beta-1}e^{-x^\beta}\left(1-e^{-\alpha x^\beta}\right)\times \\ & \left[\frac{\alpha+1}{\alpha}\left\{\left(1-e^{-x^\beta}\right)-\frac{1}{\alpha+1}\left(1-e^{-(1+\alpha)x^\beta}\right)\right\}^{a-1}\right]^s \times \\ & \left[1-\frac{\alpha+1}{\alpha}\left\{\left(1-e^{-x^\beta}\right)-\frac{1}{\alpha+1}\left(1-e^{-(1+\alpha)x^\beta}\right)\right\}\right]^{(b-1)s} \end{aligned} \tag{3.95}$$

$$\begin{aligned} = & \frac{1}{B(a,b)}\frac{\alpha+1}{\alpha}\beta x^{\beta-1}e^{-x^\beta}\left(1-e^{-\alpha x^\beta}\right)\times \\ & \left[\frac{\alpha+1}{\alpha}\left\{\left(1-e^{-x^\beta}\right)-\frac{1}{\alpha+1}\left(1-e^{-(1+\alpha)x^\beta}\right)\right\}^{a-1}\right]^s \times \\ & \sum_{i=0}^{(b-1)s}\binom{(b-1)s}{i}(-1)^i\left[1-\frac{\alpha+1}{\alpha}\left\{\left(1-e^{-x^\beta}\right)-\frac{1}{\alpha+1}\left(1-e^{-(1+\alpha)x^\beta}\right)\right\}\right]^i \end{aligned} \tag{3.96}$$

Now, the Renyi entropy of X can be obtained as

$$\begin{aligned} l_R(S) = & \frac{1}{1-S}\log\sum_{i=0}^{(b-1)s}R_i\int_0^\infty\left[\frac{\alpha+1}{\alpha}\beta x^{\beta-1}e^{-x^\beta}\left(1-e^{-\alpha x^\beta}\right)\right]\times \\ & \left[\frac{\alpha+1}{\alpha}\left\{\left(1-e^{-x^\beta}\right)-\frac{1}{\alpha+1}\left(1-e^{-(1+\alpha)x^\beta}\right)\right\}^{a-1}\right]^s \times \\ & \left[1-\frac{\alpha+1}{\alpha}\left\{\left(1-e^{-x^\beta}\right)-\frac{1}{\alpha+1}\left(1-e^{-(1+\alpha)x^\beta}\right)\right\}\right]^i dx \end{aligned} \tag{3.97}$$

$$l_R(S) = \frac{1}{1-S}\log\left[\sum_{i=0}^{(b-1)s}R_i\int_0^\infty\left[f_X(x;\alpha,\beta)\right]^s\left[S_X(x;\alpha,\beta)\right]^i dx\right] \tag{3.98}$$

Equation (3.79) becomes the Renyi entropy of the BWW distribution.

where $R_i = \frac{1}{B(a,b)^s}\binom{(b-1)s}{i}(-1)^i$, $f_X(x;\alpha,\beta)$ the pdf of the WW distribution in (3.26) and $S_X(x;\alpha,\beta)$ is the survival function of the WW distribution in (3.28).

3.6.11 Order Statistics

Let $X_1, X_2, \cdots, X_n$ be n independent and identically distributed (i.i.d) continuous random variables with a common density f and distribution function F. Define

$$\begin{aligned} X_{(1)} &= \min(X_1, X_2, \cdots, X_n) \\ X_{(2)} &= \min\{(X_1, X_2, \cdots, X_n)\} - \{X_{(1)}\} \\ \vdots & \qquad\qquad \vdots \\ X_{(m)} &= \max(X_1, X_2, \cdots, X_n) \end{aligned}$$

The ordered values $X_{(1)} \leq X_{(2)} \leq, \cdots, X_{(n)}$ are known as the order statistics. The largest order statistic of the developed beta-Weighted Weibull distribution is given by

$$\begin{aligned} G_{\max}(x) = G_{(n)}(x) &= P(X_1 \leq x, X_2 < x, \cdots, X_n \leq x) \\ &= G_1(x).G_2(x) \cdots G_n(x) \\ &= G^n(x) \end{aligned}$$

$$g_{BWWD(\max)}(x) = \frac{d}{dx}(G_{BWWD}(x))^n = n(G_{BWWD}(x))^{n-1} g_{BWWD}(x) \tag{3.99}$$

where $G_{BWWD}(x) = \frac{1}{B(a,b)} \int_0^x K^{a-1}(1-K)^{b-1} dK = \frac{B(K;a,b)}{B(a,b)}$ and K is the expression in equation (3.45) and $g_{BWWD}(x) = \frac{1}{B(a,b)} K^{a-1}(1-K)^{b-1} \frac{dK}{dx}$; Putting the expression in (3.99) above, we get

$$g_{BWWD(\max)}(x) = n \left[\frac{B(K;a,b)}{B(a,b)}\right]^{n-1} \frac{1}{B(a,b)} K^{a-1}(1-K)^{b-1} \frac{dK}{dx}$$

$$g_{BWWD(\max)}(x) = \frac{1}{B(a,b)} \int_0^x n \left[K^{a-1}(1-K)^{b-1}\right]^{n-1} K^{a-1}(1-K)^{b-1} \frac{dK}{dx} \tag{3.100}$$

where

$$K = \left[\frac{\alpha+1}{\alpha}\left\{\left(1-e^{-x^\beta}\right) - \frac{1}{\alpha+1}\left(1-e^{-(1+\alpha)x^\beta}\right)\right\}\right] \quad \text{(expression 3.45)}$$

$$g_{BWWD(max)}(x) = \frac{1}{B(a,b)} \int_0^x n \left[\frac{\alpha+1}{\alpha}\left\{\left(1-e^{-x^\beta}\right) - \frac{1}{\alpha+1}\left(1-e^{-(1+\alpha)x^\beta}\right)\right\}\right]^{a-1} \times$$
$$\left[1 - \frac{\alpha+1}{\alpha}\left\{\left(1-e^{-x^\beta}\right) - \frac{1}{\alpha+1}\left(1-e^{-(1+\alpha)x^\beta}\right)\right\}^{b-1}\right]^{n-1}$$

$$\frac{1}{B(a,b)} \int_0^x n \left[\frac{\alpha+1}{\alpha}\left\{\left(1-e^{-x^\beta}\right) - \frac{1}{\alpha+1}\left(1-e^{-(1+\alpha)x^\beta}\right)\right\}\right]^{a-1} \times \tag{3.101}$$
$$\left[1 - \frac{\alpha+1}{\alpha}\left\{\left(1-e^{-x^\beta}\right) - \frac{1}{\alpha+1}\left(1-e^{-(1+\alpha)x^\beta}\right)\right\}\right]^{b-1}$$
$$\frac{\alpha+1}{\alpha} \beta x^{\beta-1} e^{-x^\beta} e^{-\alpha x^\beta}$$

Also, the smallest order statistic of the BWW distribution is given as

$$\begin{aligned} G_{\min}(x) &= G_1(x) = 1 - P(X_{\min} > x) \\ &= 1 - P(X_1 > x, X_2 > x, \cdots, X_n > x) \\ &= 1 - (1 - G_1(x))(1 - G_2(x)) \cdots (1 - G_n(x)) \\ &= 1 - (1 - G(x))^n \end{aligned}$$

$$g_{BWWD(\min)}(x) = -\frac{d}{dx}(1 - G_{BWWD}(x))^n = n(1 - G_{BWWD}(x))^{n-1} g_{BWWD}(x) \tag{3.102}$$

Therefore,

$$g_{BWWD(\min)}(x) = n\left[1 - \frac{B(K;a,b)}{B(a,b)}\right]^{n-1} \frac{K^{a-1}(1-K)^{b-1}}{B(a,b)} \frac{dK}{dx} \tag{3.103}$$

$$= \frac{1}{B(a,b)}\left[n(1 - B(K;a,b))^{n-1} K^{a-1}(1-K)^{b-1} \frac{dK}{dx}\right] \tag{3.104}$$

$$g_{BWWD(\min)}(x) = \frac{\left[n\left\{1 - \frac{\alpha+1}{\alpha}\left(1 - e^{-x^\beta}\right) - \frac{1}{\alpha+1}\left(1 - e^{-(1+\alpha)x^\beta}\right)\right\}\right]^{a-1}}{B(a,b)} \times \tag{3.105}$$

$$\left[1 - \frac{\alpha+1}{\alpha}\left\{\left(1 - e^{-x^\beta}\right) - \frac{1}{\alpha+1}\left(1 - e^{-(1+\alpha)x^\beta}\right)\right\}^{b-1}\right]^{a-1} \times$$

$$\left[\frac{\alpha+1}{\alpha}\left\{\left(1 - e^{-x^\beta}\right) - \frac{1}{\alpha+1}\left(1 - e^{-(1+\alpha)x^\beta}\right)\right\}\right]^{a-1} \times$$

$$\left[1 - \frac{\alpha+1}{\alpha}\left\{\left(e^{-x^\beta}\right) - \frac{1}{\alpha+1}\left(1 - e^{-(1+\alpha)x^\beta}\right)\right\}\right]^{b-1} \times$$

$$\frac{\alpha+1}{\alpha}\beta x^{\beta-1} e^{x^\beta}\left(1 - e^{-\alpha x^\beta}\right)$$

3.7 The Alternative Generalised Weighted Weibull (AG-WW) Distribution

This work is extended to Alternative Generalised Weighted Weibull (AGWW) distribution defined by logarithm of the beta weighted Weibull (BWW) random variable to have a better fitting of survival data set than CRM models. The BWW density function with four parameters a, b, α, β all greater than 0 is given as follows for $t > 0$

$$g(t) = \frac{1}{B(a,b)}\left[\frac{\alpha+1}{\alpha}\beta t^{\beta-1} e^{-t^\beta}\left(1 - e^{-\alpha t^\beta}\right)\right] \tag{3.106}$$

$$\left[\frac{\alpha+1}{\alpha}\left\{\left(1 - e^{-\alpha t^\beta}\right) - \frac{1}{\alpha+1}\left(1 - e^{-(1+\alpha)t^\beta}\right)\right\}\right]^{a-1}$$

$$\left[1 - \frac{\alpha+1}{\alpha}\left\{\left(1 - e^{-\alpha t^\beta}\right) - \frac{1}{\alpha+1}\left(1 - e^{-(1+\alpha)t^\beta}\right)\right\}\right]^{b-1}$$

where $B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ is the beta function. Then, a and b are initial shape parameters to the Weighted Weibull (WW) distribution for non-normal data. Expression (3.26) can be re-written, for the sake of simplicity, in another simplified Weibull form as

$$f(x) = \frac{\gamma+1}{\gamma}\left(\frac{\alpha}{\beta}\right)^{\alpha} x^{\alpha-1} e^{-\left(\frac{x}{\beta}\right)^{\alpha}} \left(1 - e^{-\gamma\left(\frac{x}{\beta}\right)^{\alpha}}\right) \tag{3.107}$$

Then, we re-parameterized (3.109) to have the Log-Weighted Weibull (LWW) distribution by letting $y = \log(x)$ i.e. $x = e^{y}$, $\alpha = \frac{1}{\sigma}$ and $\mu = \log(\beta)$ i.e. $\beta = e^{\mu}$. Substituting appropriately into (3.109), we obtain

$$\begin{aligned} f(y) &= \frac{\gamma+1}{\gamma}\left(\frac{\frac{1}{\sigma}}{e^{\mu}}\right)^{\frac{1}{\sigma}} (e^{y})^{\frac{1}{\sigma}-1} e^{-\left(\frac{e^{y}}{e^{\mu}}\right)^{\frac{1}{\sigma}}} \left(1 - e^{-\gamma\left(\frac{e^{y}}{e^{\mu}}\right)^{\frac{1}{\sigma}}}\right) \\ &= \frac{\gamma+1}{\gamma}\frac{1}{\sigma}(e^{\mu})^{\frac{1}{\sigma}}\frac{e^{y}}{\sigma} - y + ye^{-\left(\frac{e^{y}}{\sigma} - \frac{e^{\mu}}{\sigma}\right)} \left(1 - e^{-\gamma\left(\frac{e^{y}}{\sigma} - \frac{e^{\mu}}{\sigma}\right)}\right) \end{aligned}$$

$$f(y) = \frac{\gamma+1}{\gamma} e^{\frac{y-\mu}{\sigma}} e^{-e^{\frac{y-\mu}{\sigma}}} \left(1 - e^{-\gamma\left(\frac{e^{y}}{\sigma} - \frac{e^{\mu}}{\sigma}\right)}\right) \tag{3.108}$$

Equation (3.108) is the pdf of the Log-Weighted Weibull (LWW) distribution; and we can equally write the Beta Weighted Weibull (BWW) distribution by convoluting the beta function in equation (3.108) to have the following

$$\begin{aligned} g(t) = &\left[\frac{\gamma+1}{\gamma}\left(\frac{\alpha}{\beta}\right)^{\alpha} t^{\alpha-2} e^{-\left(\frac{t}{\beta}\right)^{\alpha}} \left(1 - e^{-\gamma\left(\frac{t}{\beta}\right)^{\alpha}}\right)\right] \\ &\left[\frac{\gamma+1}{\gamma}\left\{\left(1 - e^{-\gamma\left(\frac{t}{\beta}\right)^{\alpha}}\right) - \frac{1}{\gamma+1}\left(1 - e^{-(1+\gamma)\left(\frac{t}{\beta}\right)^{\alpha}}\right)\right\}\right]^{a-1} \\ &\left[1 - \frac{\gamma+1}{\gamma}\left\{\left(1 - e^{-\gamma\left(\frac{t}{\beta}\right)^{\alpha}}\right) - \frac{1}{\gamma+1}\left(1 - e^{-(1+\gamma)\left(\frac{t}{\beta}\right)^{\alpha}}\right)\right\}\right]^{a-1} \end{aligned} \tag{3.109}$$

$t \sim BWW(a, b, \alpha, \beta, \gamma)$ Distribution; where γ is the weight parameter, α is the shape parameter, β is the scale parameter and a and b are initial shape parameters added to the existing WW distribution; equation (3.109) becomes a five-parameter BWW distribution.

The BWW distribution contains sub-models such as the Weighted Weibull (WW) where $a = b = \beta = 1$, $\gamma = \alpha$ and $\alpha = \beta$ (Shahbaz *et al.* 2010), Weighted Weibull (WW) where $a = b = 1$, $\alpha = \beta$, $\gamma = \alpha^{\beta}$ (Ramadan, 2013), Weighted Exponential (WE) where $a = b = \alpha = \beta = 1$(Gupta and Kundu, 2009), the Weighted Extreme Value (WEV) $a = b = \alpha = \beta = 1$ and $\gamma = \alpha^{\beta}$ (Ramadan 2013) among others. Moreover, the corresponding cumulative distribution function $g(t)$, the survival rate function $S(t)$ and hazard rate function $h(t)$ are respectively given as follows

$$g(t) = \frac{1}{B(a,b)} \int_{0}^{F(t)} K^{a-1}(1-K)^{b-1} dK = I_{F(t)}(a,b)$$

where K and $F(t)$ is the baseline distribution and cumulative distribution function respectively

$$S(t) = 1 - \frac{1}{B(a,b)} \int_0^{F(t)} K^{a-1}(1-K)^{b-1} dK = 1 - I_{F(t)}(a,b)$$

and

$$h(t) = \frac{K^{a-1}(1-K)^{b-1}K'}{B(a,b)[1 - I_{F(t)}(a,b)]}$$

respectively; and where K' is the pdf of the baseline distribution (3.26).
$I_x(a,b) = \frac{1}{B(a,b)} \int_0^x K^{a-1}(1-K)^{b-1} dK$ is the incomplete beta function ratio.

Let X be a random variable having the beta weighted Weibull density function in (3.108). We investigate some of the properties of the Alternative Generalised Weighted Weibull (AGWW) distribution also defined by the random variable $Y = \log(X)$ ((Ortega *et al.* (2009), (2013), Cancho *et al.* (2009), Pescim *et al.* (2013)). Again, the density function of Y is transformed in terms of$\alpha = \frac{1}{\sigma}$ and $\mu = \log(\beta)$. The density function of Y is expressed as

$$\begin{aligned} g_{AGWW}(y; a, b, \gamma, \mu, \sigma) = & \frac{\gamma+1}{\sigma\gamma B(a,b)} e^{\frac{y-\mu}{\sigma}} e^{-e^{\frac{y-\mu}{\sigma}}} \left(e^{-\gamma e^{\frac{y-\mu}{\sigma}}} \right) \\ & \left[\frac{\gamma+1}{\gamma} \left\{ \left(1 - e^{-e^{\frac{y-\mu}{\sigma}}} \right) - \frac{1}{\gamma+1} \left(1 - e^{-e^{-(1+\gamma)\left(\frac{y-\mu}{\sigma}\right)}} \right) \right\} \right]^{a-1} \\ & \left[1 - \frac{\gamma+1}{\gamma} \left\{ \left(1 - e^{-e^{\frac{y-\mu}{\sigma}}} \right) - \frac{1}{\gamma+1} \left(1 - e^{-e^{-(1+\gamma)\left(\frac{y-\mu}{\sigma}\right)}} \right) \right\} \right]^{b-1} \end{aligned} \tag{3.110}$$

where $-\infty < y < \infty,\ \sigma > 0$ and $-\infty < \mu < \infty$.
Expression (3.110) is the new Alternative Generalised Weighted Weibull (AGWW) distribution; we refer to y as an AGWW $(\mu, \sigma, \gamma, a, b)$ random variate where, μ is the location parameter, σ is a dispersion parameter, γ is the weighted parameter and a and b are shape parameters. Therefore, we hold that if $X \sim BWW(\alpha, \beta, \gamma, a, b)$.

In figure 3.23 below, as the value of the parameter a increases and b is constant, the skewness of the AGWW distribution increases to the left (negative skewed) and the kurtosis decreases. Parameter ($a = 200$) and b at different values, the skewness of the AGWW is positive skewed and the kurtosis reduces in figure 3.24. Also, at different values of a, b is constant at 10, and μ is constant therefore, the graph in 3.25 is positive skewed and kurtosis also reduces.

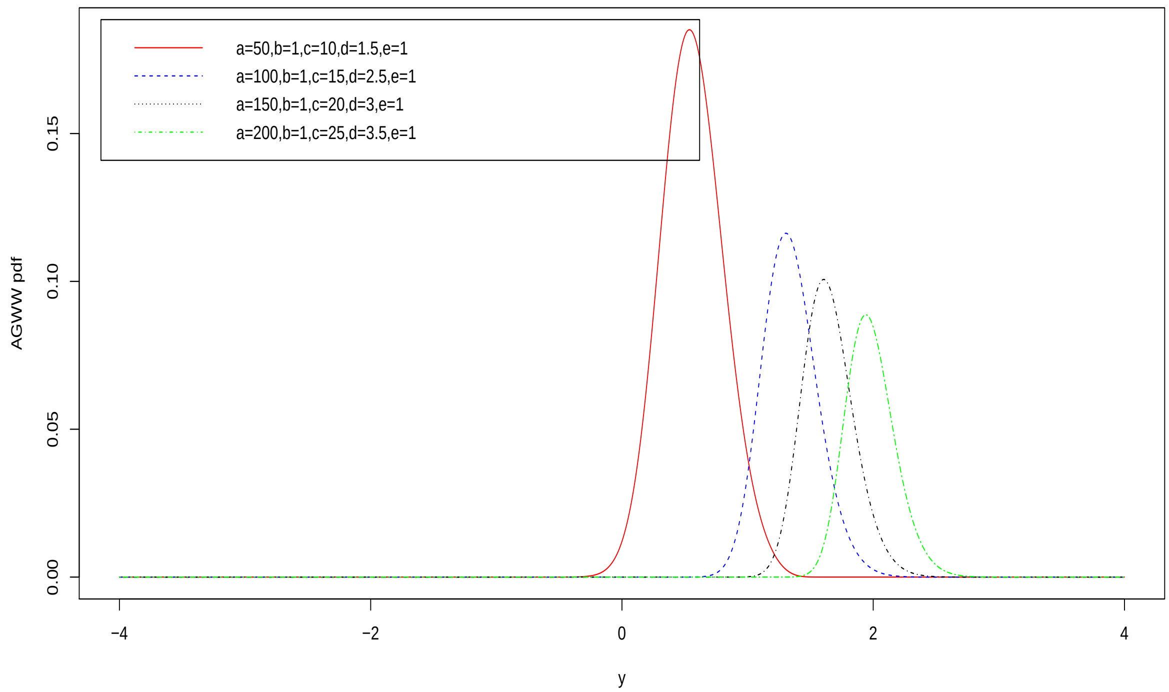

Figure 3.23: The curve of the AGWW density function (see equation 3.112) for some combinations $(a, b, \gamma, \sigma, \mu)$ from $b = 1$ and $e = \sigma = 1$ and $a = 50, 100, 150, 200$; $c = \gamma = 10, 15, 20, 25$ and $d = \mu = 1.5, 2.5, 3, 3.5$, respectively.

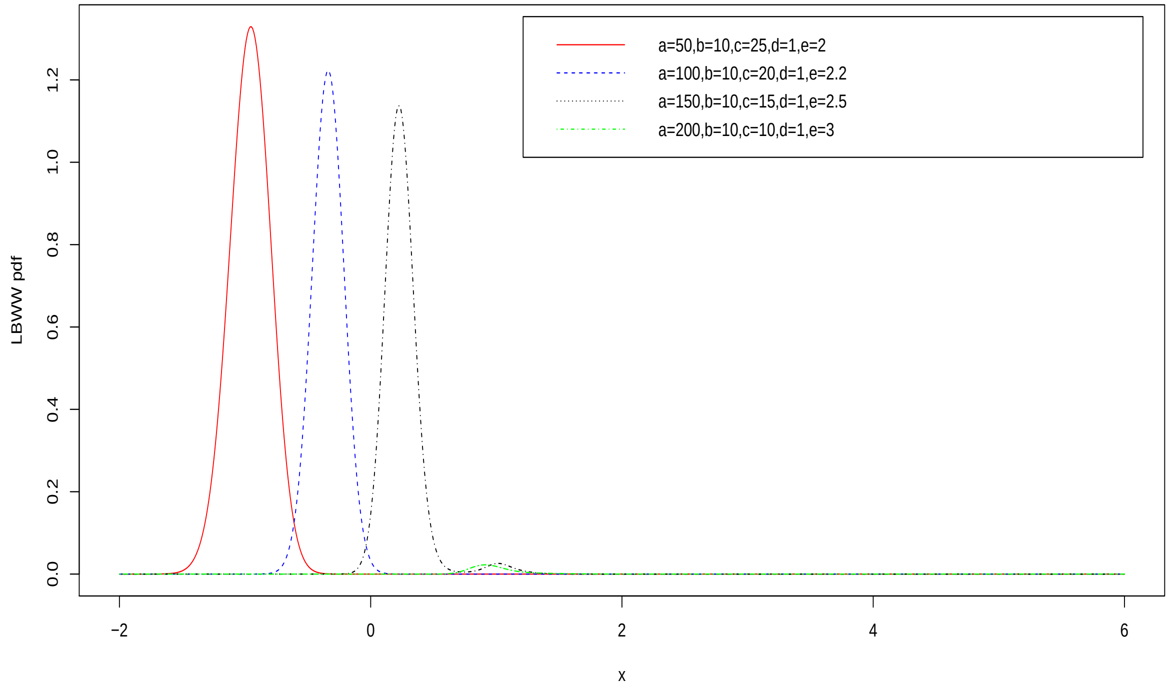

Figure 3.24: The curve of the AGWW density function (see equation 3.112) for some combinations of $(a, b, \gamma, \sigma, \mu)$ from $a = 200$ and $d = \mu = 0.1$ and $e = \sigma = 2, 2.2, 2.5, 3,\ b = 10, 8, 6, 5$ and $c = \gamma = 25, 20, 15, 10, 5,$ respectively.

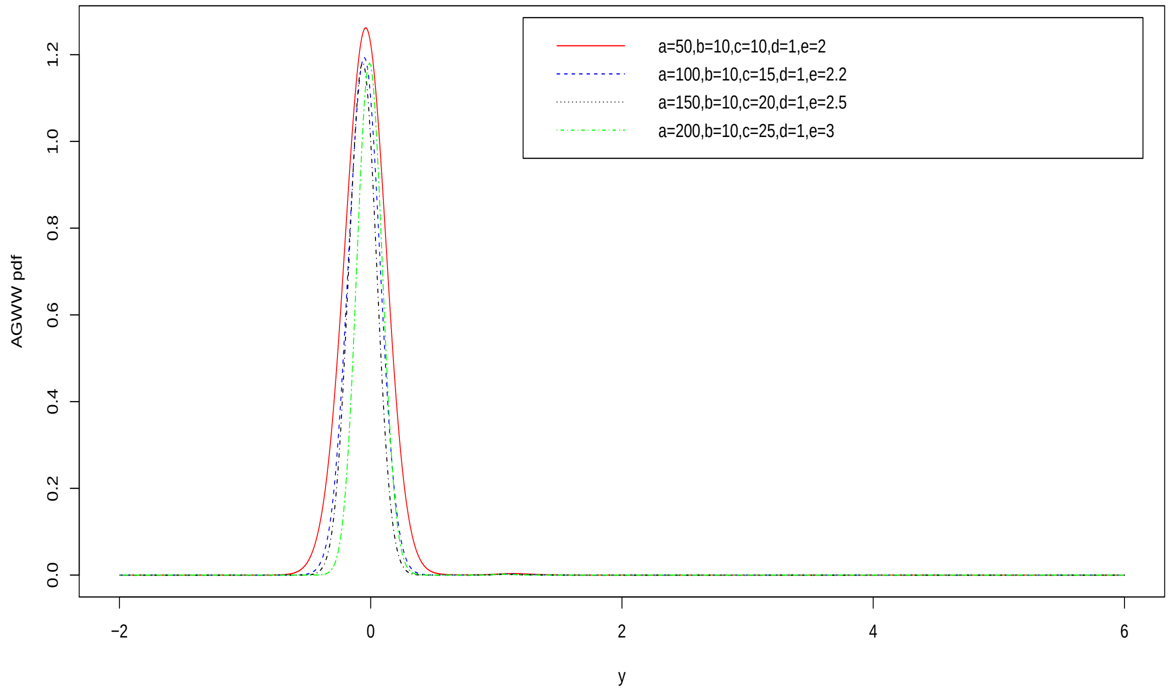

Figure 3.25: The curve of the AGWW density function (see equation 3.112) for some combinations $(a, b, \gamma, \sigma, \mu)$ from $b = 10$ and $d = \mu = 1$ and $a = 50, 100, 150, 200$; $c = \gamma = 10, 15, 20, 25$ and $e = \sigma = 2, 2.2, 2.5, 3$, respectively.

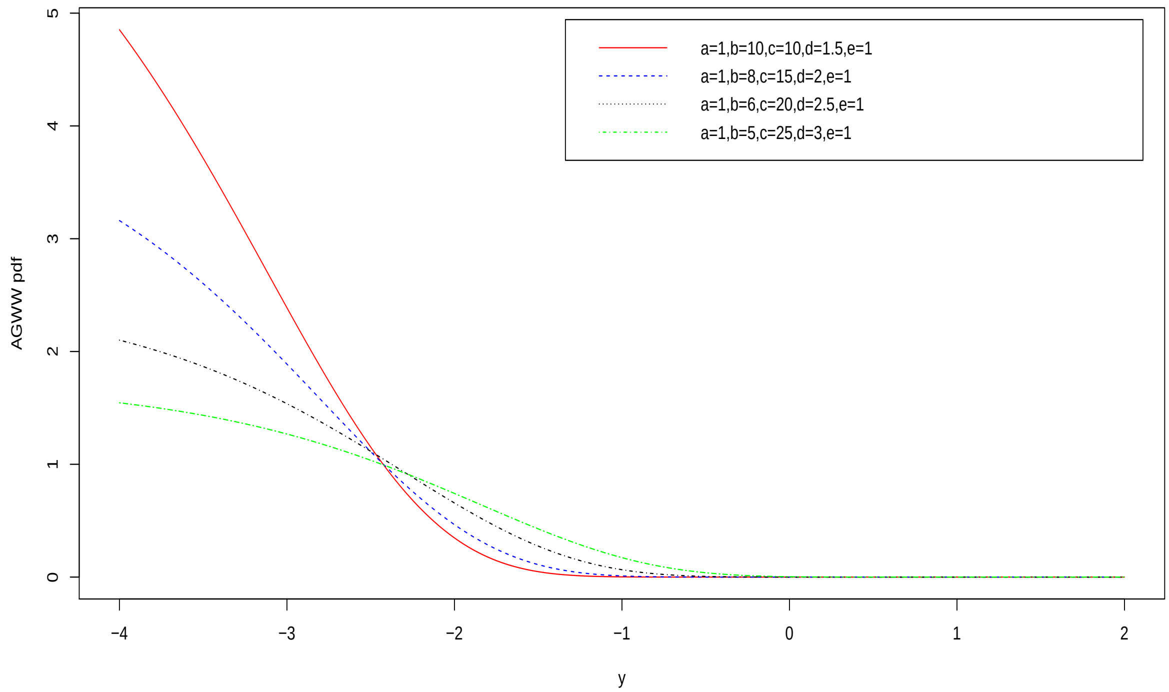

Figure 3.26: The curve of the AGWW density function (see equation 3.112) for some combinations $(a, b, \gamma, \sigma, \mu)$ from $a = 1$ and $e = \sigma = 1$ and $c = \gamma = 10, 15, 20, 25$, $d = \mu = 1.5, 2.5, 3, 3.5$ and $b = 10, 8, 6, 5$, respectively.

Again, the corresponding survival rate function to (3.110) is given as

$$S(y) = 1 - \frac{1}{B(a,b)} \int_0^{F(y)} K^{a-1}(1-K)^{b-1} dk = 1 - I_{F(t)}(a,b) \tag{3.111}$$

where $F_{(y)} = \left[\frac{\gamma+1}{\gamma}\left\{\left(1-e^{-e^{\left(\frac{y-\mu}{\sigma}\right)}}\right) - \frac{1}{\gamma+1}\left(1-e^{-(1+\gamma)e^{\left(\frac{y-\mu}{\sigma}\right)}}\right)\right\}\right]$

3.7.1 Some Properties of the AGWW Distribution

Furthermore, some of the properties of the standardised AGWW random variable defined by $Z = \frac{Y-\mu}{\sigma}$ were also studied. The density function of Z becomes

$$\eta(z;a,b,\gamma) = \frac{1}{B(a,b)}\left[\frac{\gamma+1}{\gamma}\left\{e^z e^{-e^z}\left(-e^{-\gamma e^z}\right)\right\}\right] \tag{3.112}$$
$$\left[\frac{\gamma+1}{\gamma}\left\{\left(1-e^{-e^z}\right) - \frac{1}{\gamma+1}\left(1-e^{-(1+\gamma)e^z}\right)\right\}\right]^{a-1}$$
$$\left[1-\frac{\gamma+1}{\gamma}\left\{\left(1-e^{-e^z}\right) - \frac{1}{\gamma+1}\left(1-e^{-(1+\gamma)e^z}\right)\right\}\right]^{a-1}$$

The corresponding cumulative distribution function (cdf) is

$$F_Z(Z) = I_{\left[\frac{\gamma+1}{\gamma}\left\{\left(1-e^{-e^{\left(\frac{y-\mu}{\sigma}\right)}}\right) - \frac{1}{\gamma+1}\left(1-e^{-(1+\gamma)e^{\left(\frac{y-\mu}{\sigma}\right)}}\right)\right\}\right]}(a,b)$$

The condition $a = b$ =1 corresponds to the standardised weighted Weibull distribution.

3.7.2 Expansion of Binomial

Expanding the binomial term in (3.112), as used in Ortega *et al.* (2013), we have

$$\eta(z;a,b,\gamma) = \frac{1}{B(a,b)}\sum_{i=0}^{\infty}(-1)^i\binom{b-1}{i}\left[\frac{\gamma+1}{\gamma}\left\{e^z e^{-e^z}\left(-e^{-\gamma e^z}\right)\right\}\right] \tag{3.113}$$
$$\left[\frac{\gamma+1}{\gamma}\left\{\left(1-e^{-e^z}\right) - \frac{1}{\gamma+1}\left(1-e^{-(1+\gamma)e^z}\right)\right\}\right]^{a(i+1)-1}$$

The density function $h_a = (a-1)\left[\frac{\gamma+1}{\gamma}\left\{e^z e^{-e^z}\left(-e^{-\gamma e^z}\right)\right\}\right]$ for $a > 0$ gives Lehmann type II weighted Weibull (Alexander *et al.* 2011) and its corresponding cumulative function is

$$H_b(x) = 1 - \left[1-\frac{\gamma+1}{\gamma}\left\{\left(1-e^{-e^z}\right) - \frac{1}{\gamma+1}\left(1-e^{-(1+\gamma)e^z}\right)\right\}\right]^b$$

Then, $\eta(z;a,b,\gamma) = \sum_{i=0}^{\infty} k_i h_{a(i+1)}(z)$,
where the coefficients are

$$k_i = \frac{(-1)^i\binom{b-1}{i}}{a(j+1)B(a,b)}$$

Moreover, the AGWW distribution function can be expressed as a linear combination of LWW densities. For $a = 1$, the AGWW reduces to the Log-Lehmann Type II Weighted Weibull (LLWW) and for $b = 1$, it becomes the Log-Exponentiated Weighted Weibull(LEWW) distribution; and for $a = b = 1$, it becomes Log Weighted Weibull distribution which are the new models defined here. The AGWW random variable z can be generated directly from the beta variate V with parameter $a > 0$ and $b > 0$ by $Z = \log[-\log(1-v) + (-\log(1-v))]$.

3.7.3 Moments and Generating Function

The rth ordinary moment of the AGWW distribution (3.112) is

$$\mu'_{AGWW(r)} = E(Z^r) = \frac{1}{B(a,b)} \int_{-\infty}^{\infty} Z^r \left[\frac{\gamma+1}{\gamma}\left\{e^z e^{-e^z}\left(-e^{-\gamma e^z}\right)\right\}\right] \left[\frac{\gamma+1}{\gamma}\left\{\left(1-e^{-e^z}\right) - \frac{1}{\gamma+1}\left(1-e^{-(1+\gamma)e^z}\right)\right\}\right]^{a-1} \left[1 - \frac{\gamma+1}{\gamma}\left\{\left(1-e^{-e^z}\right) - \frac{1}{\gamma+1}\left(1-e^{-(1+\gamma)e^z}\right)\right\}\right]^{b-1} dZ \tag{3.114}$$

We expanding the binomial term and setting $u = e^z$, we get

$$\mu'_{AGWW(r)} = \frac{1}{B(a,b)} \sum_{i=0}^{\infty} (-1)^i \binom{b-1}{i} \int_0^{\infty} \log(u)^r \times \left[\frac{\gamma+1}{\gamma} e^u e^{-e^u}\left(-e^{-\gamma e^u}\right)\right] \times \left[\frac{\gamma+1}{\gamma}\left\{\left(1-e^{-e^u}\right) - \frac{1}{\gamma+1}\left(1-e^{-(1+\gamma)e^u}\right)\right\}\right]^{a(i+1)-1} udu$$

$I_{(r,(i+1))} = \left(\frac{\partial}{\partial q}\right)^r [a(i+1)^{-q}\Gamma(q)]_{q=1}$ (Ortega *et al.* and Pascoa *et al.* 2013) and thus

$$\mu'_r = \frac{1}{B(a,b)} \sum_{i=0}^{\infty} (-1)^i \binom{b-1}{i} I_{(r,(i+1))} \tag{3.115}$$

Equation (3.115) gives the moments of the AGWW distribution. Automatically, the measures are mainly controlled by the parameters of a and b.

The moment generating function (mgf) of Z, it implies that $M(t) = E(e^{tz})$, follows from

(3.112) as

$$M_{AGWW}(t) = \frac{1}{B(a,b)} \sum_{i=0}^{\infty} (-1)^i \binom{b-1}{i} \int_0^{\infty} U^t$$
$$\left[\frac{\gamma+1}{\gamma} e^u e^{-e^u} \left(-e^{-\gamma e^u} \right) \right] \times$$
$$\left[\frac{\gamma+1}{\gamma} \left\{ \left(1 - e^{-e^u}\right) - \frac{1}{\gamma+1} \left(1 - e^{-(1+\gamma)e^u}\right) \right\} \right]^{a(i+1)-1} udu$$

Then,

$$M(t) = \frac{\Gamma(t+1)}{B(a,b)} \sum_{i=0}^{\infty} (-1)^i \binom{b-1}{i} [(a(i+1)-1)]^{-(t+1)} \tag{3.116}$$

Meanwhile, the moment (3.116) can be derived from (3.112) using differentiation.

Nevertheless, the first four moments, the skewness and kurtosis of the AGWW distribution are derived using the rth ordinary moment of the AGWW as expressed in equation (3.116).

$$\mu'_{AGWW(r)} = \int_0^{\infty} \log U^r \left[\frac{\gamma+1}{\gamma\sigma B(a,b)} [M(u)]^{a-1} [1 - M(u)]^{b-1} dm(u) \right] \tag{3.117}$$

where,

$$M(u) = \frac{\omega\left(1 - e^{-\gamma e^{p(u)}}\right) - \omega^{-1}\left(1 - e^{-\omega(p(u))}\right)}{\gamma}; \quad \omega = \gamma + 1, \;\; p(u) = e^{\frac{y-\mu}{\sigma}}$$

where,

$$\mu'_{AGWW(r)} = \left[\frac{\sigma(\omega) - \omega^{-\frac{r}{\mu}} \Gamma\left(1 + \frac{r}{\mu}\right)}{\gamma\sigma B(a,b)} \right] \sum_{i=0}^{\infty} (-1)^i \binom{b-1}{i} \tag{3.118}$$
$$\left\{ \int_0^{\infty} \log \left[\frac{\omega\left(1 - e^{-\gamma e^{p(u)}}\right) - \omega^{-1}\left(1 - e^{-\omega(p(u))}\right)}{\gamma} \right]^{a(i+1)-1} du \right\}$$

where

$$L = \frac{\sum_{i=0}^{\infty} (-1)^i \binom{b-1}{i} \int_0^{\infty} \log \left[\omega\left(1 - e^{-\gamma e^{p(u)}}\right) - \omega^{-1}\left(1 - e^{-\omega(p(u))}\right) \right]^{a(i+1)-1} du}{\gamma\sigma B(a,b)}$$

Therefore, the first four non-central moments μ'_r by substituting for $r = 1, 2, 3$ and 4 respectively in equation (3.118) is given as follows

$$\mu'_{AGWW(r)} = E_{AGWW}(u) = \left[\frac{\sigma(\omega) - \omega^{-\frac{1}{\mu}\gamma\left(1+\frac{1}{\mu}\right)}}{\gamma\sigma B(a,b)} \right] \sum_{i=0}^{\infty} (-1)^i \binom{b-1}{i} \tag{3.119}$$

The first moment of the AGWW as obtained from equations (3.119) and (3.68) respectively, also hold here.

Consequently, the mean, second, third and fourth moments of the AGWW distribution are given as

$$\mu = \mu_1'$$
$$\mu_2 = \mu_2' - \mu^2$$
$$\mu_3 = \mu_3 - 3\mu\mu_2' + 2\mu^3 \text{ and}$$
$$\mu_4 = \mu_4' - 4\mu\mu_3' + 6\mu^2\mu_2' - 3\mu^4$$

where

$$\mu_1' = L\left[p\sigma\left\{\omega - \omega^{-\frac{1}{\mu}}\right\}\Gamma\left(1+\frac{1}{\mu}\right)\right] \tag{3.120}$$
$$\mu_2' = L\left[p\sigma\left\{\omega - \omega^{-\frac{2}{\mu}}\right\}\Gamma\left(1+\frac{2}{\mu}\right)\right] \tag{3.121}$$
$$\mu_3' = L\left[p\sigma\left\{\omega - \omega^{-\frac{3}{\mu}}\right\}\Gamma\left(1+\frac{3}{\mu}\right)\right] \tag{3.122}$$
$$\mu_4' = L\left[p\sigma\left\{\omega - \omega^{-\frac{4}{\mu}}\right\}\Gamma\left(1+\frac{4}{\mu}\right)\right] \tag{3.123}$$

Moments measures of Skewness ξ_1 and excess kurtosis, ξ_2 are respectively given as

$$\xi_1 = \frac{\mu_3}{\mu_2^{\frac{3}{2}}} \tag{3.124}$$
$$\xi_2 = \frac{\mu_4}{\mu_2^2} - 3 \tag{3.125}$$

The skewness curves of the LBWW are shown below for different values of parameters a and b.

In figure 3.27, parameters (a, b) increase, the skewness tends towards zero in (i and ii). Figure 3.28 contain the plots of skewness of the AGWW as parameter a held fixed and the values of b increases in (i); and the values of a increases and b is held fixed in (ii) then, the skewness tends to zero.

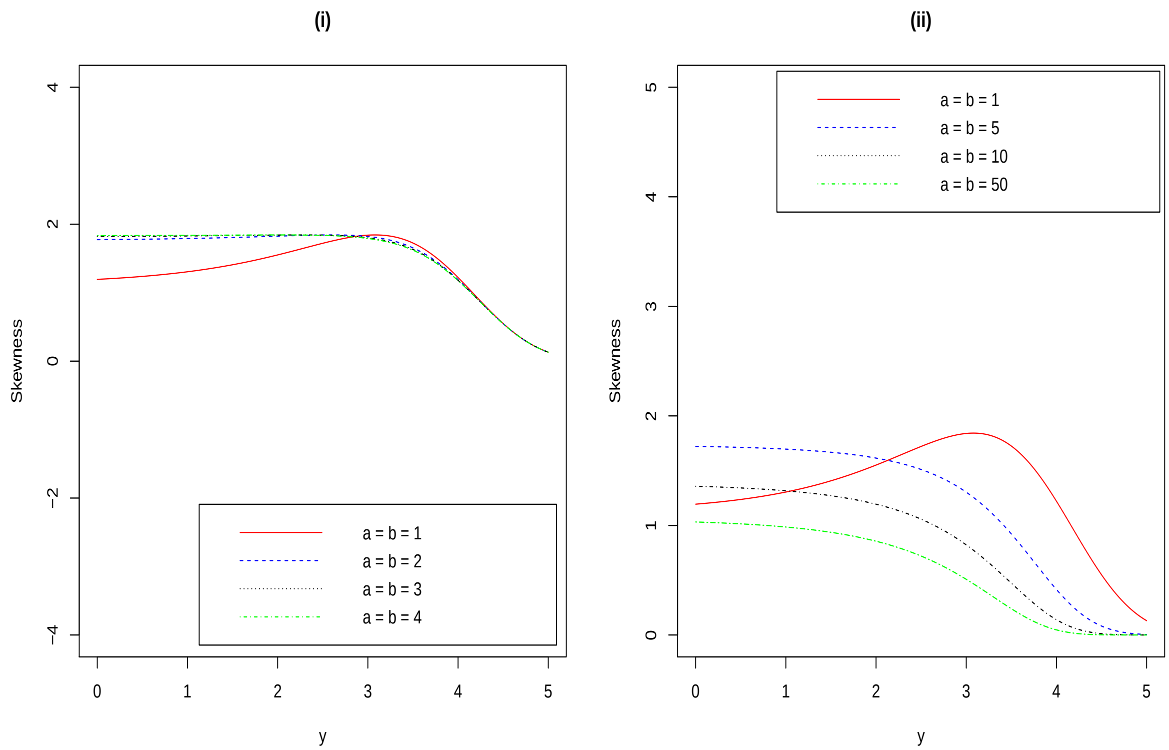

Figure 3.27: Skewness of the AGWW distribution (see equation 3.125): (i) for values of a and b : $a = b = 1, 2, 3$ and 4 and (ii) for values of a and b : $a = b = 1, 5, 10$ and 50, respectively.

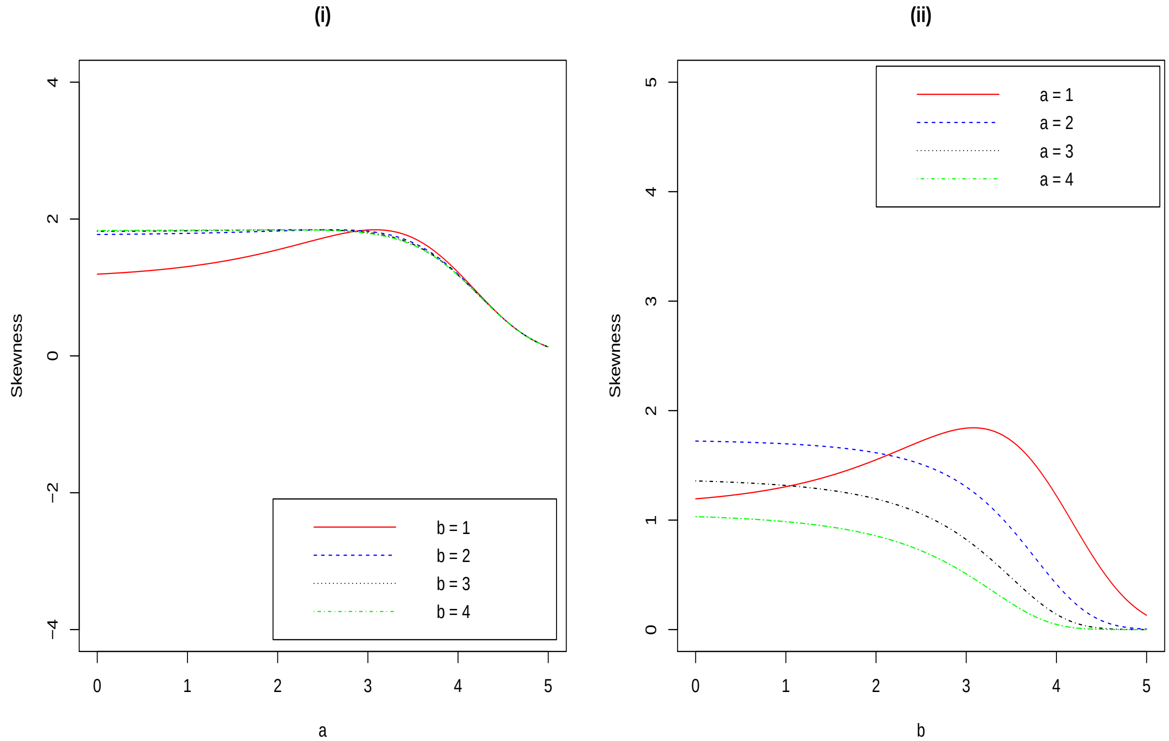

Figure 3.28: Skewness of the AGWW distribution (see equation 3.125): (i) function of a for some values of $b : b = 1, 2, 3$ and 4 and (ii) function of b for some values of $a : a = 1, 2, 3$ and 4, respectively.

Again, the plots of excess of kurtosis $\xi = \frac{\mu_4}{\mu_2^2} - 3$ of the AGWW are shown below for different values of a and b is shown below:

In figure 3.29, parameters (a, b) increase, the kurtosis increases in (i and ii). Figure 3.30 contain the plots of kurtosis of the AGWW as parameter a held fixed and the values of b increases in (i); and the values of a increases and b is held fixed in (ii) then, the kurtosis increases.

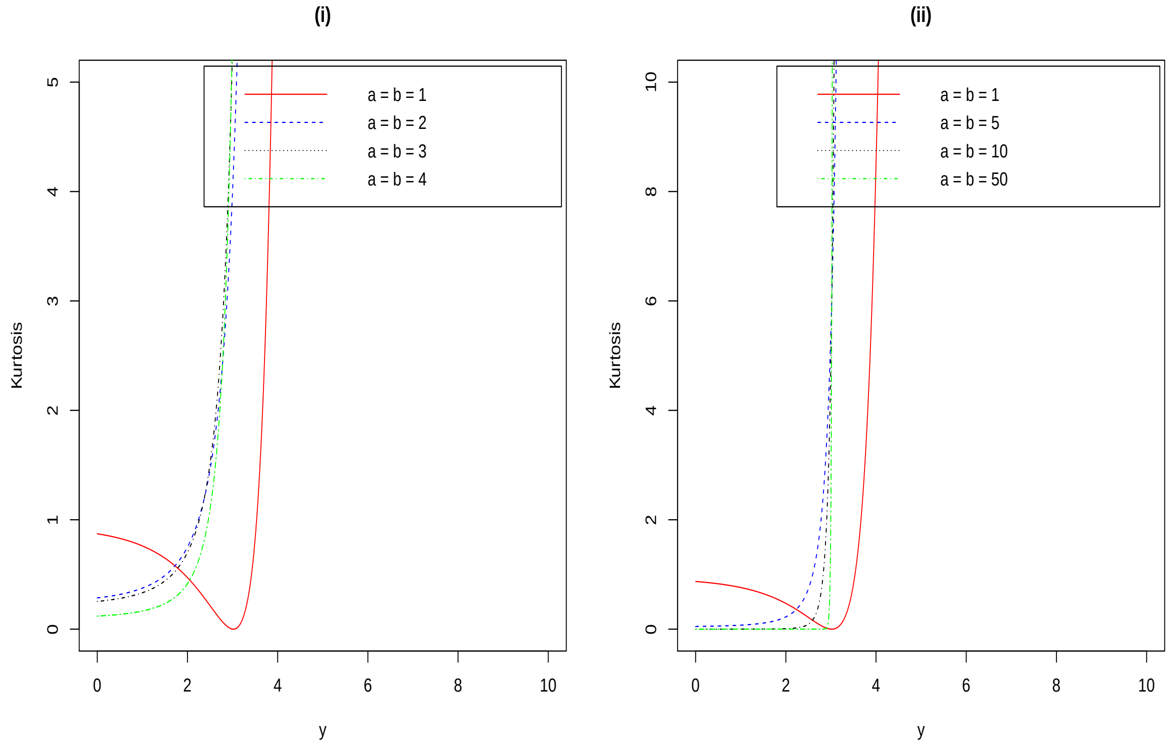

Figure 3.29: Excess of Kurtosis of the AGWW distribution (see equation 3.126): (i) for values of a and b : $a = b = 1, 2, 3$ and 4 and (ii) for values of a and b : $a = b = 1, 5, 10$ and 50, respectively.

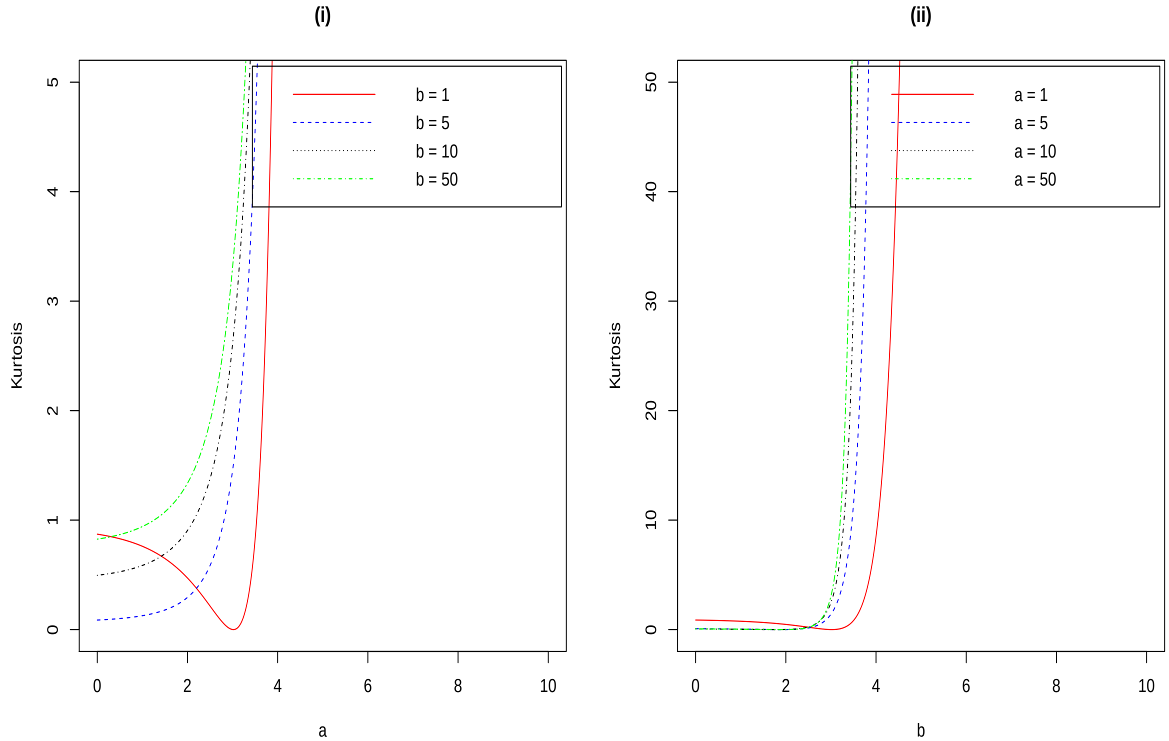

Figure 3.30: Excess of Kurtosis of the AGWW distribution (see equation 3.126): (i) function of a for values of $b : b = 1, 5, 10$ and 50 and (ii) function of b for values of $a : a = 1, 5, 10$ and 50, respectively.

Meanwhile, the R codes for figure 1 to 30 are in appendix A.

3.7.4 Quartile Function

The quantile function has been successfully obtained for various distributions in the literature by inverting the parent distribution function of the said distribution; it is used for purposes of simulation. In the manner of Alexander *et al.* 2011 and Ortega *et al.* 2013, we provide an expansion for quantile function of the AGWW distribution $q = F^{-1}(u)$, i.e. $u = F(q)$ to obtain $I_m(a, b)$ being an incomplete Beta (a, b) function with index m where $m = \frac{\gamma+1}{\gamma}\left[\left(1 - e^{-e^q}\right) - \frac{1}{\gamma+1}\left(1 - e^{-(1+\gamma)e^q}\right)\right]$
Again, it is likely to obtain m as function of u by inverting the incomplete beta function

$$
\begin{aligned}
m &= I_u^{-1}(a, b), \\
&= p + \frac{b-1}{a+1}p^2 + \frac{(b-1)[a^2 + 3ab - a + 5b - 4]}{2[(a^2 + 3a + 2)(a+1)]}p^3 + \\
&\frac{(b-1)[a^4 + (6a^3b - 1) + (b+2)(ba^2b - 5a^2) + (33ab^2 - 30ab + 4a) + (c^*) + 18]}{3[(a+1)^2(a^2 + 3a + 2)]}p^4 + \\
&0\left(p^{\frac{5}{a}}\right),
\end{aligned}
$$

where
$c^* = 31b^2 - 47b$
$p = [a, uB(a, b)]^{\frac{1}{a}}$ for $a > 0$.
Then, $q = \log[-\log(1-m)]$; the above expression also defines the AGWW quantile function.

3.7.5 Order Statistics: AGWW Distribution

An important concept in statistical theory and application is the order statistics being transformation $(z_1, \cdots, z_n)$ of independent random sample $(x_1, \cdots, x_n)$ where $z_i = x_{(i)}, x_{(1)} < x_{(2)} < \cdots < x_{(n)}$ The joint density function $f_{i:n}(z)$ of the order statistics $(z_{i:n}) = (z_i, z_n)$ for $i = 1, \cdots, n$ where $(x_1, \cdots, x_n)$ is a random sample i.e. independently identically distributed (i.i.d) AGWW random variables is expressed as

$$
f_{i,n}(z) = \frac{f(z)}{B(i, n-j+1)} \sum_{i=0}^{n-j} (-1)^i \binom{n-j}{i} F(z)^{i+j-1}
$$

Pascoa *et al.* (2013)

$$
f_{i,n}(z) = \frac{\eta(z; a, b, \gamma)}{B(i, n-j+1)} \sum_{i=0}^{n-j} (-1)^i \binom{n-j}{i} I_{\frac{\gamma+1}{\gamma}\left[\left(1-e^{-e^z}\right) - \frac{1}{\gamma+1}\left(1-e^{-(1+\gamma)e^z}\right)\right]}(a, b)^{i+j-1} \quad (3.126)
$$

Now, we obtain an expresion for the density function of the AGWW order statistics. For instance, we make use of the incomplete beta function expansion for $r > 0$, a real non-integer

$$I_{\frac{\gamma+1}{\gamma}\left[\left(1-e^{-e^z}\right)-\frac{1}{\gamma+1}\left(1-e^{-(1+\gamma)e^z}\right)\right]}(a,b) = \frac{B(a,b)]^{-1}\sum_{m=0}^{\infty}(1-\gamma)_m g_z[D]^{a+m}[1-D]^{b+m}}{[(a+m)(b+m)]m!}$$

where

$$g_z = \frac{\gamma+1}{\gamma}\left\{\left(e^{-e^z}\right)\left(-e^{-(1+\gamma)e^z}\right)\right\}$$
$$D = \left[\frac{\gamma+1}{\gamma}\left\{\left(1-e^{-e^z}\right)-\frac{1}{\gamma+1}\left(1-e^{-(1+\gamma)e^z}\right)\right\}\right]$$

and $f(w) = \frac{\Gamma(f+w)}{\Gamma(f)}$ is the ascending factorial. We then get

$$I_{\frac{\gamma+1}{\gamma}\left[\left(1-e^{-e^z}\right)-\frac{1}{\gamma+1}\left(1-e^{-(1+\gamma)e^z}\right)\right]}(a,b) = \sum_{w=0}^{\infty} dw I_{\frac{\gamma+1}{\gamma}\left[\left(e^{-we^z}\right)-\frac{1}{\gamma+1}\left(1-e^{-(1+\gamma)e^z}\right)\right]}(a,b) \tag{3.127}$$

where the coefficient dw (for $w = 0, 1, \cdots$) is given as

$$dw = \frac{(-1)^w}{B(a,b)}\sum_{m=0}^{\infty}\frac{(1-\gamma)_m\binom{a+m}{w}\binom{b+m}{w}}{[(a+m)(b+m)]m!}$$

We use the identity $\left(\sum_{w=0}^{\infty}a_w x^w\right)^n = \sum_{i=0}^{\infty}C_{n,w}x^w$ for positive integer n in

$$I_{\frac{\gamma+1}{\gamma}\left[\left(1-e^{-e^z}\right)-\frac{1}{\gamma+1}\left(1-e^{-(1+\gamma)e^z}\right)\right]}(a,b)^{i+j-1}$$

Again, we obtain

$$I_{\frac{\gamma+1}{\gamma}\left[\left(1-e^{-e^z}\right)-\frac{1}{\gamma+1}\left(1-e^{-(1+\gamma)e^z}\right)\right]}(a,b)^{i+j-1} = \tag{3.128}$$
$$\sum_{w=0}^{\infty}C_{i+j-1,w}\frac{\gamma+1}{\gamma}\left[\left(e^{-we^z}\right)-\frac{1}{\gamma+1}\left(1-e^{-(1+\gamma)e^z}\right)\right]$$

where $C_{i+j-1,0} = d_o^{i+j-1}$ and for $w = 1, 2, \cdots,$ Also, according to (Ortega *et al.* 2013), we have

$$C_{i+j-1,w} = (Wd_0)^{-1}\sum_{r=1}^{w}[(i+j)r-w]d_i C_{i+j-1,w-r} \tag{3.129}$$

Substituting (3.123) and (3.129), we get

$$f_{i:n}(z) = \sum_{m,w=0}^{\infty}(-1)^m\binom{b-1}{m}g_w\left[\frac{\gamma+1}{\gamma}\left\{e^z e^{-z}\left(-e^{-\gamma e^z}\right)\right\}\right] \tag{3.130}$$
$$\left[\frac{\gamma+1}{\gamma}\left\{\left(1-e^{-e^z}\right)-\frac{1}{\gamma+1}\left(1-e^{-(1+\gamma)e^z}\right)\right\}\right]^{a(m+w+1)-1}$$

where

$$g_w = \frac{i = 0\sum^{n-j}(-1)^i\binom{n-j}{i}C_{i+j-1,w}}{B(i, n-i+1)B(a,b)}$$

The moments, MGF, mean deviations of the AGWW order statistics can be derived from (3.130) by using the compensation for these quantities of the AGWW distribution. For instance, the rth ordinary moment of $Z_{i:n}$ is simply expressed as

$$E(X_{i:n}^r) = \sum_{m,w=0}^{\infty}(-1)^m\binom{b-1}{m}g_w I(r, m+w)$$

where $(r, m+w)$ has been defined just before equation (3.124)

3.8 The Alternative Generalised Weighted Weibull Regression Model.

Here, using AGWW distribution function, we study location-scale regression model linking the response variable y_i and vector $X_i^T = (x_{i1}, \cdots, x_{ip})$ of explanatory variables X; we follow model (3.15) which has been used in literature, e.g. Ortega et al, 2009; Cancho et al, 2009; Pescim *et al.*, 2013 and Ortega *et al.*, 2013 where the random error z_i has density function (3.112) with parameters $\beta = (\beta_1, \cdots, \beta_p)^T, \sigma > 0, a > 0, b > 0$ and $\gamma > 0$ are unknown parameters. The parameter $\mu_i = X_i^T\beta$ is the location of y_i. The location parameter vector $\mu = (\mu_1, \cdots, \mu_n)^T$ is represented by a model $\mu = X^T\beta$ where $X = (X_1, X_2, \cdots, X_n)^T$ is a known model matrix. The AGWW regression model (3.15) allows and opens now possibilities for fitting many difficult types of data. The developed model also contains sub-models which are new regression models. For $a = b = 1$ we obtain log-weighted-Weibull regression model, for $b = 1$, It becomes the log-exponentiated weighted Weibull regression model. If $\sigma = 1$ in addition to $b = 1$, the AGWWRM reduces to log-exponentiated weighted regression, for $\sigma = 1$, and the model called the log beta exponential weighted regression model.

3.8.1 Maximum Likelihood Estimation

Consider a sample $(y_1, x_1), \cdots, (y_n, x_n)$ of n independent observations, where each random response is defined by $y_i = \min\{\log(t_i), log(c_i)\}$. We assume non-informative censoring such that the observed lifetimes and censoring times are independent. Let F and C be the sets of individuals for which y_i is the log-lifetime and log-censoring, respectively. We can then apply conventional likelihood estimation techniques here. The likelihood function for the vector of parameters $\theta = (a, b, \gamma, \sigma, \beta^T)^T$ from model (3.15) has the form

$l(\theta) = \sum_{i\in F}\log[f(y_i)] + \sum_{i\in C}\log[S(y_i)]$, where $f(y_i)$ is the density function (3.112) and $S(y_i)$ is the survival function (3.113) of Y_i.

The log-likelihood function for θ reduces to

$$\begin{aligned} l(\theta) &= -r\log\{\log(\sigma)+\log[B(a,b)]\}+ \\ &\left[\frac{\gamma+1}{\gamma}\left\{\sum_{i\in F}(z_i)+\sum_{i\in F}\left(-e^{z_i}\left(1-e^{-\gamma e^{z_i}}\right)\right)\right\}\right]+ \\ &(a-1)\sum_{i\in F}\log\left[\frac{\gamma+1}{\gamma}\sum_{i\in F}\left\{\left(1-e^{-e^{z_1}}\right)-\frac{1}{\gamma+1}\sum_{i\in F}\left(1-e^{-(1+\gamma)e^{z_i}}\right)\right\}\right]+ \\ &(b-1)\sum_{i\in F}\log\left[1-\frac{\gamma+1}{\gamma}\sum_{i\in F}\left\{\left(1-e^{-e^{z_1}}\right)-\frac{1}{\gamma+1}\sum_{i\in F}\left(1-e^{-(1+\gamma)e^{z_i}}\right)\right\}\right]+ \\ &\sum_{i\in C}\log\left\{1-I_{\left[\frac{\gamma+1}{\gamma}\sum_{i\in F}\left\{\left(1-e^{-e^{z_i}}\right)-\frac{1}{\gamma+1}\sum_{i\in F}\left(1-e^{-(1+\gamma)e^{z_i}}\right)\right\}\right]}\right\}^{a,b)} \end{aligned} \tag{3.131}$$

where r is the number of uncensored observations (failures) and $z_i = \frac{(y_i - x_i^T\beta)}{\sigma}$.

The MLE $\hat{\theta}$ of the vector θ of unknown parameters can be calculated by maximising the likelihood function in (3.131). The fitted AGWW model gives the estimated survival function of Y for any individual with explanatory vector x

$$S(y;\hat{a},\hat{b},\hat{\gamma},\hat{\sigma},\hat{\beta}^T) = 1 - I_{\left[\frac{\hat{\gamma}+1}{ga\hat{m}ma}\left\{\left(1-e^{-e^{\frac{y-X^T\hat{\beta}}{\hat{\sigma}}}}\right)-\frac{1}{\hat{\gamma}+1}\left(1-e^{-(1+\hat{\gamma})e^{\frac{y-X^T\hat{\beta}}{\hat{\sigma}}}}\right)\right\}\right]^{(\hat{a},\hat{b})}} \tag{3.132}$$

The invariance property of the MLEs yields the survival function for $V = e^Y$

$$S(y;\hat{a},\hat{b},\hat{\alpha},\hat{\beta},\hat{\lambda}) = 1 - I_{\left[\frac{\hat{\alpha}+1}{\hat{\alpha}}\left\{\left(1-e^{-\left(\frac{t}{\hat{\lambda}}\right)^{\hat{\beta}}}\right)-\frac{1}{\hat{\alpha}+1}\left(1-e^{-(1+\hat{\alpha})\left(\frac{t}{\hat{\lambda}}\right)^{\hat{\beta}}}\right)\right\}\right]^{(\hat{a},\hat{b})}} \tag{3.133}$$

where $\hat{\sigma} = \frac{1}{\hat{\alpha}}$ and $\hat{\beta} = e^{X^T\hat{\beta}}$.

3.9 Model selection using information criteria

In this section, we studied relevant traditional information-theoretic criteria including Akaike Information Criterion (AIC) (Akaike 1973) and Bayesian Information Criterion (BIC) (Schwarz 1978); and lesser-known criteria such as Consistent Akaike Information Criterion (CAIC) which have been used for the purpose of identifying the correct asymmetric model and address model selection problems. Model selection refers to the problem of using the data to select one model from the list of competing models. It also involves the use of a model selection criterion to find the best fitting model to the data (Wasserman 2000). Model selection using information criteria has been developed to summarize data evidence in favour of a model. Meanwhile, information criteria techniques emphasize minimizing the amount of information required to express the data and model. This leads to selection of models that constitute an efficient representation of the data. Fortunately, several works on beta generalised modelling have used information criteria to select the best model among competing models. Famoye *et al.* 2005, Carrasco *et al.* 2008, Ortega *et al.* 2009, Hashimoto *et al.* 2010, Alexander *et al.* 2012, Ortega *et al.* 2013, Mahmoud *et al.* 2015 are instances. Therefore, we employed information criteria in our study to select the most appropriate model between the developed and competing models based on the model with consistently lower values.

3.9.1 Akaike Information Criterion (AIC)

Akaike Information Criterion is one of the first model selection methods introduced; it is also the most commonly used information criterion. AIC is based on the idea that a chosen model is correct if it can sufficiently describe any future data with the same distribution and therefore AIC can be regarded as a hypothetical cross validation method (Acquah 2013). It selects a model that minimizes the expected error of the new observation with the same distribution as the data used for fitting the model. It is defined as:

$$AIC = -2LogLik + 2r$$

where LogLik refers to the likelihood under the fitted model and r is the number of parameters in the model.

The model with minimum AIC value is chosen to be the best model among competing models. More importantly, AIC is aimed at finding the best approximating model to the unknown true data generating process and its applications.

3.9.2 Bayesian information criterion (BIC)

Bayesian information criterion is another widely used information criterion. Unlike Akaike Information Criterion, BIC is derived within a Bayesian framework as an estimate of the Bayes factor for two competing models (Schwarz 1978, Kass and Rafftery 1995). Thus, BIC is defined as:

$$BIC = -2LogLik + r(\log n)$$

where LogLik refers to the likelihood under the fitted model, r is the number of parameters in the model and $Log(n)$ is the logarithm of n(= number of sample size).
Automatically, BIC differs from AIC only in the second term which now depends on sample size n. Models that minimize the Bayesian Information Criteria are selected. BIC is designed to identify the true model while AIC does not depend directly on sample size. Bozdogan (1987) noted that because of this, AIC lacks certain properties of asymptotic consistency. From a Bayesian perspective, BIC is designed to find the most probable model given the data and this led the adoption of Bayesian information criterion.

3.9.3 Consistent Akaike Information Criterion (CAIC)

Bozdogan (1987) proposed a corrected version of AIC in an attempt to overcome the tendency of the AIC to overestimate the complexity of the underlying model; and again observed that Akaike Information Criterion (AIC) does not directly depend on sample size and as a result lacks certain properties of asymptotic consistency. In formulating CAIC, a correction factor based on the sample size is employed to compensate for the overestimating nature of AIC. CAIC, which reflects sample size and has properties of asymptotic consistency is defined as:

$$CAIC = -2LogLik + r[(Logn) + 1$$

where LogLik refers to the likelihood under the fitted model, r is the number of parameters in the model and n is the logarithm of sample size.
Invariably, AIC differs from CAIC in the second term which now takes into account sample size n. Models that minimize the Consistent Akaike Information Criterion are selected.

Chapter 4

ANALYSIS OF DATA

4.1 Real Data Set

We have a data on time from commencement to completion of PhD programme at University of Ibadan as survival data obtained from Olubusoye and Olusoji (2014) to provide data support for this work. The data had 187 candidates who had completed their PhD programme at the university. The variables involved in this study included;

$y =$ duration in years to complete the PhD programme,

$x_1 =$ rank of supervisor (Prof = 3, Reader = 2, S/L = 1, N/A = 0),

$x_2 =$ employment status (fully employed = 2, self-employed = 1, N/A = 0),

$x_3 =$ marital status (married = 2, single = 1, N/A = 0),

$x_4 =$ age of candidate and

$x_5 =$ faculty of candidate (Arts = 1, Agriculture/Forestry = 2, FBMS= 3, Institute of Education = 4, Pharmacy = 5, Public Health = 6, Science = 7, Social Science.= 8, Technology = 9 and Veterinary Medicine = 10).

The newly developed regression model explained the study variable y observed on 187 candidates i.e. $i = 1, 2, \cdots, n = 187$ in terms of explanatory variables $x_h,\ h = 1, 2, \cdots, 5$ as defined above. The purpose of the analysis was to investigate the significance of the model and the relevance of the parameters to the phenomenon represented by the study variable y, i.e. whether the estimated parameters were each positively or negatively significant, the standard error, the p-values, the AIC, BIC and CAIC values were determined; the minimum and maximum numbers of years to complete PhD programme would be estimated. The performance of the model would be evaluated through investigation of the parameters.

However, some exploratory analyses of the data (EDA) especially through graphs were done. The AGWW regression model is given as

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i3} + \beta_4 x_{i4} + \beta_5 x_{i5} + \sigma z_i, \qquad i = 1, \cdots, 187 \tag{4.1}$$

where the random variable y_i has the AGWW distribution in (3.110); and R code software would be used for the analysis of data. The scatter plots of time to completion of the PhD programme against each of the exploratory variables are given below.

4.1.1 Exploratory Data Analysis (EDA)

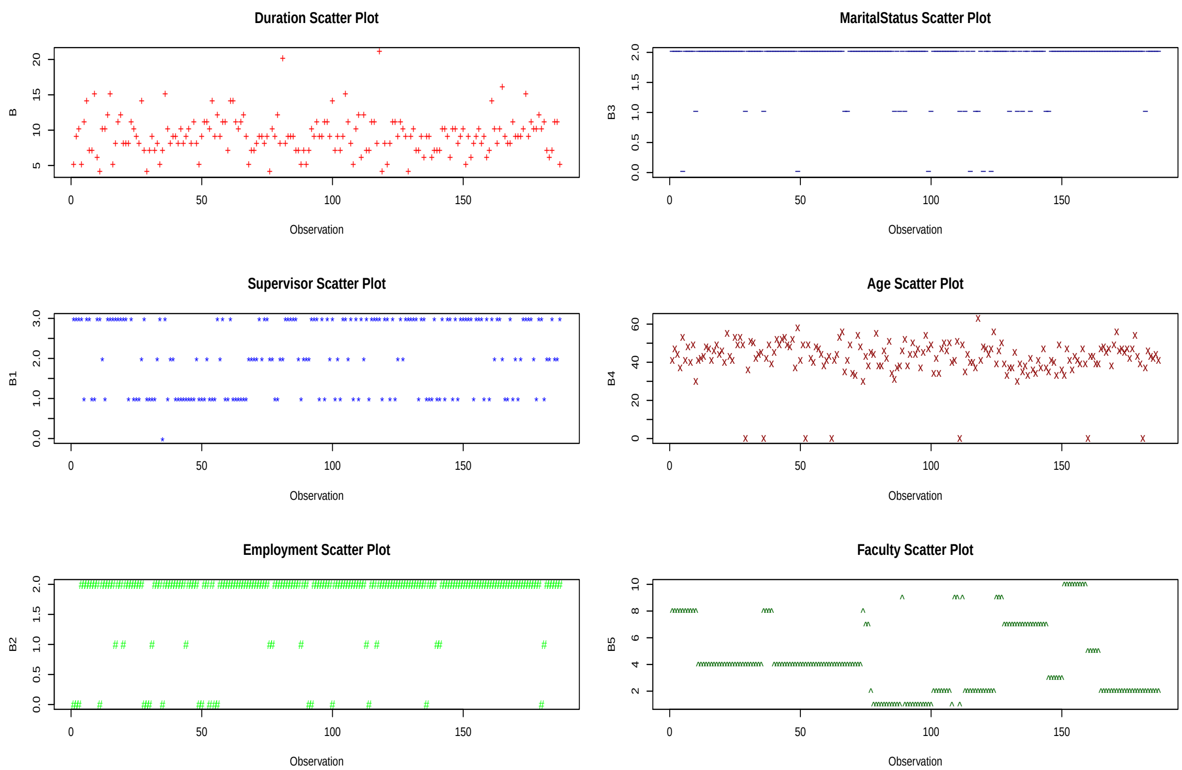

Figure 4.1: Scatter plots of study variable, i.e. duration (time to completion of PhD programme), against explanatory variables, i.e. Supervisor, Employment, Marital-Status, Age and Faculty

Figure 4.2 shows line plots of the study variable against each of the explanatory variables

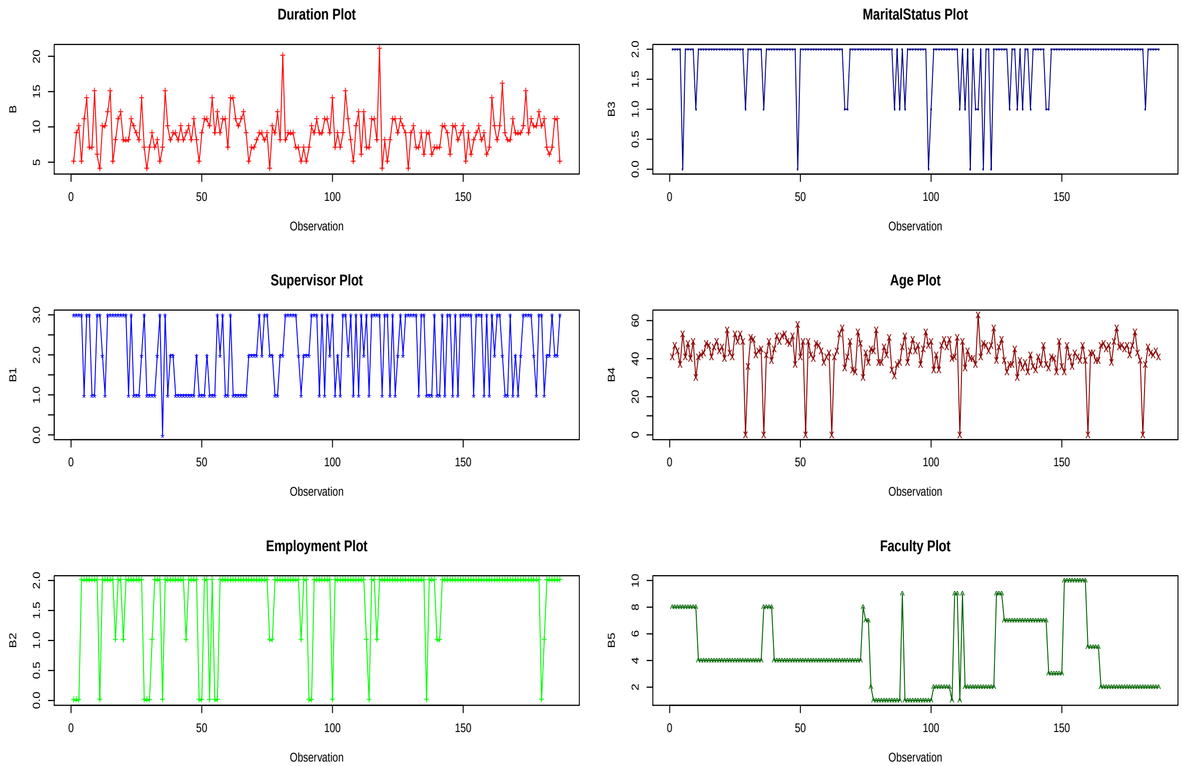

Figure 4.2: The line plot of Duration (time to completion), Supervisor, Employment, MaritalStatus, Age and Faculty

Figure 4.3 shows some relevant distribution plots of time to completion of PhD degree programme over 187 candidates

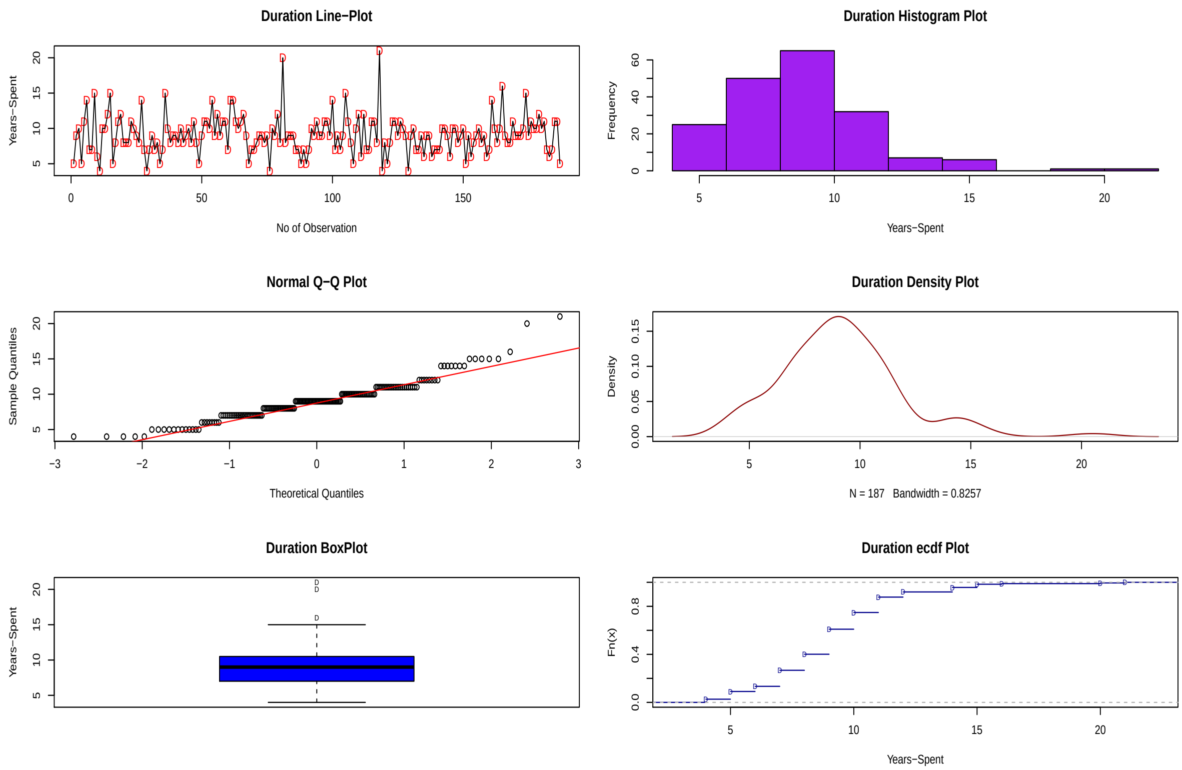

Figure 4.3: Plots of Time Spent by the Candidates to complete their PhD programme

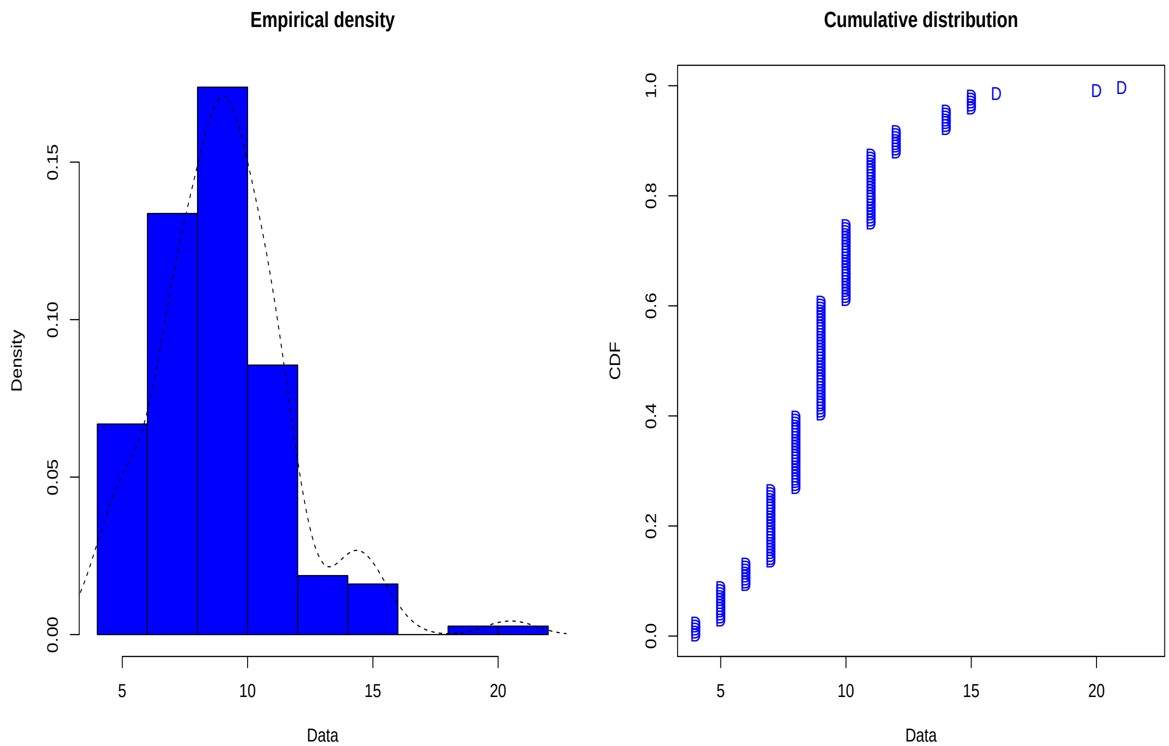

Figure 4.4: Plots of Time Spent by the Candidates to complete their PhD programme

Table 4.1: Descriptive Statistics of the Duration Data (Time to Completion)

Min	Q_1	Median	Mean	Q_3	Max	Skewness	Kurtosis
4.00	7.000	9.000	9.112	10.500	21.000	0.9373726	5.4713

Figure 4.4 below contain the histogram and theoretical densities of Weibull, Log-Normal, Gamma and Beta Weighted Weibull distribution using time spent (Duration data) by the candidates to Complete their PhD programme; at different values of parameters $(a, b) = (7,7), (5,5), (3.5,3.5), (2.5,2.5)$; and $\alpha, \beta = (3, 3.5)$ are fixed. In figure 4.4 (i), as the value of parameters a and b increase and α, β held fixed, the skewness of the BWW distribution is balanced and the level of the kurtosis increases, weibull is negative skewed while, log-normal and gamma are positive skewed and the level of their kurtosis are constant. Then, the graph in figure 4.4 (ii), indicate that as the value of parameters a and b decrease and α, β held fixed, the skewness of the BWW distribution is balanced and the level of kurtosis decreased, weibull, log-normal and gamma distribution skewness and kurtosis are constant since the values of their parameters are fixed.

The histogram in figure 4.4 (iii) and (iv), show that as the value of parameters a and b decrease and α, β held fixed, the skewness of the BWW distribution still balanced and the level of the kurtosis also decreases, but others remain the same. Therefore, if the values of parameters (a and b) increase or decrease the BWW distribution would still maintain its shape, but the level of the kurtosis is decreasing as the values decrease.

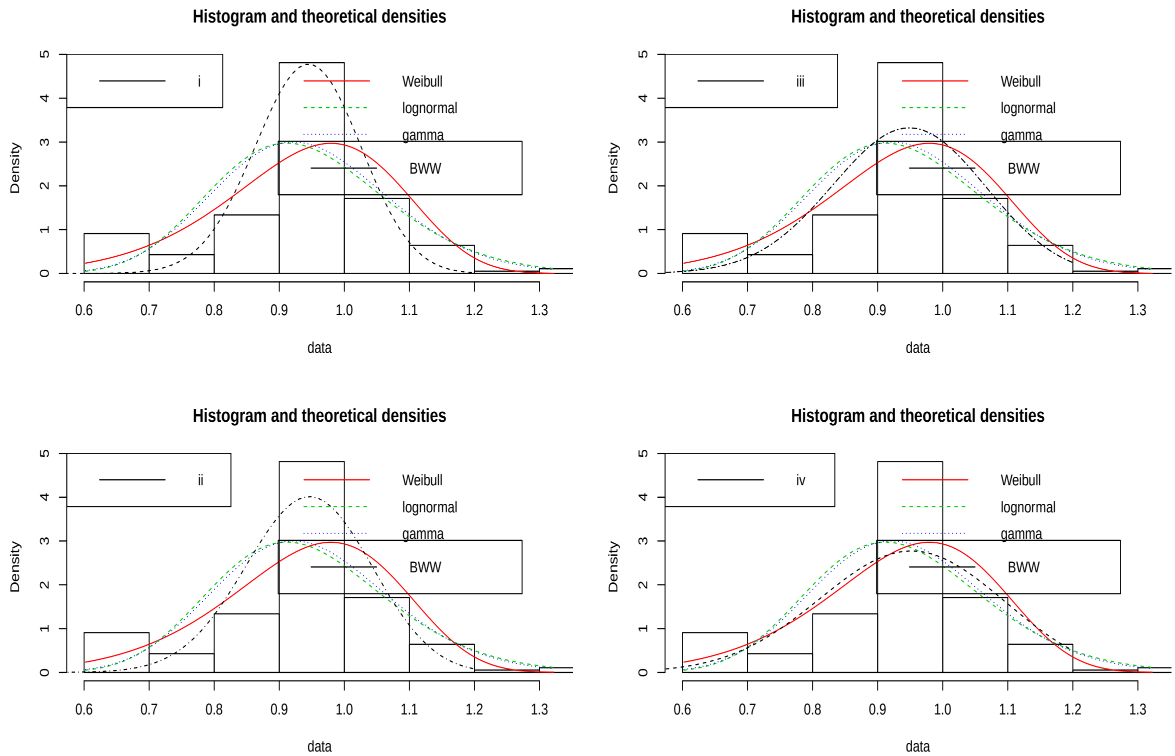

Figure 4.5: Plots of histogram and theoretical densities of Weibull, Lognormal, Gamma and Beta Weighted Weibull (BWW) distribution. The R code for the analysis is in appendix B.

Plot of the Beta Weighted Weibull (BWW) Distribution

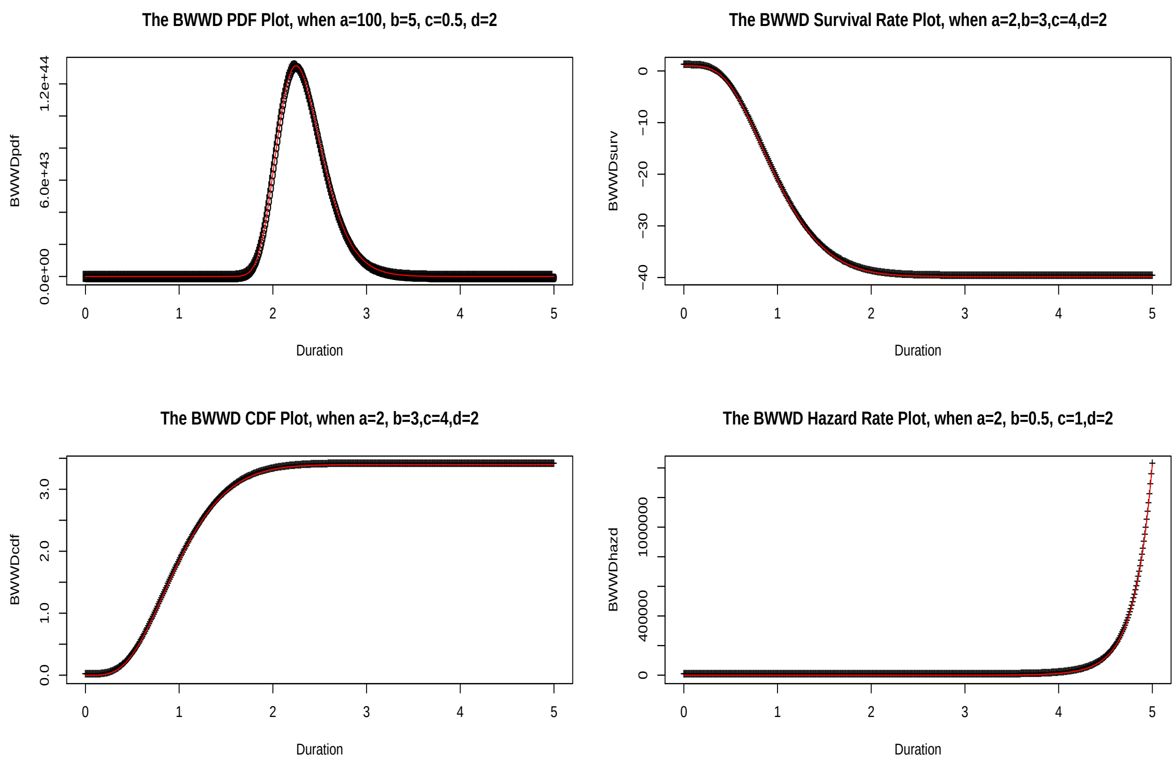

Figure 4.6: Combine graphs of the PDF, CDF, Survival rate and Hazard rate function of the BWW distribution. The R code that generate figure 4.1 to 4.4 is in appendix B.

4.1.2 Summary of the Analysis

Table 4.2: MLEs of the parameters from the developed and conventional regression models fitted to the completion of PhD data set, the corresponding SEs (given in parentheses) and p-value in [.].

Parameter	^{a}AGWWRM	^{b}LLWWRM	^{c}LEWWRM	^{d}LWWRM
α	4.150		1.004	
	(0.013)	1	(2.901e-01)	1
	$[< 2e-16^{***}]$		$[< 2e-16^{***}]$	
b	1.025	1.004		
	(0.019)	(2.901e-01)	1	1
	$[< 2e-16^{***}]$	$[< 2e-16^{***}]$		
γ	5.120	2.099	2.099	0.127
	(4.03e-03)	(3.259e-03)	(3.259e-03)	0.006416
	$[< 2e-16^{***}]$	$[< 2e-16^{***}]$	$[< 2e-16^{***}]$	$[< 2e-16^{***}]$
σ	3.210	5.947	5.947	5.070
	(1.006e-07)	(7.188e-02)	(7.188e-02)	(1.172110)
	$[< 2e-16^{***}]$	$[< 2e-16^{***}]$	$[< 2e-16^{***}]$	$[< 2e-16^{***}]$
β_o	2.550	-2.830	-2.830	3.401
	(9.270e-06)	(2.729e-02)	(2.729e-02)	(0.623829)
	$[< 2e-16^{***}]$	$[< 2e-16^{***}]$	$[< 2e-16^{***}]$	$[4.99e-08^{***}]$
β_1	3.250	-2.825	-2.825	3.165
	(9.259e-06)	(6.373e-02)	(6.373e-02)	(0.214746)
	$[< 2e-16^{***}]$	$[< 2e-16^{***}]$	$[< 2e-16^{***}]$	$[< 2e-16^{***}]$
β_2	1.250	-2.819	-2.825	3.213
	(1.065e-05)	(3.470e-02)	(3.470e-2)	(0.301351)
	$[< 2e-16^{***}]$	$[< 2e-16^{***}]$	$[< 2e-16^{***}]$	$[< 2e-16^{***}]$
β_3	4.150	-2.821	-2.821	4.868
	(1.080e-05)	(6.924e-02)	(6.924e-02)	(0.513446)
	$[< 2e-16^{***}]$	$[< 2e-16^{***}]$	$[< 2e-16^{***}]$	$[< 2e-16^{***}]$
β_4	1.310	2.847	2.847	1.168
	(1.752e-06)	(1.027e-01)	(1.027e-01)	(0.050523)
	$[< 2e-16^{***}]$	$[< 2e-16^{***}]$	$[< 2e-16^{***}]$	$[< 2e-16^{***}]$
β_5	5.150	6.045	6.045	5.104
	(7.978e-06)	(1.507e-01)	(1.507e-01)	(0.289001)
	$[< 2e-16^{***}]$	$[< 2e-16^{***}]$	$[< 2e-16^{***}]$	$[< 2e-16^{***}]$

Note: The full name of each modelin table 4.2

^{a}AGWWRM: Alternative Generalised Weighted Weibull Regression Model

^{b}LLWWRM: Log-Lehmann Type II Weighted Weibull Regression Model

^{c}LEWWRM: Log-Exponentiated Weighted Weibull Regression Model

^{d}LWWRM: Log-Weighted Weibull Regression Model

Table 4.3: Table 4.2 Continued

Parameter/model	^{e}LBWRM	^{f}LWRM	^{g}LBNRM	^{h}LNRM
α	2.359		2.031	
	(1.854e-01)	1	(0.0054406)	1
	$[< 2e-16^{***}]$		$[< 2e-16^{***}]$	
b	0.0006		9.070	
	(1.709e-05)	1	(0.0025655)	1
	$[< 2e-16^{***}]$		$[< 2e-16^{***}]$	
γ	1	1	1	1
σ	1.610	4.250	0.838	0.005
	(3.760e-04)	(0.003958)	(0.0004735)	(3.288e-08)
	$[< 2e-16^{***}]$	$[< 2e-16^{***}]$	$[< 2e-16^{***}]$	$[< 2e-16^{***}]$
β_o	-2.796	1.500	3.280	2.363
	(3.931e-04)	(0.080680)	(0.1079018)	(4.377e-04)
	$[< 2e-16^{***}]$	$[< 2e-16^{***}]$	$[< 2e-16^{***}]$	$[< 2e-16^{***}]$
β_1	-4.407	1.450	2.589	-1.077e06
	(1.854e-01)	(0.249595)	(0.1311005)	(6.812e06)
	$[< 2e-16^{***}]$	$[< 2e-16^{***}]$	$[< 2e-16^{***}]$	0.87434
β_2	-3.953	3.100	4.473	-1.182
	(7.059e-01)	(0.160956)	(0.0915384)	(2.212e-04)
	$[2.15e-08^{***}]$	$[< 2e-16^{***}]$	$[< 2e-16^{***}]$	$[< 2e-16^{***}]$
β_3	-1.879	3.500	0.366	1.265e05
	(1.854e-01)	(0.338418)	(0.0656831)	(9.223e-06)
	$[< 2e-16^{***}]$	$[< 2e-16^{***}]$	$[< 2e-16^{***}]$	$[< 2e-16^{***}]$
β_4	-1.764	1.020	6.824	1.540e-06
	(7.298e-01)	(0.021796)	(0.0116190)	(4.967e-07)
	$[< 2e-16^{***}]$	$[< 2e-16^{***}]$	$[< 2e-16^{***}]$	[0.00193]
β_5	-2.480	0.510	0.526	-4.408e-07
	(1.709e-05)	(0.224879)	(0.0502601)	(2.144e-06)
	$[< 2e-16^{***}]$	$[0.0233^{*}]$	$[< 2e-16^{***}]$	[0.83709]

Note: The full name of each model in table 4.2 continued

^{e}LBWRM: Log-Beta Weibull Regression Model; ^{f}LWRM: Log-Weibull Regression Model

^{g}LBNRM: Log-Beta Normal Regression Model; ^{h}LNRM: Log-Normal Regression Model

Table 4.4: Table 4.2 Continued

Parameter/Model	$^{i}LBLLRM$	$^{j}LLRM$	$^{k}LWRM$
α	0.005		
	(0.0001292)	1	1
	$[< 2e-16^{***}]$		
b	5.300		
	(0.153)	1	1
	$[< 2e-16^{***}]$		
γ			0.024
	1	1	(1.105e-04)
			$[< 2e-16^{***}]$
σ	9.554	0.019	8.091
	(0.314)	(0.0006)	(0.803)
	$[< 2e-16^{***}]$	$[< 2e-16^{***}]$	$[< 2e-16^{***}]$
β_0	3.143	-5.606	3.099
	(2.844)	(0.272)	(1.034)
	[0.269]	$[< 2e-16^{***}]$	$[0.00272^{**}]$
β_1	0.323	2.276	-1.396
	(1.827)	(0.072)	(0.683)
	[0.859]	$[< 2e-16^{***}]$	$[< 2e-16^{***}]$
β_2	4.422	0.450	1.142
	(4.053)	(0.250)	(1.098)
	[0.275]	[0.071]	$[< 2e-16^{***}]$
β_3	2.218	-0.350	1.062
	(2.758)	(0.107)	(0.431)
	[0.421]	$[0.001065^{**}]$	$[0.01370^{*}]$
β_4	6.083	-0.106	1.213
	(2.065)	(0.199)	(0.095)
	$[0.00323^{**}]$	[0.594]	$[< 2e-16^{***}]$

Note: The full name of each model in table 4.2 continued and significant codes:

^{i}LBLLRM: Log-Beta Log-Logistic Regression Model;

^{j}LLRM: Log-Logistic Regression Model

^{k}LWRM: Log-Weighted Regression Model

Significant codes: *** significant at .001%, ** significant at 0.1%, * significant at 1%, . significant at 5%, significant at 10% and 1 significant at 100%.

Table 4.5: Log-Likelihood (LogL), Akaike Information Criterion (AIC), Bayesian Information Criterion and Consistent Akaike Information Criterion (CAIC)

Model Selection Criterion	LogL	AIC	BIC	CAIC
[a]AGWWRM	4556387*	-9112754*	-9112751*	-9112751.1*
[b]LLWWRM	795888.7	-1591759	-1591757	-1591756.9
[c]LEWWRM	795888.7	-1591759	-1591757	-1591756.9
[d]LWWRM	55305.57	-110595.1	-110593	-110592.95
[e]LBWRM	237148.3	-474278.6	-474249.5	-474249.5
[f]LWRM	243722.2	-487430.4	-487428.5	-487428.5
[g]LBNRM	1538126	-3076234	-3076205	-3076205
[h]LNRM	777.5911	-1541.182	-1518.564	-1518.527
[i]LBLLRM	2115.622	-4213.244	-4184.164	-4184.116
[j]LLRM	558.3308	-1102.662	-1080.044	-1080.007
[k]LWRM	126411.5	-252807	-252804.8	-3076205

Note: Full name of each model fitted in table 4.3 are stated below:

[a]AGWWRM: Alternative Generalised Weighted Weibull Regression Model

[b]LLWWRM: Log-Lehmann Type II Weighted Weibull Regression Model

[c]LEWWRM: Log-Exponentiated Weighted Weibull Regression Model

[d]LWWRM: Log-Weighted Weibull Regression Model

[e]LBWRM: Log-Beta Weibull Regression Model;

[f]LWRM: Log-Weibull Regression Model

[g]LBNRM: Log-Beta Normal Regression Model;

[h]LNRM: Log-Normal Regression Model [i]LBLLRM: Log-Beta Log-Logistic Regression Model;

[j]LLRM: Log-Logistic Regression Model

[k]LWRM: Log-Weighted Regression Model

Furthermore, we calculated both the skewness and excess kurtosis of the BWW distribution using equations (3.73 and 3.74) above (i.e the third and fourth noncentral moments). The first four ordinary moments, variance, skewness and kurtosis were listed in the table below for selected parameter values of the BWW distribution $(a, b, (\alpha, \beta);\ (\alpha, \beta)$ fixed at $(2, 10)$.

Table 4.6: Moments of the BWW distribution for some parameter values; $\alpha = 2$ and $\beta = 10$

	The BWW $(a, b, 2, 10)$ distribution			
μ'_s	a=b=0.5	a=2,b=0.5	a=1,b=0.5	a=b=2
μ'_1	0.31858	0.75063	0.50042	1.00084
μ'_2	0.32108	0.75655	0.50436	1.00873
μ'_3	0.32578	0.76760	0.57173	1.02338
μ'_4	0.33264	0.78377	0.52251	1.04502
Variance	0.21960	0.19310	0.25394	0.00704
Skewness	0.81212	-1.06276	0.04049	0.35622
Kurtosis	-1.29702	0.74040	-2.91964	0.23832

Again, the skewness and excess kurtosis of the developed AGWW distribution are computed using equations (3.124 and 3.125) above (i.e the third and fourth noncentral moments). The first four ordinary moments, variance, skewness and kurtosis were listed in the table below for selected parameter values of the AGWW distribution $(a, b, \gamma, \mu, \sigma)$; (γ, μ, σ) fixed at (0.5, 3, 1).

Table 4.7: Moments of the AGWW distribution for some parameter values; $\gamma = 0.5,\ \mu = 3$ and $\sigma = 1$

	The AGWW (a,b,0.5,3,1) distribution			
μ'_s	a=b=0.5	a=2,b=0.5	a=1,b=0.5	a=b=2
μ'_1	0.35611	0.83907	0.55938	1.11876
μ'_2	0.42348	0.99779	0.66519	1.33039
μ'_3	0.53052	1.25000	0.83333	1.66667
μ'_4	0.69554	1.63882	1.09255	2.18509
Variance	0.29667	0.29375	0.35228	0.07877
Skewness	1.04230	-0.50360	0.48318	0.09240
Kurtosis	-0.57058	1.01410	-1.52476	-0.11905

Table 4.8: The first four moments of the AGWW, LBW, LW and LWeighted distribution

Parameter	(a, b, 1.5, 3, 1)			
Distribution/Moment	AGWW	LBWD	LWD	LWeightedD
μ_1'	1.04967	1.07720	0.17953	0.23333
μ_2'	1.17785	1.23680	0.20613	0.52000
μ_3'	1.40000	1.50000	0.25000	1.62400
μ_4'	1.75046	1.90877	0.31813	6.59840

AGWW is the Alternative Generalised weighted Weibull Distribution

LBW is the Log-Beta Weibull Distribution,

LW is the Log Weibull Distribution and

LWeighted is the Log-weighted Distribution.

Table 4.9: Table 4.7: The first four moments of the LBN, LN, LLog and LWW distribution

Parameter	(a, b, 1.5, 3, 1)			
Distribution/Moment	LBND	LND	LLogD	LWWD
μ'_1	1.07720	0.17953	0.17953	0.17494
μ'_2	1.23680	0.20613	0.20613	0.19631
μ'_3	1.50000	0.25000	0.25000	0.23333
μ'_4	1.90877	0.31813	0.31813	0.29174

LBN is the Log-Beta Normal Distribution, LN is the Log-Normal Distribution, LLog is the Log Logistic Distribution and LWW is the Log-weighted Weibull Distribution.

Table 4.10: The variance, skewness and kurtosis of the AGWW, LBW, LW, LWeighted, LBN, LN, LLog and LWW distribution

Distribution	Parameter	Variance	Skewness	Kurtosis
AGWWD	$a = 2, b = 2, \gamma = 1.5, \mu = 3, \sigma = 1$	0.07606	0.19099	-0.06967
LBWD	$a = 2, b = 2, \gamma = 1, \mu = 3, \sigma = 1$	0.07644	0.12838	0.03943
LWD	$a = 1, b = 1, \gamma = 1, \mu = 3, \sigma = 1$	0.17390	2.07607	2.79825
LWeightedD	$a = 1, b = 1, \gamma = 1.5, \mu = 1, \sigma = 1$	0.46556	4.04649	21.19263
LBND	$a = 2, b = 2, \gamma = 1, \mu = 3, \sigma = 1$	0.07644	0.12838	0.03943
LND	$a = 1, b = 1, \gamma = 1, \mu = 3, \sigma = 1$	0.17390	2.07607	2.79825
LLogD	$a = 1, b = 1, \gamma = 1, \mu = 3, \sigma = 1$	0.17390	2.07607	2.79825
LWWD	$a = 1, b = 1, \gamma = 1.5, \mu = 3, \sigma = 1$	0.16570	2.09058	2.88941

The R code for the analysis is in appendix B.

4.2 Discussion

Figure (4.1.1) contains scatter plots of exploratory data analysis against each of the explanatory variables to describes the nature of the dependence of the former on the latter. Figure 4.2 contains the duration (time to completion) plot against the individuals, completion line plot, Normal QQ-plot, Box-plot, histogram plot, density plot, ecdf plot, empirical density plots and cumulative distribution plot respectively.

Table 4.1 again describes the nature of the data (non-normal data) and may guide the choice of which distribution among a set of parametric distributions. The skewness and kurtosis results are especially useful for this purpose. A non-zero skewness reveals a lack of symmetry of the empirical distribution (i.e. in a symmetric distribution) as in the case of normal distribution, the coefficient of skewness of a normal distribution is 0), while the kurtosis value quantifies the weight of tails in comparison to the normal distribution for which the kurtosis equals 3 (i.e. in a normal distribution, the coefficient of kurtosis is 3) (Ezequiel 2013 and Delignette-Muller and Dutang 2014); also, figure shows the plots of the probability density function (pdf), the distribution function (cdf), the survival rate function and the hazard rate function of the BWW distribution for different values of the parameters; the plots also feature indication of the level of flexibility, versatility and robustness of the distribution including its ability to accommodate varying types of risk functions.

Moreover, Table 4.2 contains the maximum likelihood estimates of the parameters of the newly developed and conventional models including Alternative Generalised Weighted Weibull Regression Model (AGWWRM), Log-Lehmann Type II Weighted Weibull Regression Model(LLWWRM), Log-Exponentiated Weighted Weibull Regression Model (LEWWRM), Log-Weighted Weibull Regression Model (LWWRM), Log-Beta Weibull Regression Model (LBWRM), Log-Weibull Regression Model (LWRM), Log-Beta Normal Regression Model (LBNRM), Log-Normal Regression Model (LNRM), Log-Beta Log Logistic Regression Model (LBLLRM), Log-Logistic Regression Model (LLRM) and Log-Weighted Regression Model (LWRM); the table also has estimates of basic model parameters and the regression coefficients of factors that influence time to complete PhD programme in the University of Ibadan. The estimates of the model parameters and the factors of the developed models are positive and the p-values under new developed model and its sub-models are of high but varying level of significance; significant levels were as high as 0.000 for some parameters.

Hence, Table 4.3 compared the developed models with extant models using log-likelihood model selection criteria, the Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC) and Consistent Akaike Information Criterion (CAIC). The log-likelihood of the models are all positive impacting negatively on the differential ability of the selection criteria. Meanwhile, the lower the AIC or BIC or CAIC of a model, the more efficient is the model. According to Bozdogan (1987) quoted by Acquah (2010 and 2013), the AIC does not link directly with sample size and it lacks certain properties of asymptotic consistency. It is necessary to formulate a correction factor based on sample size to compensate for the overestimating nature of AIC. BIC and CAIC reflect sample size and have properties of asymptotic consistency. The regression component of the model was estimated as

$$y = 2.550 + 3.250x_1 + 1.250x_2 + 4.150x_3 + 1.310x_4 + 5.150x_5 \tag{4.2}$$

Equation (4.2) could be interpreted as follows:
The parameters of the regression model (estimated $_s$) of the developed model in table 2 above are positive. There will be an increase of 3.250 in y for a unit change in x_1 when variables x_2 to x_5 are held fixed and an increase of 1.250 in y for a unit change in x_2 when variables x_1, x_3 to x_5 are held fixed. Again, there would be an increase of 4.150 in y for a unit change in x_3 when variables x_1, x_2, x_4 and x_5 are held fixed, an increase of 1.310 in y for a unit change in x_4 when variables x_1, x_2, x_3 and x_5 are held fixed, and an increase of 5.150 in y for a unit change in x_5 when variables x_1 to x_4 are held fixed.

We also use dummy trapping (coding) for the fifth factor (x_5) to select the level of the faculty i.e faculty is nominal scale with ten levels taken Arts as baseline Dummy coding is probably the most commonly used coding scheme. It compares each level of the categorical variable to a reference level. The coding by dummy trapping gives:
Faculty of candidate (Arts = 0, Agriculture/Forestry = 7, FBMS = 4, Institute of Education = 2, Pharmacy = 8, Public Health = 6, Science = 5, Social Science.= 9, Technology = 3 and Veterinary Medicine = 1). Therefore, the regression component of the model is

$$y = 1.200 + 1.300x_1 + 0.100x_2 + 1.300x_3 + 0.100x_4 + 0.600x_5 \tag{4.3}$$

The log-likelihood = 336175.4, $AIC = -672330.8$, $BIC = -672298.5$ and $CAIC = -672297.5$. The dummy coding is in appendix B using R.

In addition, table 4.4 consist the calculated results of the skewness and kurtosis of the beta weighted weibull (BWW) distribution. We assigned some selected values to the parameter a and b being positive shape parameters, which control skewness through the relative tail weights and fixed other parameters in distribution.Therefore, when $a = b < 1$, the BWW distribution had positive skewness, if $a > b$, it has negative skewness, for $a < b$, it had positive skewness and negative skewness when $a = b > 1$.In the vein, for $a = b < 1$, $a > b$ and $a < b$ it had negative excess kurtosis; and for $a = b > 1$, it had positive excess kurtosis.

However, table 4.5 showed the computed results of skewness and kurtosis of the alternative Generalised weighted Weibull (AGWW) distribution. We also assigned some selected values to the parameter a and b and fixed other parameters in distribution. The developed AGWW distribution had positive skewness when $a = b < 1$ and $a = b > 1$; and for $a < b$ and $a > b$, it had negative skewness, while it had negative excess kurtosis for $a < b$, $a > b$, $a = b < 1$ and $a = b > 1$.

Hence, the variance, skewness and kurtosis of the AGWW and the six competing distributions were shown in table 4.8 above. AGWW has the smallest variance, then equation (3.124) and (3.125) were used to compute skewness and kurtosis of the AGWW distribution; and compared with the skewness and kurtosis of LBW, LW, LWeighted, LBN, LN, LLog and LWW distribution. Therefore, from the result, the AGWW distribution minimised the skewness (positive) and has negative kurtosis while competing distributions have positive skewness and kurtosis.

Chapter 5

SUMMARY, CONCLUSION AND CONTRIBUTION TO KNOWLEDGE

5.1 Summary

The values of AIC, BIC and CAIC in the proposed Alternative Generalised Weighted Weibull Regression Model were respectively less than existing generalised weighted Weibull regression models. Therefore, the developed AGWWRM regression model provided a better fit than extant models; it has lowest AIC, BIC and CAIC respectively. Hence, the study has contributed a generalised composite of the Beta Weighted Weibull in the Alternative Generalised Weighted Weibull distributions and generated an Alternative Generalized Weighted Weibull Regression Model that is more flexible, versatile and uniformly more efficient than some existing Generalized Weighted Weibull Regressions Models.

5.2 Conclusion

In literature, many/different forms of classical regression models and skewed regression models have been developed to model residual terms that are not normal and heteroscedastic; but these appear to be inappropriate or not valid for several real life phenomena that exhibit a high level of skewness due to inability to accommodate the four types of failure functions (Ortega *et al.* (2009) and Pescim *et al.* (2013)). Moreover, in survival analysis, survival data requires more flexible and robust model that would adequately accommodate and isolate intractable distribution characteristics such that would engender complex skewness and kurticity , this is the reason for Alternative Generalised Weighted Weibull distribution that can embrace or accommodate the four types of function failures,. We studied the performance of the developed model and compared this with some extant models including Log-Weibull Regression Model (LWRM), Log-Normal Regression Model (LNRM), Log-Logistic Regression Model (LLRM), Log-Weighted Regression Model (LWRM), Log-Beta Weibull Regression Model (LBWRM), Log-Beta Normal Regression Model (LBNRM) and Log-Beta Log Logistic Regression Model (LBLLRM), using data on completion of PhD programme with the help of R software code. The results showed that our new Alternative Generalised Weighted Weibull Regression Model (AGWWRM) was an improved tool for statistical modelling and inference.

5.3 Contribution to Knowledge

The limitations of the conventional models included inability to capture the underlying shape of the data discussed by Silva *et al.* (2009) and to accommodate the types of functional failures observed by Ortega *et al.* (2009) and Paul (2011). Also there are an array of alternative parametric models like the Exponential, Weibull, Gompertz and generalised gamma among others; even these models do not often adequately capture the underlying shape of the distribution particularly of a data with extremes and complex tails. The Alternative Generalised Weighted Weibull (AGWW) distribution can more adequately take care of data with these complexities including those with increasing, decreasing, bathtub and unimodal shape and with difficult hazard and survival rate functions. In this wise, this study has contributed meaningfully to advances and development in survival analysis

5.4 Suggestion for Further Study

Some suggested areas for further study:

(a) Extension of the alternative generalised weighted Weibull regression model to data from other fields such as Biology (survival time), Medicine (treatment of effects or the efficacy of drugs), Quality control (lifetime of component), Engineering (failure time) and Credit risk modelling in finance (default time of a firm), among others

(b) Testing for heteroscedasticity and other assumption failures beyond AIC, BIC and CAIC

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APPENDIX A

R Code

The code in R program to generate graphs of the Beta Weighted Weibull (BWW) Distribution and Log-Beta Weighted Weibull (LBWW) Distribution which include: PDF, CDF, SURVIVAL, HAZARD, SKEWNESS and KURTOSIS of the BWWD and PDF, SKEWNESS and KUTOSIS of the LBWWD.

```
a=200

b=10

c=12

d=1.3

x=seq(0,5,0.01)

bww1.pdf=function(x,a,b,c,d){

k1=(1-exp(-x^d))

k2=(1-exp(-(1+c)*x^d))/(c+1)

k3=((c+1)/c)*k1

k6=(1-exp(-c*x^d))

k7=((c+1)/c)*d*x^(d-1)*exp(-x^d)

k8=(k3-k2)^(a-1)
```

```
k9=(1-(k3-k2))^(b-1)

k4=beta(a,b)

k5=1/k4

bww1.pdf=k5*k8*k9*k7*k6

}

plot(x,bww1.pdf(x,200,10,12,1.3),col="red",ylim=c(0,3.5),type="p",

xlab="x",ylab="pdf BWWD")

lines(x,bww1.pdf(x,150,8,12,1.3),col="blue",lty=2)

lines(x,bww1.pdf(x,100,6,12,1.3),col="black",lty=4)

lines(x,bww1.pdf(x,10,5,12,1.3),col="green",lty=6)

legend("topright",inset=0.02,col=c("red","blue","black","green"),

legend=c("a=200,b=10,c=12,d=1.3","a=150,b=8,c=12,d=1.3",

"a=100,b=6,c=12,d=1.3","a=10,b=5,c=12,d=1.3"),

lty=1:2:4)
```

When $a = 1$

```
a=1

b=10

c=12

d=1.3

x=seq(0,5,0.01)

bww1.pdf=function(x,a,b,c,d){

k1=(1-exp(-x^d))

k2=(1-exp(-(1+c)*x^d))/(c+1)

k3=((c+1)/c)*k1
```

```
k6=(1-exp(-c*x^d))

k7=((c+1)/c)*d*x^(d-1)*exp(-x^d)

k8=(k3-k2)^(a-1)

k9=(1-(k3-k2))^(b-1)

k4=beta(a,b)

k5=1/k4

bww1.pdf=k5*k8*k9*k7*k6

}

plot(x,bww1.pdf(x,1,10,12,1.3),col="red",ylim=c(0,3.5),type="p",

xlab="x",ylab="pdf LWWD (when a = 1)")

lines(x,bww1.pdf(x,1,8,12,1.3),col="blue",lty=2)

lines(x,bww1.pdf(x,1,6,12,1.3),col="black",lty=4)

lines(x,bww1.pdf(x,1,5,12,1.3),col="green",lty=6)

legend("topright",inset=0.02,col=c("red","blue","black","green"),

legend=c("a=1,b=10,c=12,d=1.3","a=1,b=8,c=12,d=1.3",

"a=1,b=6,c=12,d=1.3","a=1,b=5,c=12,d=1.3"),

lty=1:2:4)
```

When $b = 1$

```
a=200

b=1

c=12

d=1.3

x=seq(0,5,0.01)

bww1.pdf=function(x,a,b,c,d){
```

```
k1=(1-exp(-x^d))

k2=(1-exp(-(1+c)*x^d))/(c+1)

k3=((c+1)/c)*k1

k6=(1-exp(-c*x^d))

k7=((c+1)/c)*d*x^(d-1)*exp(-x^d)

k8=(k3-k2)^(a-1)

k9=(1-(k3-k2))^(b-1)

k4=beta(a,b)

k5=1/k4

bww1.pdf=k5*k8*k9*k7*k6

}

plot(x,bww1.pdf(x,200,1,12,1.3),col="red",ylim=c(0,3.5),

type="p",xlab="x",ylab="pdf EWWD (when b = 1)")

lines(x,bww1.pdf(x,150,1,12,1.3),col="blue",lty=2)

lines(x,bww1.pdf(x,100,1,12,1.3),col="black",lty=4)

lines(x,bww1.pdf(x,10,1,12,1.3),col="green",lty=6)

legend("topleft",inset=0.02,col=c("red","blue","black","green"),

legend=c("a=200,b=1,c=12,d=1.3","a=150,b=1,c=12,d=1.3",

"a=100,b=1,c=12,d=1.3","a=10,b=1,c=12,d=1.3"),

lty=1:2:4)
```

When $a = b = 1$

```
a=1

b=1

c=12
```

```
d=1.3

x=seq(0,5,0.01)

bww1.pdf=function(x,a,b,c,d){

k1=(1-exp(-x^d))

k2=(1-exp(-(1+c)*x^d))/(c+1)

k3=((c+1)/c)*k1

k6=(1-exp(-c*x^d))

k7=((c+1)/c)*d*x^(d-1)*exp(-x^d)

k8=(k3-k2)^(a-1)

k9=(1-(k3-k2))^(b-1)

k4=beta(a,b)

k5=1/k4

bww1.pdf=k5*k8*k9*k7*k6

}

plot(x,bww1.pdf(x,1,1,12,1.3),col="red",ylim=c(0,3.5),

type="p",xlab="x",ylab="pdf WW (when a = b = 1)")

lines(x,bww1.pdf(x,1,1,12,1.3),col="blue",lty=2)

lines(x,bww1.pdf(x,1,1,12,1.3),col="black",lty=4)

lines(x,bww1.pdf(x,1,1,12,1.3),col="green",lty=6)

legend("topleft",inset=0.02,col=c("red","blue","black","green"),

legend=c("a=1,b=1,c=12,d=1.3","a=1,b=1,c=12,d=1.3",

"a=1,b=1,c=12,d=1.3","a=1,b=1,c=12,d=1.3"),

lty=1:2:4)
```

Combine

```
a=200

b=10

c=12

d=1.3

x=seq(0,5,0.01)

bww1.pdf=function(x,a,b,c,d){

k1=(1-exp(-x^d))

k2=(1-exp(-(1+c)*x^d))/(c+1)

k3=((c+1)/c)*k1

k6=(1-exp(-c*x^d))

k7=((c+1)/c)*d*x^(d-1)*exp(-x^d)

k8=(k3-k2)^(a-1)

k9=(1-(k3-k2))^(b-1)

k4=beta(a,b)

k5=1/k4

bww1.pdf=k5*k8*k9*k7*k6

}

plot(x,bww1.pdf(x,200,10,12,1.3),col="red",ylim=c(0,3.5),

type="p",xlab="x",ylab="pdf")

lines(x,bww1.pdf(x,1,10,12,1.3),col="blue",lty=2)

lines(x,bww1.pdf(x,200,1,12,1.3),col="black",lty=4)

lines(x,bww1.pdf(x,200,10,12,1),col="purple",lty=6)

lines(x,bww1.pdf(x,1,1,12,1.3),col="green",lty=8)

legend("topright",inset=0.02,
```

```
col=c("red","blue","black","purple","green"),
legend=c("a=200,b=10 (BWWD)","a=1,b=10 (LWWD)","a=200,b=1 (EWWD)",
"a=200,b=10,d=1 (BWED)","a=1,b=1 (WWD)"),lty=1:2:4)
```

CDF

```
a=200
b=10
c=4
d=2.5
x=seq(0,5,0.01)
bww1.cdf=function(x,a,b,c,d){
beta(a,b)
k1=(c+1/c)
k2=(1-exp(-x^d))
k3=(1-exp(-(1+c)*x^d))/(c+1)
k4=k1*(k2-k3)
k5=k4
k6=beta(a,b)
k7=k5*k6
bww1.cdf=k7/k6
}
plot(x,bww1.cdf(x,200,10,4,2.5),col="red",ylim=c(0,3.5),
type="l",xlab="x",ylab="cdf (BWWD)")
lines(x,bww1.cdf(x,150,8,3,2),col="blue",lty=2)
lines(x,bww1.cdf(x,100,6,3.5,1.5),col="black",lty=4)
```

```
lines(x,bww1.cdf(x,10,5,2.5,1.3),col="green",lty=6)

legend("bottomright",inset=0.02,col=c("red","blue","black","green"),

legend=c("a=200,b=10,c=4,d=2.5,","a=150,b=8,c=3,d=2,",

"a=100,b=6,c=3.5,d=1.5,","a=10,b=5,c=2.5,d=1.3,"),

lty=1:2:4)
```

When $a = 1$

```
a=1

b=10

c=4

d=2.5

x=seq(0,5,0.01)

bww1.cdf=function(x,a,b,c,d){

beta(a,b)

k1=(c+1/c)

k2=(1-exp(-x^d))

k3=(1-exp(-(1+c)*x^d))/(c+1)

k4=k1*(k2-k3)

k5=k4

k6=beta(a,b)

k7=k5*k6

bww1.cdf=k7/k6

}

plot(x,bww1.cdf(x,200,10,4,2.5),col="red",ylim=c(0,3.5),

type="l",xlab="x",ylab="cdf LWWD (when a = 1)")
```

```
lines(x,bww1.cdf(x,1,8,3,2),col="blue",lty=2)

lines(x,bww1.cdf(x,1,6,3.5,1.5),col="black",lty=4)

lines(x,bww1.cdf(x,1,5,2.5,1.3),col="green",lty=6)

legend("bottomright",inset=0.02,col=c("red","blue","black","green"),

legend=c("a=1,b=10,c=4,d=2.5,","a=1,b=8,c=3,d=2,",

"a=1,b=6,c=3.5,d=1.5,","a=1,b=5,c=2.5,d=1.3,"),

lty=1:2:4)
```

When $b = 1$

```
a=200

b=1

c=4

d=2.5

x=seq(0,5,0.01)

bww1.cdf=function(x,a,b,c,d){

beta(a,b)

k1=(c+1/c)

k2=(1-exp(-x^d))

k3=(1-exp(-(1+c)*x^d))/(c+1)

k4=k1*(k2-k3)

k5=k4

k6=beta(a,b)

k7=k5*k6

bww1.cdf=k7/k6

}
```

```
plot(x,bww1.cdf(x,200,1,4,2.5),col="red",ylim=c(0,3.5),
type="l",xlab="x",ylab="cdf EWWD (when b = 1)")
lines(x,bww1.cdf(x,150,1,3,2),col="blue",lty=2)
lines(x,bww1.cdf(x,100,1,3.5,1.5),col="black",lty=4)
lines(x,bww1.cdf(x,10,1,2.5,1.3),col="green",lty=6)
legend("bottomright",inset=0.02,col=c("red","blue","black","green"),
legend=c("a=200,b=1,c=4,d=2.5,","a=200,b=1,c=3,d=2,",
"a=200,b=1,c=3.5,d=1.5,","a=200,b=1,c=2.5,d=1.3,"),
lty=1:2:4)
```

When $a = 1, b = 1$

```
a=1
b=1
c=4
d=2.5
x=seq(0,5,0.01)
bww1.cdf=function(x,a,b,c,d){
beta(a,b)
k1=(c+1/c)
k2=(1-exp(-x^d))
k3=(1-exp(-(1+c)*x^d))/(c+1)
k4=k1*(k2-k3)
k5=k4
k6=beta(a,b)
k7=k5*k6
```

```
bww1.cdf=k7/k6

}

plot(x,bww1.cdf(x,1,1,4,2.5),col="red",ylim=c(0,3.5),

type="l",xlab="x",ylab="cdf WWD (when a = b = 1)")

lines(x,bww1.cdf(x,1,1,3,2),col="blue",lty=2)

lines(x,bww1.cdf(x,1,1,3.5,1.5),col="black",lty=4)

lines(x,bww1.cdf(x,1,1,2.5,1.3),col="green",lty=6)

legend("bottomright",inset=0.02,col=c("red","blue","black","green"),

legend=c("a=1,b=1,c=4,d=2.5,","a=1,b=1,c=3,d=2,",

"a=1,b=1,c=3.5,d=1.5,","a=1,b=1,c=2.5,d=1.3,"),

lty=1:2:4)
```

Combine CDF

```
a=200

b=10

c=4

d=2.5

x=seq(0,5,0.01)

bww1.cdf=function(x,a,b,c,d){

beta(a,b)

k1=(c+1/c)

k2=(1-exp(-x^d))

k3=(1-exp(-(1+c)*x^d))/(c+1)

k4=k1*(k2-k3)

k5=k4
```

```
k6=beta(a,b)

k7=k5*k6

bww1.cdf=k7/k6

}

plot(x,bww1.cdf(x,200,10,4,2.5),col="red",ylim=c(0,3.5),

type="p",xlab="x",ylab="cdf")

lines(x,bww1.cdf(x,1,10,3,2),col="blue",lty=2)

lines(x,bww1.cdf(x,200,1,3.5,1.5),col="black",lty=4)

lines(x,bww1.cdf(x,1,1,2.5,1.3),col="green",lty=6)

legend("bottomright",inset=0.02,col=c("red","blue","black","green"),

legend=c("a=200,b=10,c=4,d=2.5, (BWWD)","a=1,b=10,c=4,d=2.5,

(LWWD)","a=200,b=1,c=4,d=2.5, (EWWD)",

"a=1,b=1,c=2.5,d=1.3, (WWD)"),lty=1:2:4)
```

Survival Rate

```
a=2

b=3

c=4

d=2

x=seq(0,10,0.01)

bww1.sur=function(x,a,b,c,d){

beta(a,b)

k1=(c+1/c)

k2=(1-exp(-x^d))

k3=(1-exp(-(1+c)*x^d))/(c+1)
```

```
k4=k1*(k2-k3)

k5=k4

k6=beta(a,b)

k7=k5*k6

k8=(k6-k5)

bww1.sur=k8/k6

}

plot(x,bww1.sur(x,2,3,4,2),col="red",ylim=c(-40,0),type="l",

xlab="x",ylab="survival (BWWD)")

lines(x,bww1.sur(x,1.8,3.5,3,3),col="blue",lty=2)

lines(x,bww1.sur(x,1.5,4,3,2.5),col="black",lty=4)

lines(x,bww1.sur(x,1.6,5,2.5,3),col="green",lty=6)

legend("topright",inset=0.02,col=c("red","blue","black","green"),

legend=c("a=2,b=3,c=4,d=2","a=1.8,b=3.5,c=3,d=3",

"a=1.5,b=4,c=3,d=2.5","a=1.6,b=5,c=2.5,d=3"),

lty=1:2:4)
```

When $a = 1$

```
a=1

b=3

c=4

d=2

x=seq(0,10,0.01)

lww1.sur=function(x,a,b,c,d){

beta(a,b)
```

```
k1=(c+1/c)

k2=(1-exp(-x^d))

k3=(1-exp(-(1+c)*x^d))/(c+1)

k4=k1*(k2-k3)

k5=k4

k6=beta(a,b)

k7=k5*k6

k8=(k6-k5)

lww1.sur=k8/k6

}

plot(x,lww1.sur(x,1,3,4,2),col="red",ylim=c(-40,0),

type="l",xlab="x",ylab="survival LWWD (when a = 1)")

lines(x,lww1.sur(x,1,3.5,3,3),col="blue",lty=2)

lines(x,lww1.sur(x,1,4,3,2.5),col="black",lty=4)

lines(x,lww1.sur(x,1,5,2.5,3),col="green",lty=6)

legend("bottomright",inset=0.02,col=c("red","blue","black","green"),

legend=c("a=1,b=3,c=4,d=2","a=1,b=3.5,c=3,d=3",

"a=1,b=4,c=3,d=2.5","a=1,b=5,c=2.5,d=3"),

lty=1:2:4)
```

When $b = 1$

```
a=2

b=1

c=4

d=2
```

```
x=seq(0,10,0.01)

eww1.sur=function(x,a,b,c,d){

beta(a,b)

k1=(c+1/c)

k2=(1-exp(-x^d))

k3=(1-exp(-(1+c)*x^d))/(c+1)

k4=k1*(k2-k3)

k5=k4

k6=beta(a,b)

k7=k5*k6

k8=(k6-k5)

eww1.sur=k8/k6

}

plot(x,eww1.sur(x,2,1,4,2),col="red",ylim=c(-40,0),type="l",

xlab="x",ylab="survival EWWD (when b = 1)")

lines(x,eww1.sur(x,1.8,1,3,3),col="blue",lty=2)

lines(x,eww1.sur(x,1.5,1,3,2.5),col="black",lty=4)

lines(x,eww1.sur(x,1.6,1,2.5,3),col="green",lty=6)

legend("bottomright",inset=0.02,col=c("red","blue","black","green"),

legend=c("a=2,b=1,c=4,d=2","a=1.8,b=1,c=3,d=3",

"a=1.5,b=1,c=3,d=2.5","a=1.6,b=1,c=2.5,d=3"),

lty=1:2:4)
```

Survival Rate

```
a=2
```

```
b=3
c=4
d=2
x=seq(0,10,0.01)
ww1.sur=function(x,a,b,c,d){
beta(a,b)
k1=(c+1/c)
k2=(1-exp(-x^d))
k3=(1-exp(-(1+c)*x^d))/(c+1)
k4=k1*(k2-k3)
k5=k4
k6=beta(a,b)
k7=k5*k6
k8=(k6-k5)
ww1.sur=k8/k6
}
plot(x,ww1.sur(x,1,1,4,2),col="red",ylim=c(-40,0),type="p",
xlab="x",ylab="survival WWD (when a = b = 1)")
lines(x,ww1.sur(x,1,1,3,3),col="blue",lty=2)
lines(x,ww1.sur(x,1,1,3,2.5),col="black",lty=4)
lines(x,ww1.sur(x,1,1,2.5,3),col="green",lty=6)
legend("bottomright",inset=0.02,col=c("red","blue","black","green"),
legend=c("a=1,b=1,c=4,d=2","a=1,b=1,c=3,d=3",
"a=1,b=1,c=3,d=2.5","a=1,b=1,c=2.5,d=3"),lty=1:2:4)
```

Combined Survival

```
a=2

b=3

c=4

d=2

x=seq(0,10,0.01)

ww1.sur=function(x,a,b,c,d){

beta(a,b)

k1=(c+1/c)

k2=(1-exp(-x^d))

k3=(1-exp(-(1+c)*x^d))/(c+1)

k4=k1*(k2-k3)

k5=k4

k6=beta(a,b)

k7=k5*k6

k8=(k6-k5)

ww1.sur=k8/k6

}

plot(x,ww1.sur(x,2,3,4,2),col="red",ylim=c(-40,0),type="p",

xlab="x",ylab="survival (BWWD, LWWD, EWWD and WWD)")

lines(x,ww1.sur(x,1,3.5,3,3),col="blue",lty=2)

lines(x,ww1.sur(x,1.5,1,3,2.5),col="black",lty=4)

lines(x,ww1.sur(x,1,1,2.5,3),col="green",lty=6)
```

```
legend("bottomright",inset=0.02,col=c("red","blue","black","green"),
legend=c("a=2,b=3,c=4,d=2,(BWWD)","a=1,b=3.5,c=3,d=3,(LWWD)",
"a=1.5,b=1,c=3,d=2.5,(EWWD)","a=1,b=1,c=2.5,d=3,(WWD)"),lty=1:2:4)
```

Harzard Rate

```
a=1.3
b=0.6
c=1
d=2
x=seq(0,5,0.01)
bww1.haz=function(x,a,b,c,d){
beta(a,b)
k1=(c+1/c)
k2=(1-exp(-x^d))
k3=(1-exp(-(1+c)*x^d))/(c+1)
k4=k1*(k2-k3)
k5=k4
k6=beta(a,b)
k7=k5*k6
k8=(k6-k5)
k9=(d*x^(d-1))*(exp(-x^d))*(1-exp(-c*x^d))
k10=k1*k9
k11=1/k6
k12=(k5)^(a-1)
k13=(1-k5)^(b-1)
```

```
k14=(k6-k7)

bww1.haz=(k12*k13*k10)/k14

}

plot(x,bww1.haz(x,1.3,0.6,1,2),col="red",

type="p",xlab="x",ylab="hazard")

lines(x,bww1.haz(x,1,0.6,1,2),col="blue",lty=2)

lines(x,bww1.haz(x,1.3,1,1,2),col="black",lty=4)

lines(x,bww1.haz(x,1,1,1,2),col="green",lty=6)

legend("topleft",inset=0.02,col=c("red","blue","black","green"),

legend=c("a=1.3,b=0.6,c=1,d=2, (BWWD)","a=1,b=0.6,c=1,

d=2, (LWWD)","a=1.3,b=1,c=1,d=2, (EWWD)",

"a=1,b=1,c=1,d=2, (WWD)"),lty=1:2:4)
```

When $a = 1$

```
a=2

b=0.6

c=1

d=2

x=seq(0,5,0.01)

bww1.haz=function(x,a,b,c,d){

beta(a,b)

k1=(c+1/c)

k2=(1-exp(-x^d))

k3=(1-exp(-(1+c)*x^d))/(c+1)

k4=k1*(k2-k3)
```

```
k5=k4

k6=beta(a,b)

k7=k5*k6

k8=(k6-k5)

k9=(d*x^(d-1))*(exp(-x^d))*(1-exp(-c*x^d))

k10=k1*k9

k11=1/k6

k12=(k5)^(a-1)

k13=(1-k5)^(b-1)

k14=(k6-k7)

bww1.haz=(k12*k13*k10)/k14

}

plot(x,bww1.haz(x,2,0.6,1,2),col="red",main="The HAZARD Plot of

BWWD",ylim=c(0,1200000),type="l",xlab="x",ylab="haz")

lines(x,bww1.haz(x,18,4,2.2,0.4),col="blue",lty=2)

lines(x,bww1.haz(x,20,4,2.1,0.4),col="black",lty=4)

lines(x,bww1.haz(x,20.2,4,2.1,0.4),col="darkblue",lty=6)

legend("topleft",inset=0.02,col=c("red","blue","black","darkblue"),

legend=c("a=17,b=4,c=2.2,d=0.4","a=18,b=4,c=2.2,d=0.4",

"a=20,b=4,c=2.1,d=0.4.5","a=20.2,b=4,c=2.1,d=0.4"),lty=1:2:4)
```

Skewness

```
a=5

b=5

c=0.5
```

```
d=3
i=1
x=seq(0,5,0.01)
m=exp(-x^d)
k=gamma
bww1.sk=function(x,a,b,c,d,i){
k1=(1+c(1-m)-n^(-1)*(1-m^(1+c)))^a*(i+1)-1/c*beta(a,b)
k2=(k1*(1+c-(1+c)^(-3/d)*k(1+(3/d)))-
3*(1+c-(1+c)^(-1/d)*k(1+(1/d)))*
(1+c-(1+c)^(-2/d)*k(1+(2/d)))+
2*(1+c-(1+c)^(-1/d)*k(1+(1/d)))^3)
k3=(k1*((1+c-(1+c)^(-2/d)*k(1+(2/d)))-
(1+c-(1+c)^(-1/d)*k(1+(1/d)))^3))
k4=k2/k3
bww1.sk=k4
}
plot(x,bww1.sk(x,5,5,0.5,3,1),col="red",main="(i)",ylim=c(-1,5),
type="l",xlab="x",ylab= "Skewness")
lines(x,bww1.sk(x,10,10,30,2.5,1),col="blue",lty=2)
lines(x,bww1.sk(x,20,20,20,2.5,1),col="black",lty=4)
lines(x,bww1.sk(x,50,50,30,2.5,1),col="green",lty=6)
legend("topright",inset=0.02,col=c("red","blue","black","green"),
legend=c("a=b=5","a=b=10","a=b=20","a=b=50"),lty=1:2:4)
a=2
```

```
b=2

c=0.5

d=3

i=1

x=seq(0,5,0.01)

m=exp(-x^d)

k=gamma

bww1.sk=function(x,a,b,c,d,i){

k1=(1+c(1-m)-n^(-1)*(1-m^(1+c)))^a*(i+1)-1/c*beta(a,b)

k2=(k1*(1+c-(1+c)^(-3/d)*k(1+(3/d)))-3*(1+c-(1+c)^(-1/d)*

k(1+(1/d)))*(1+c-(1+c)^(-2/d)*k(1+(2/d)))+

2*(1+c-(1+c)^(-1/d)*k(1+(1/d)))^3)

k3=(k1*((1+c-(1+c)^(-2/d)*k(1+(2/d)))-

(1+c-(1+c)^(-1/d)*k(1+(1/d)))^3))

k4=k2/k3

bww1.sk=k4

}

plot(x,bww1.sk(x,2,2,0.5,3,1),col="red",main="(ii)",

ylim=c(-1,5),type="l",xlab="x",ylab= "Skewness")

lines(x,bww1.sk(x,3,3,30,2.5,1),col="blue",lty=2)

lines(x,bww1.sk(x,4,4,20,2.5,1),col="black",lty=4)

lines(x,bww1.sk(x,5,5,30,2.5,1),col="green",lty=6)

legend("topright",inset=0.02,col=c("red","blue","black","green"),

legend=c("a = b = 2","a = b = 3","a = b = 4","a = b = 5"),lty=1:2:4)
```

```
a=1

b=2

c=0.5

d=3

i=1

x=seq(0,5,0.01)

m=exp(-x^d)

k=gamma

bww1.sk=function(x,a,b,c,d,i){

k1=(1+c(1-m)-n^(-1)*(1-m^(1+c)))^a*(i+1)-1/c*beta(a,b)

k2=(k1*(1+c-(1+c)^(-3/d)*k(1+(3/d)))-3*(1+c-(1+c)^(-1/d)*k(1+(1/d)))*

(1+c-(1+c)^(-2/d)*k(1+(2/d)))+2*(1+c-(1+c)^(-1/d)*k(1+(1/d)))^3)

k3=(k1*((1+c-(1+c)^(-2/d)*k(1+(2/d)))-

(1+c-(1+c)^(-1/d)*k(1+(1/d)))^3))

k4=k2/k3

bww1.sk=k4

}

plot(x,bww1.sk(x,1,2,0.5,3,1),col="red",main="(i)",

ylim=c(-1,5),type="l",xlab="x",ylab= "Skewness")

lines(x,bww1.sk(x,1,3,30,2.5,1),col="blue",lty=2)

lines(x,bww1.sk(x,1,4,20,2.5,1),col="black",lty=4)

lines(x,bww1.sk(x,1,5,30,2.5,1),col="green",lty=6)

legend("topright",inset=0.02,col=c("red","blue","black","green"),

legend=c("b = 2","b = 3","b = 4","b = 5"),lty=1:2:4)
```

```
a=2

b=1

c=0.5

d=3

i=1

x=seq(0,5,0.01)

m=exp(-x^d)

k=gamma

bww1.sk=function(x,a,b,c,d,i){

k1=(1+c(1-m)-n^(-1)*(1-m^(1+c)))^a*(i+1)-1/c*beta(a,b)

k2=(k1*(1+c-(1+c)^(-3/d)*k(1+(3/d)))-3*(1+c-(1+c)^(-1/d)*k(1+(1/d)))*

(1+c-(1+c)^(-2/d)*k(1+(2/d)))+2*(1+c-(1+c)^(-1/d)*k(1+(1/d)))^3)

k3=(k1*((1+c-(1+c)^(-2/d)*k(1+(2/d)))-

(1+c-(1+c)^(-1/d)*k(1+(1/d)))^3))

k4=k2/k3

bww1.sk=k4

}

plot(x,bww1.sk(x,2,1,0.5,3,1),col="red",main="(ii)",

ylim=c(-1,5),type="l",xlab="x",ylab= "Skewness")

lines(x,bww1.sk(x,3,1,30,2.5,1),col="blue",lty=2)

lines(x,bww1.sk(x,4,1,20,2.5,1),col="black",lty=4)

lines(x,bww1.sk(x,5,1,30,2.5,1),col="green",lty=6)

legend("topright",inset=0.02,col=c("red","blue","black","green"),

legend=c("a = 2","a = 3","a = 4","a = 5"),lty=1:2:4)
```

Kurtosis

```
a=5

b=5

c=0.5

d=3

i=1

x=seq(0,5,0.01)

m=exp(-x^d)

k=gamma

bww1.kt=function(x,a,b,c,d,i){

k1=(1+c(1-m)-(1+c)^(-1)*(1-m^(1+c)))^a*(i+1)-1/c*beta(a,b)

k2=k1*(1+c-(1+c)^(-4/d)*k(1+(4/d)))

k3=4*(k1*(1+c-(1+c)^(-1/d)*k(1+(1/d))))

k4=k1*(1+c-(1+c)^(-3/d)*k(1+(3/d)))

k5=6*((1+c-(1+c)^(-1/d)*k(1+(1/d))))^2

k6=k1*(1+c-(1+c)^(-2/d)*k(1+(2/d)))

k7=3*((1+c-(1+c)^(-1/d)*k(1+(1/d))))^4

k8=k2-(k3*k4)+(k5*k6)-k7

k9=(k1*((1+c-(1+c)^(-2/d)*k(1+(2/d)))-

(1+c-(1+c)^(-1/d)*k(1+(1/d)))^2))

k10=(k8/k9)-3

bww1.kt=k10

}
```

```
plot(x,bww1.kt(x,5,5,0.5,3,1),col="red",ylim=c(5,30),
main="(i)",type="l",xlab="x",ylab= "Kurtosis")
lines(x,bww1.kt(x,10,10,0.4,3,1),col="blue",lty=2)
lines(x,bww1.kt(x,20,20,0.3,3,1),col="black",lty=4)
lines(x,bww1.kt(x,50,50,0.2,3,1),col="green",lty=6)
legend("topright",inset=0.02,col=c("red","blue","black","darkblue"),
legend=c("a = b = 5,","a = b =10,",
"a = b = 20,","a = b = 50"),lty=1:2:4)
a=2
b=2
c=0.5
d=3
i=1
x=seq(0,5,0.01)
m=exp(-x^d)
k=gamma
bww1.kt=function(x,a,b,c,d,i){
k1=(1+c(1-m)-(1+c)^(-1)*(1-m^(1+c)))^a*(i+1)-1/c*beta(a,b)
k2=k1*(1+c-(1+c)^(-4/d)*k(1+(4/d)))
k3=4*(k1*(1+c-(1+c)^(-1/d)*k(1+(1/d))))
k4=k1*(1+c-(1+c)^(-3/d)*k(1+(3/d)))
k5=6*((1+c-(1+c)^(-1/d)*k(1+(1/d))))^2
k6=k1*(1+c-(1+c)^(-2/d)*k(1+(2/d)))
k7=3*((1+c-(1+c)^(-1/d)*k(1+(1/d))))^4
```

```
k8=k2-(k3*k4)+(k5*k6)-k7

k9=(k1*((1+c-(1+c)^(-2/d)*k(1+(2/d)))-

(1+c-(1+c)^(-1/d)*k(1+(1/d)))^2))

k10=(k8/k9)-3

bww1.kt=k10

}

plot(x,bww1.kt(x,2,2,0.5,3,1),col="red",ylim=c(5,30),main="(ii)",

type="l",xlab="x",ylab= "Kurtosis")

lines(x,bww1.kt(x,3,3,0.4,3,1),col="blue",lty=2)

lines(x,bww1.kt(x,4,4,0.3,3,1),col="black",lty=4)

lines(x,bww1.kt(x,5,5,0.2,3,1),col="green",lty=6)

legend("bottomright",inset=0.02,

col=c("red","blue","black","darkblue"),

legend=c("a = b = 2,","a = b =3,","a = b = 4,","a = b = 5"),

lty=1:2:4)


a=1

b=2

c=0.5

d=3

i=1

x=seq(0,5,0.01)

m=exp(-x^d)

k=gamma
```

```
bww1.kt=function(x,a,b,c,d,i){

k1=(1+c(1-m)-(1+c)^(-1)*(1-m^(1+c)))^a*(i+1)-1/c*beta(a,b)

k2=k1*(1+c-(1+c)^(-4/d)*k(1+(4/d)))

k3=4*(k1*(1+c-(1+c)^(-1/d)*k(1+(1/d))))

k4=k1*(1+c-(1+c)^(-3/d)*k(1+(3/d)))

k5=6*((1+c-(1+c)^(-1/d)*k(1+(1/d))))^2

k6=k1*(1+c-(1+c)^(-2/d)*k(1+(2/d)))

k7=3*((1+c-(1+c)^(-1/d)*k(1+(1/d))))^4

k8=k2-(k3*k4)+(k5*k6)-k7

k9=(k1*((1+c-(1+c)^(-2/d)*k(1+(2/d)))-

(1+c-(1+c)^(-1/d)*k(1+(1/d)))^2))

k10=(k8/k9)-3

bww1.kt=k10

}

plot(x,bww1.kt(x,1,2,0.5,3,1),col="red",main="(iii)",

type="l",xlab="x",ylab= "Kurtosis")

lines(x,bww1.kt(x,1,3,0.4,3,1),col="blue",lty=2)

lines(x,bww1.kt(x,1,4,0.3,3,1),col="black",lty=4)

lines(x,bww1.kt(x,1,5,0.2,3,1),col="green",lty=6)

legend("topright",inset=0.02,col=c("red","blue","black","darkblue"),

legend=c("b=2","b=3","b=4","b=5"),

lty=1:2:4)

a=2

b=1
```

```
c=0.5

d=3

i=1

x=seq(0,5,0.01)

m=exp(-x^d)

k=gamma

bww1.kt=function(x,a,b,c,d,i){

k1=(1+c(1-m)-(1+c)^(-1)*(1-m^(1+c)))^a*(i+1)-1/c*beta(a,b)

k2=k1*(1+c-(1+c)^(-4/d)*k(1+(4/d)))

k3=4*(k1*(1+c-(1+c)^(-1/d)*k(1+(1/d))))

k4=k1*(1+c-(1+c)^(-3/d)*k(1+(3/d)))

k5=6*((1+c-(1+c)^(-1/d)*k(1+(1/d))))^2

k6=k1*(1+c-(1+c)^(-2/d)*k(1+(2/d)))

k7=3*((1+c-(1+c)^(-1/d)*k(1+(1/d))))^4

k8=k2-(k3*k4)+(k5*k6)-k7

k9=(k1*((1+c-(1+c)^(-2/d)*k(1+(2/d)))-

(1+c-(1+c)^(-1/d)*k(1+(1/d)))^2))

k10=(k8/k9)-3

bww1.kt=k10

}

plot(x,bww1.kt(x,2,1,0.5,3,1),col="red",

main="(iv)",type="l",xlab="x",ylab= "Kurtosis")

lines(x,bww1.kt(x,3,1,0.4,3,1),col="blue",lty=2)

lines(x,bww1.kt(x,4,1,0.3,3,1),col="black",lty=4)
```

```
lines(x,bww1.kt(x,5,1,0.2,3,1),col="green",lty=6)

legend("topright",inset=0.02,col=c("red","blue","black","darkblue"),

legend=c("a=2","a=3","a=4","a=5"),

lty=1:2:4)

a=1

b=1

c=2

d=10

i=1

x=seq(0,10,0.01)

m=exp(-x^d)

k=gamma

bww1.kt=function(x,a,b,c,d,i){

k1=(1+c(1-m)-(1+c)^(-1)*(1-m^(1+c)))^a*(i+1)-1/c*beta(a,b)

k2=k1*(1+c-(1+c)^(-4/d)*k(1+(4/d)))

k3=4*(k1*(1+c-(1+c)^(-1/d)*k(1+(1/d))))

k4=k1*(1+c-(1+c)^(-3/d)*k(1+(3/d)))

k5=6*((1+c-(1+c)^(-1/d)*k(1+(1/d))))^2

k6=k1*(1+c-(1+c)^(-2/d)*k(1+(2/d)))

k7=3*((1+c-(1+c)^(-1/d)*k(1+(1/d))))^4

k8=k2-(k3*k4)+(k5*k6)-k7

k9=(k1*((1+c-(1+c)^(-2/d)*k(1+(2/d)))-

(1+c-(1+c)^(-1/d)*k(1+(1/d)))^2))

k10=(k8/k9)-3
```

```
bww1.kt=k10
}
plot(x,bww1.kt(x,1,1,2,10,1),col="red",ylim=c(0,5),
main="(i)",type="l",xlab="x",ylab= "Kurtosis")
lines(x,bww1.kt(x,0.5,50,2,10,1),col="blue",lty=2)
lines(x,bww1.kt(x,0.8,1,2,10,1),col="black",lty=4)
lines(x,bww1.kt(x,0.6,5,2,10,1),col="green",lty=6)
legend("topright",inset=0.02,col=c("red","blue","black","green"),
legend=c("a=1,b=1,c=2,d=10","a=0.5,b=50,c=2,d=10",
"a=0.8,b=1,c=2,d=10","a=0.6,b=5,c=2,d=10"),lty=1:2:4)
a=1
b=1
c=2
d=10
i=1
x=seq(0,10,0.01)
m=exp(-x^d)
k=gamma
bww1.kt=function(x,a,b,c,d,i){
k1=(1+c(1-m)-(1+c)^(-1)*(1-m^(1+c)))^a*(i+1)-1/c*beta(a,b)
k2=k1*(1+c-(1+c)^(-4/d)*k(1+(4/d)))
k3=4*(k1*(1+c-(1+c)^(-1/d)*k(1+(1/d))))
k4=k1*(1+c-(1+c)^(-3/d)*k(1+(3/d)))
k5=6*((1+c-(1+c)^(-1/d)*k(1+(1/d))))^2
```

```
k6=k1*(1+c-(1+c)^(-2/d)*k(1+(2/d)))

k7=3*((1+c-(1+c)^(-1/d)*k(1+(1/d))))^4

k8=k2-(k3*k4)+(k5*k6)-k7

k9=(k1*((1+c-(1+c)^(-2/d)*k(1+(2/d)))-

(1+c-(1+c)^(-1/d)*k(1+(1/d)))^2))

k10=(k8/k9)-3

bww1.kt=k10

}

plot(x,bww1.kt(x,1,1,2,10,1),col="red",ylim=c(0,20),

main="(ii)",type="l",xlab="x",ylab= "Kurtosis")

lines(x,bww1.kt(x,0.5,50,2,10,1),col="blue",lty=2)

lines(x,bww1.kt(x,0.6,5,2,10,1),col="black",lty=4)

lines(x,bww1.kt(x,0.8,1,2,10,1),col="green",lty=6)

legend("topright",inset=0.02,col=c("red","blue","black","green"),

legend=c("a=1,b=1,c=2,d=10","a=0.5,b=50,c=2,d=10",

"a=0.6,b=5,c=2,d=10","a=0.8,b=1,c=2,d=10"),lty=1:2:4)
```

PDF of the LBWW Distribution

a increasing, d increasing, c increasing, $e = b = 1$

```
a=50

b=1

c=10

d=1.5

e=1

x=seq(-4,4,0.01)
```

```
bww1.pdf=function(x,a,b,c,d,e){

k1=(exp(x-d)/e)

k2=exp(-exp(x-d)/e)

k3=(exp(-c*exp(x-d)/e))

k6=(1-k3)

k7=(1/(c+1))*(1-exp(-exp((1+c)*(x-d)/e)))

k8=(k6-k7)^(a-1)

k9=(1-(k6-k7))^(b-1)

k4=(c*d)*beta(a,b)

k5=(c+1)/k4

bww1.pdf=k5*k2*k3*k8*k9

}

plot(x,bww1.pdf(x,50,1,10,1.5,1),col="red",type="l",

xlab="x",ylab="LBWW pdf")

lines(x,bww1.pdf(x,100,1,15,2.5,1),col="blue",lty=2)

lines(x,bww1.pdf(x,150,1,20,3,1),col="black",lty=4)

lines(x,bww1.pdf(x,200,1,25,3.5,1),col="green",lty=6)

legend("topleft",inset=0.02,col=c("red","blue","black","green"),

legend=c("a=50,b=1,c=10,d=1.5,e=1","a=100,b=1,c=15,d=2.5,e=1",

"a=150,b=1,c=20,d=3,e=1","a=200,b=1,c=25,

d=3.5,e=1"),lty=1:2:4)

a=200, b and c decreasing, d = 0.1 and  e increasing

a=200

b=10
```

```
c=25
d=0.1
e=2
x=seq(-2,4,0.01)
bww1.pdf=function(x,a,b,c,d,e){
k1=(exp(x-d)/e)
k2=exp(-exp(x-d)/e)
k3=(exp(-c*exp(x-d)/e))
k6=(1-k3)
k7=(1/(c+1))*(1-exp(-exp((1+c)*(x-d)/e)))
k8=(k6-k7)^(a-1)
k9=(1-(k6-k7))^(b-1)
k4=(c*d)*beta(a,b)
k5=(c+1)/k4
bww1.pdf=k5*k2*k3*k8*k9
}
plot(x,bww1.pdf(x,200,10,25,0.1,2),col="red",
type="l",xlab="x",ylab="LBWW pdf")
lines(x,bww1.pdf(x,200,8,20,0.1,2.2),col="blue",lty=2)
lines(x,bww1.pdf(x,200,6,15,0.1,2.5),col="black",lty=4)
lines(x,bww1.pdf(x,200,5,10,0.1,3),col="green",lty=6)
legend("topright",inset=0.02,col=c("red","blue","black","green"),
legend=c("a=200,b=10,c=25,d=0.1,e=2","a=200,b=8,c=20,d=0.1,e=2.2",
"a=200,b=6,c=15,d=0.1,e=2.5","a=200,b=10,c=5,d=0.1,e=3"),lty=1:2:4)
```

a increasing, b = 10, d = 1, c and e increasing

```
a=50

b=10

c=10

d=0

e=2

x=seq(-2,6,0.01)

bww1.pdf=function(x,a,b,c,d,e){

k1=(exp(x-d)/e)

k2=exp(-exp(x-d)/e)

k3=(exp(-c*exp(x-d)/e))

k6=(1-k3)

k7=(1/(c+1))*(1-exp(-exp((1+c)*(x-d)/e)))

k8=(k6-k7)^(a-1)

k9=(1-(k6-k7))^(b-1)

k4=(c*d)*beta(a,b)

k5=(c+1)/k4

bww1.pdf=k5*k2*k3*k8*k9 }

plot(x,bww1.pdf(x,50,10,10,1,2),

col="red",type="l",xlab="x",ylab="LBWW pdf")

lines(x,bww1.pdf(x,100,10,15,1,2.2),col="blue",lty=2)

lines(x,bww1.pdf(x,150,10,20,1,2.5),col="black",lty=4)

lines(x,bww1.pdf(x,200,10,25,1,3),col="green",lty=6)

legend("topright",inset=0.02,col=c("red","blue","black","green"),
```

```
legend=c("a=50,b=10,c=10,d=1,e=2","a=100,b=10,c=15,d=1,
e=2.2","a=150,b=10,c=20,d=1,e=2.5",
"a=200,b=10,c=25,d=1,e=3"),lty=1:2:4)
a and e = 1, b decreasing, c and d= increasing
a=1
b=10
c=10
d=1.5
e=1
x=seq(-4,2,0.01)
bww1.pdf=function(x,a,b,c,d,e){
k1=(exp(x-d)/e)
k2=exp(-exp(x-d)/e)
k3=(exp(-c*exp(x-d)/e))
k6=(1-k3)
k7=(1/(c+1))*(1-exp(-exp((1+c)*(x-d)/e)))
k8=(k6-k7)^(a-1)
k9=(1-(k6-k7))^(b-1)
k4=(c*d)*beta(a,b)
k5=(c+1)/k4
bww1.pdf=k5*k2*k3*k8*k9
}
plot(x,bww1.pdf(x,1,10,10,1.5,1),col="red",
type="l",xlab="x",ylab="LBWW pdf")
```

```
lines(x,bww1.pdf(x,1,8,15,2,1),col="blue",lty=2)
lines(x,bww1.pdf(x,1,6,20,2.5,1),col="black",lty=4)
lines(x,bww1.pdf(x,1,5,25,3,1),col="green",lty=6)
legend("topright",inset=0.02,col=c("red","blue","black","green"),
legend=c("a=1,b=10,c=10,d=1.5,e=1","a=1,b=8,c=15,d=2,e=1",
"a=1,b=6,c=20,d=2.5,e=1","a=1,b=5,c=25,d=3,e=1"),lty=1:2:4)
```

Skewness plots of the LBWW distribution

```
a=1
b=1
c=0.5
d=3
e=1
i=1
x=seq(0,5,0.01)
m=exp(exp((x-d)/e))
k=gamma
bww1.sk=function(x,a,b,c,d,e,i){
k1=log(1+c(1-(c*m))-(1+c)^(-1)*(1-m*(1+c)))^a*(i+1)-1/(c*e)*beta(a,b)
k2=(k1*log((e)*1+c-(1+c)^(-3/d)*k(1+(3/d)))-3*log((e)*1+
c-(1+c)^(-1/d)*k(1+(1/d)))*log((e)*1+
c-(1+c)^(-2/d)*k(1+(2/d)))+2*log((e)*1+
c-(1+c)^(-1/d)*k(1+(1/d)))^3)
k3=(k1*log(((e)*1+c-(1+c)^(-2/d)*k(1+(2/d)))-
```

```
log((e)*1+c-(1+c)^(-1/d)*k(1+(1/d)))^2))
k4=k2/((k3)^(3/2))
bww1.sk=k4
}
plot(x,bww1.sk(x,1,1,0.5,3,1,1),col="red",ylim=c(0,4),
main="(i)",type="l",xlab="y",ylab= "Skewness")
lines(x,bww1.sk(x,2,2,0.5,3,1,1),col="blue",lty=2)
lines(x,bww1.sk(x,3,3,0.5,3,1,1),col="black",lty=4)
lines(x,bww1.sk(x,4,4,0.5,3,1,1),col="green",lty=6)
legend("bottomright",inset=0.02,col=c("red","blue","black","green"),
legend=c("a = b = 1","a = b = 2","a = b = 3","a = b = 0"),lty=1:2:4)
a=1
b=1
c=0.5
d=3
e=1
i=1
x=seq(0,5,0.01)
m=exp(exp((x-d)/e))
k=gamma
bww1.sk=function(x,a,b,c,d,e,i){
k1=log(1+c(1-(c*m))-(1+c)^(-1)*(1-m*(1+c)))^a*(i+1)-1/(c*e)*beta(a,b)
k2=(k1*log((e)*1+c-(1+c)^(-3/d)*k(1+(3/d)))-3*log((e)*1+
c-(1+c)^(-1/d)*k(1+(1/d)))*log((e)*1+c-(1+c)^(-2/d)*k(1+(2/d)))+
```

```
2*log((e)*1+c-(1+c)^(-1/d)*k(1+(1/d)))^3)

k3=(k1*log(((e)*1+c-(1+c)^(-2/d)*k(1+(2/d)))-

log((e)*1+c-(1+c)^(-1/d)*k(1+(1/d)))^2))

k4=k2/((k3)^(3/2))

bww1.sk=k4

}

plot(x,bww1.sk(x,1,1,0.5,3,1,1),col="red",ylim=c(-4,1),

main="(ii)",type="l",xlab="y",ylab= "Skewness")

lines(x,bww1.sk(x,5,5,0.5,3,1,1),col="blue",lty=2)

lines(x,bww1.sk(x,10,10,0.5,3,1,1),col="black",lty=4)

lines(x,bww1.sk(x,50,50,0.5,3,1,1),col="green",lty=6)

legend("topright",inset=0.01,col=c("red","blue","black","green"),

legend=c("a=b = 1","a=b=5","a=b=10","a=b=50"),lty=1:2:4)

a=1

b=1

c=0.5

d=3

e=1

i=1

x=seq(0,5,0.01)

m=exp(exp((x-d)/e))

k=gamma

bww1.sk=function(x,a,b,c,d,e,i){

k1=(1+c(1-(c*m))-(1+c)^(-1)*(1-m*(1+c)))^a*(i+1)-1/(c*e)*beta(a,b)
```

```
k2=(k1*((e)*1+c-(1+c)^(-3/d)*k(1+(3/d)))-
3*((e)*1+c-(1+c)^(-1/d)*k(1+(1/d)))*((e)*1+
c-(1+c)^(-2/d)*k(1+(2/d)))+2*((e)*1+c-(1+c)^(-1/d)*k(1+(1/d)))^3)
k3=(k1*(((e)*1+c-(1+c)^(-2/d)*k(1+(2/d)))-
((e)*1+c-(1+c)^(-1/d)*k(1+(1/d)))^2))
k4=k2/((k3)^(3/2))
bww1.sk=k4
}
plot(x,bww1.sk(x,1,1,0.5,3,1,1),col="red",ylim=c(-4,4),
main="(i)",type="l",xlab="a",ylab= "Skewness")
lines(x,bww1.sk(x,1,2,0.5,3,1,1),col="blue",lty=2)
lines(x,bww1.sk(x,1,3,0.5,3,1,1),col="black",lty=4)
lines(x,bww1.sk(x,1,4,0.5,3,1,1),col="green",lty=6)
legend("bottomright",inset=0.02,col=c("red","blue","black","green"),
legend=c("b = 1","b = 2","b = 3","b = 4"),lty=1:2:4)
a=5
b=5
c=0.5
d=3
e=1
i=1
x=seq(0,5,0.01)
m=exp(exp((x-d)/e))
k=gamma
```

```
bww1.sk=function(x,a,b,c,d,e,i){
k1=(1+c(1-(c*m))-(1+c)^(-1)*(1-m*(1+c)))^a*(i+1)-1/(c*e)*beta(a,b)
k2=(k1*((e)*1+c-(1+c)^(-3/d)*k(1+(3/d)))-3*((e)*1+
c-(1+c)^(-1/d)*k(1+(1/d)))*((e)*1+
c-(1+c)^(-2/d)*k(1+(2/d)))+2*((e)*1+c-(1+c)^(-1/d)*k(1+(1/d)))^3)
k3=(k1*(((e)*1+c-(1+c)^(-2/d)*k(1+(2/d)))-((e)*1+
c-(1+c)^(-1/d)*k(1+(1/d)))^2))
k4=k2/((k3)^(3/2))
bww1.sk=k4
}
plot(x,bww1.sk(x,1,1,0.5,3,1,1),col="red",ylim=c(0,5),
main="(ii)",type="l",xlab="b",ylab= "Skewness")
lines(x,bww1.sk(x,2,1,0.5,3,1,1),col="blue",lty=2)
lines(x,bww1.sk(x,3,1,0.5,3,1,1),col="black",lty=4)
lines(x,bww1.sk(x,4,1,0.5,3,1,1),col="green",lty=6)
legend("topright",inset=0.01,col=c("red","blue","black","green"),
legend=c("a = 1","a = 2","a = 3","a = 4"),lty=1:2:4)
Kurtosis plots of the LBWW distribution
a=1
b=1
c=0.5
d=3
e=1
i=1
```

```
x=seq(0,10,0.01)

m=exp(exp((x-d)/e))

k=gamma

bww1.kt=function(x,a,b,c,d,e,i){

k1=log(1+c(1-(c*m))-(1+c)^(-1)*(1-m*(1+c)))^a*(i+1)-1/(c*e)*beta(a,b)

k2=k1*log((e)*1+c-(1+c)^(-4/d)*k(1+(4/d)))

k3=4*(k1*log((e)*1+c-(1+c)^(-1/d)*k(1+(1/d))))

k4=k1*log((e)*1+c-(1+c)^(-3/d)*k(1+(3/d)))

k5=6*log(((e)*1+c-(1+c)^(-1/d)*k(1+(1/d))))^2

k6=k1*log((e)*1+c-(1+c)^(-2/d)*k(1+(2/d)))

k7=3*log(((e)*1+c-(1+c)^(-1/d)*k(1+(1/d))))^4

k8=k2-(k3*k4)+(k5*k6)-k7

k9=(k1*log(((e)*1+c-(1+c)^(-2/d)*k(1+(2/d)))-((e)*1+

c-(1+c)^(-1/d)*k(1+(1/d)))^2))

k10=(k9)^2

k11=(k8/k10)-3

bww1.kt=k10

}

plot(x,bww1.kt(x,1,1,0.5,3,1,1),col="red",ylim=c(0,5),

main="(i)",type="l",xlab="y",ylab= "Kurtosis")

lines(x,bww1.kt(x,2,2,0.5,3,1,1),col="blue",lty=2)

lines(x,bww1.kt(x,3,3,0.5,3,1,1),col="black",lty=4)

lines(x,bww1.kt(x,4,4,0.5,3,1,1),col="green",lty=6)

legend("topright",inset=0.01,col=c("red","blue","black","green"),
```

```
legend=c("a = b = 1","a = b = 2","a = b = 3","a = b = 4"),lty=1:2:4)

a=1

b=1

c=0.5

d=3

e=1

i=1

x=seq(0,10,0.01)

m=exp(exp((x-d)/e))

k=gamma

bww1.kt=function(x,a,b,c,d,e,i){

k1=log(1+c(1-(c*m))-(1+c)^(-1)*(1-m*(1+c)))^a*(i+1)-1/(c*e)*beta(a,b)

k2=k1*log((e)*1+c-(1+c)^(-4/d)*k(1+(4/d)))

k3=4*(k1*log((e)*1+c-(1+c)^(-1/d)*k(1+(1/d))))

k4=k1*log((e)*1+c-(1+c)^(-3/d)*k(1+(3/d)))

k5=6*log(((e)*1+c-(1+c)^(-1/d)*k(1+(1/d))))^2

k6=k1*log((e)*1+c-(1+c)^(-2/d)*k(1+(2/d)))

k7=3*log(((e)*1+c-(1+c)^(-1/d)*k(1+(1/d))))^4

k8=k2-(k3*k4)+(k5*k6)-k7

k9=(k1*log(((e)*1+c-(1+c)^(-2/d)*k(1+(2/d)))-

((e)*1+c-(1+c)^(-1/d)*k(1+(1/d)))^2))

k10=(k9)^2

k11=(k8/k10)-3

bww1.kt=k10
```

```
}

plot(x,bww1.kt(x,1,1,0.5,3,1,1),col="red",ylim=c(0,10),

main="(ii)",type="l",xlab="y",ylab= "Kurtosis")

lines(x,bww1.kt(x,5,5,0.5,3,1,1),col="blue",lty=2)

lines(x,bww1.kt(x,10,10,0.5,3,1,1),col="black",lty=4)

lines(x,bww1.kt(x,50,50,0.5,3,1,1),col="green",lty=6)

legend("topright",inset=0.01,col=c("red","blue","black","green"),

legend=c("a=b=1","a=b=5","a=b=10","a=b=50"),lty=1:2:4)


a=1

b=1

c=0.5

d=3

e=1

i=1

x=seq(0,10,0.01)

m=exp(exp((x-d)/e))

k=gamma

bww1.kt=function(x,a,b,c,d,e,i){

k1=log(1+c(1-(c*m))-(1+c)^(-1)*(1-m*(1+c)))^a*(i+1)-1/(c*e)*beta(a,b)

k2=k1*log((e)*1+c-(1+c)^(-4/d)*k(1+(4/d)))

k3=4*(k1*log((e)*1+c-(1+c)^(-1/d)*k(1+(1/d))))

k4=k1*log((e)*1+c-(1+c)^(-3/d)*k(1+(3/d)))

k5=6*log(((e)*1+c-(1+c)^(-1/d)*k(1+(1/d))))^2
```

```
k6=k1*log((e)*1+c-(1+c)^(-2/d)*k(1+(2/d)))

k7=3*log(((e)*1+c-(1+c)^(-1/d)*k(1+(1/d))))^4

k8=k2-(k3*k4)+(k5*k6)-k7

k9=(k1*log(((e)*1+c-(1+c)^(-2/d)*k(1+(2/d)))-

((e)*1+c-(1+c)^(-1/d)*k(1+(1/d)))^2))

k10=(k9)^2

k11=(k8/k10)-3

bww1.kt=k10

}

plot(x,bww1.kt(x,1,1,0.5,3,1,1),col="red",ylim=c(0,5),main="(i)",

type="l",xlab="a",ylab= "Kurtosis")

lines(x,bww1.kt(x,1,2,0.5,3,1,1),col="blue",lty=2)

lines(x,bww1.kt(x,1,3,0.5,3,1,1),col="black",lty=4)

lines(x,bww1.kt(x,1,4,0.5,3,1,1),col="green",lty=6)

legend("topright",inset=0.01,col=c("red","blue","black","green"),

legend=c("b = 1","b = 2","b = 3","b = 4"),lty=1:2:4)

a=1

b=1

c=0.5

d=3

e=1

i=1

x=seq(0,10,0.01)

m=exp(exp((x-d)/e))
```

```
k=gamma
bww1.kt=function(x,a,b,c,d,e,i){
k1=log(1+c(1-(c*m))-(1+c)^(-1)*(1-m*(1+c)))^a*(i+1)-1/(c*e)*beta(a,b)
k2=k1*log((e)*1+c-(1+c)^(-4/d)*k(1+(4/d)))
k3=4*(k1*log((e)*1+c-(1+c)^(-1/d)*k(1+(1/d))))
k4=k1*log((e)*1+c-(1+c)^(-3/d)*k(1+(3/d)))
k5=6*log(((e)*1+c-(1+c)^(-1/d)*k(1+(1/d))))^2
k6=k1*log((e)*1+c-(1+c)^(-2/d)*k(1+(2/d)))
k7=3*log(((e)*1+c-(1+c)^(-1/d)*k(1+(1/d))))^4
k8=k2-(k3*k4)+(k5*k6)-k7
k9=(k1*log(((e)*1+c-(1+c)^(-2/d)*k(1+(2/d)))-((e)*1+
c-(1+c)^(-1/d)*k(1+(1/d)))^2))
k10=(k9)^2
k11=(k8/k10)-3
bww1.kt=k10
}
plot(x,bww1.kt(x,1,1,0.5,3,1,1),col="red",ylim=c(0,50),
main="(ii)",type="l",xlab="b",ylab= "Kurtosis")
lines(x,bww1.kt(x,2,1,0.5,3,1,1),col="blue",lty=2)
lines(x,bww1.kt(x,3,1,0.5,3,1,1),col="black",lty=4)
lines(x,bww1.kt(x,4,1,0.5,3,1,1),col="green",lty=6)
legend("topright",inset=0.01,col=c("red","blue","black","green"),
legend=c("a = 1","a = 2","a = 3","a = 4"),lty=1:2:4)
```

APPENDIX B

R Code for Exploratory Data Analysis (EDA)

The code in R program to generates scatter plot for each variable data i.e Exploratory Data Analysis (EDA), descriptive statistics, the graphs of the BWW distribution and the loglikehood estimation of both developed model and some extant models.

Data used for the Analysis. (Completion of PhD Data)

```
data<-read.csv("P.data5.csv",header=1)

data

X X1 X2 X3 X4 X5

1    5  3  0  2 41  8

2    9  3  0  2 47  8

3   10  3  0  2 44  8

4    5  3  2  2 37  8

5   11  1  2  0 53  8

6   14  3  2  2 41  8
```

7	7	3	2	2	48	8
8	7	1	2	2	40	8
9	15	1	2	2	49	8
10	6	3	2	1	30	8
11	4	3	0	2	41	4
12	10	2	2	2	42	4
13	10	1	2	2	43	4
14	12	3	2	2	48	4
15	15	3	2	2	47	4
16	5	3	2	2	41	4
17	8	3	1	2	46	4
18	11	3	2	2	49	4
19	12	3	2	2	44	4
20	8	3	1	2	46	4
21	8	3	2	2	40	4
22	8	1	2	2	55	4
23	11	3	2	2	43	4
24	10	1	2	2	41	4
25	9	1	2	2	53	4
26	8	1	2	2	49	4
27	14	2	2	2	53	4
28	7	3	0	2	49	4
29	4	1	0	1	0	4
30	7	1	0	2	36	4

31	9	1	1	2	51	4
32	7	1	2	2	50	4
33	8	2	2	2	42	4
34	5	3	2	2	44	4
35	7	0	0	2	45	4
36	15	3	2	1	0	8
37	10	1	2	2	42	8
38	8	2	2	2	49	8
39	9	2	2	2	39	8
40	9	1	2	2	45	4
41	8	1	2	2	52	4
42	10	1	2	2	49	4
43	8	1	2	2	52	4
44	9	1	1	2	53	4
45	10	1	2	2	49	4
46	8	1	2	2	48	4
47	11	1	2	2	52	4
48	8	2	2	2	37	4
49	5	1	0	0	58	4
50	9	1	0	2	41	4
51	11	1	2	2	49	4
52	11	2	2	2	0	4
53	10	1	0	2	49	4
54	14	1	2	2	42	4

55	9	1	0	2	40	4
56	12	3	0	2	48	4
57	9	2	2	2	47	4
58	11	3	2	2	44	4
59	11	1	2	2	38	4
60	7	1	2	2	41	4
61	14	3	2	2	43	4
62	14	1	2	2	0	4
63	11	1	2	2	41	4
64	10	1	2	2	44	4
65	11	1	2	2	53	4
66	12	1	2	2	56	4
67	9	1	2	1	35	4
68	5	2	2	1	41	4
69	7	2	2	2	49	4
70	7	2	2	2	34	4
71	8	2	2	2	33	4
72	9	3	2	2	54	4
73	9	2	2	2	48	4
74	8	3	2	2	30	8
75	9	3	2	2	43	7
76	4	2	1	2	38	7
77	10	2	1	2	45	2
78	9	1	2	2	44	1

79	12	1	2	2	55	1
80	8	2	2	2	38	1
81	20	2	2	2	38	1
82	8	3	2	2	46	1
83	9	3	2	2	42	1
84	9	3	2	2	51	1
85	9	3	2	2	34	1
86	7	3	2	1	31	1
87	7	2	2	2	37	1
88	5	1	1	1	38	1
89	7	2	2	2	46	9
90	5	2	2	1	52	1
91	7	2	0	2	38	1
92	10	3	0	2	44	1
93	9	3	2	2	50	1
94	11	3	2	2	44	1
95	9	1	2	2	47	1
96	9	3	2	2	37	1
97	11	1	2	2	45	1
98	11	3	2	2	54	1
99	9	2	2	0	47	1
100	14	3	0	1	49	1
101	7	1	2	2	34	2
102	9	2	2	2	42	2

103	7	1	2	2	34	2
104	9	3	2	2	47	2
105	15	3	2	2	50	2
106	11	2	2	2	46	2
107	8	3	2	2	50	2
108	5	1	2	2	40	1
109	10	3	2	2	41	9
110	12	1	2	2	51	9
111	6	3	2	1	0	1
112	12	2	2	2	49	9
113	7	3	1	1	35	2
114	7	1	0	2	44	2
115	11	3	2	0	40	2
116	11	3	2	2	40	2
117	8	3	1	1	37	2
118	21	3	2	1	63	2
119	4	1	2	2	41	2
120	8	3	2	0	48	2
121	5	3	2	2	47	2
122	8	1	2	2	44	2
123	11	3	2	0	47	2
124	11	1	2	2	56	2
125	9	2	2	2	39	9
126	11	3	2	2	46	9

127	10	2	2	2	50	9
128	9	3	2	2	39	7
129	4	3	2	2	33	7
130	9	3	2	1	37	7
131	10	3	2	2	37	7
132	7	3	2	2	45	7
133	7	1	2	1	30	7
134	9	3	2	2	39	7
135	6	3	2	1	35	7
136	9	1	0	2	38	7
137	9	1	2	2	33	7
138	6	1	2	1	42	7
139	7	3	2	2	36	7
140	7	1	1	2	34	7
141	7	1	1	2	41	7
142	10	3	2	2	37	7
143	10	1	2	2	47	7
144	9	3	2	1	37	7
145	6	3	2	1	35	3
146	10	1	2	2	41	3
147	10	3	2	2	40	3
148	8	1	2	2	33	3
149	9	3	2	2	49	3
150	10	3	2	2	36	3

151	5	3	2	2	33	10
152	9	3	2	2	47	10
153	6	3	2	2	41	10
154	8	1	2	2	36	10
155	9	3	2	2	43	10
156	10	3	2	2	41	10
157	8	3	2	2	39	10
158	9	1	2	2	47	10
159	6	3	2	2	39	10
160	7	1	2	2	0	5
161	14	3	2	2	43	5
162	10	2	2	2	43	5
163	8	3	2	2	39	5
164	10	3	2	2	39	5
165	16	2	2	2	47	2
166	9	1	2	2	48	2
167	8	1	2	2	45	2
168	8	3	2	2	47	2
169	11	1	2	2	38	2
170	9	2	2	2	49	2
171	9	1	2	2	56	2
172	9	2	2	2	46	2
173	10	3	2	2	47	2
174	15	3	2	2	45	2

```
175  9  3  2  2 47  2

176 11  3  2  2 42  2

177 10  2  2  2 47  2

178 10  1  2  2 54  2

179 12  3  2  2 43  2

180 10  3  0  2 39  2

181 11  1  1  2  0  2

182  7  2  2  1 37  2

183  6  2  2  2 46  2

184  7  3  2  2 43  2

185 11  2  2  2 42  2

186 11  2  2  2 44  2

187  5  3  2  2 41  2

n=187

attach(Badmus)

names(Badmus)

y=Duration

X1=Supervisor

X2=Employment

X3=MaritalStatus

X4=Age

X5=Faculty

Duration Data:

y<-c(5,9,10,5,11,14,7,7,15,6,4,10,10,12,15,5,8,11,12,8,8,8,11,10,9,8,14,7,4,7,9,7,8,5,
```

```
summary(y)

qnorm(y)

pnorm(y)

dnorm(y)

plotdist(y, histo = TRUE, demp = TRUE)

boxplot(y)

skewness(y)
```

Beta Weighted Weibull (BWW) Distribution Plot of PDF with Duration Data

```
a=100

b=5

c=0.5

d=2

Duration=seq(0,5,0.01)

k1=(1-exp(-y^d))

k2=(1-exp(-(1+c)*Duration^d))/(c+1)

k3=((c+1)/c)*k1

k6=(1-exp(-c*Duration^d))

k7=((c+1)/c)*d*Duration^(d-1)*exp(-Duration^d)

k8=(k3-k2)^(a-1)

k9=(1-(k3-k2))^(b-1)

k4=beta(a,b)

k5=1/k4

BWWDpdf=k5*k8*k9*k7*k6

plot(Duration,BWWDpdf,pch="b",col="black",
```

```
main= "The BWWD PDF Plot, when a=100, b=5, c=0.5, d=2")
```

Beta Weighted Weibull (BWW) Distribution Plot of CDF with Duration Data

```
a=2

b=3

c=4

d=2

x=seq(0,5,0.01)

beta(a,b)

k1=(c+1/c)

k2=(1-exp(-Duration^d))

k3=(1-exp(-(1+c)*Duration^d))/(c+1)

k4=k1*(k2-k3)

k5=k4

k6=beta(a,b)

k7=k5*k6

BWWDcdf=k7/k6

plot(Duration,BWWDcdf,pch="+",col="black",

main ="The BWWD CDF Plot, when a=2, b=3,c=4,d=2")
```

Beta Weighted Weibull (BWW) Distribution: Plot of Survival Rate Functio with Duration Data

```
a=2

b=3

c=4

d=2
```

```
x=seq(0,10,0.01)

beta(a,b)

k1=(c+1/c)

k2=(1-exp(-Duration^d))

k3=(1-exp(-(1+c)*Duration^d))/(c+1)

k4=k1*(k2-k3)

k5=k4

k6=beta(a,b)

k7=k5*k6

k8=(k6-k5)

BWWDsurv=k8/k6

plot(Duration,BWWDsurv,pch="+",main= "The BWWD Survival Rate Plot,

when a=2,b=3,c=4,d=2")
```

Beta Weighted Weibull (BWW) Distribution: Plot of Hazard Rate Function with Duration Data

```
a=2

b=0.5

c=1

d=2

x=seq(0,5,0.01)

beta(a,b)

k1=(c+1/c)

k2=(1-exp(-Duration^d))

k3=(1-exp(-(1+c)*Duration^d))/(c+1)
```

```
k4=k1*(k2-k3)

k5=k4

k6=beta(a,b)

k7=k5*k6

k8=(k6-k5)

k9=(d*Duration^(d-1))*(exp(-Duration^d))*(1-exp(-c*Duration^d))

k10=k1*k9

k11=1/k6

k12=(k5)^(a-1)

k13=(1-k5)^(b-1)

k14=k6-k7

BWWDhazd=k12*k13*k10/k14

plot(Duration,BWWDhazd,pch="+",col="black",

main= "The BWWD Hazard Rate Plot, when a=2, b=0.5, c=1,d=2")
```

Log-Beta Weighted Weibull (LBWW) Distribution

```
Loglik<-function(p)-n*log(beta(p[1],p[2]))+

sum(log((((p[3]+1)/(p[4]*p[3]))*exp((y-(p[5]+(p[6]*X1)+

(p[7]*X2)+(p[8]*X3)+(p[9]*X4)+(p[10]*X5))/(p[4]))*exp(-exp((y-(p[5]+

(p[6]*X1)+(p[7]*X2)+(p[8]*X3)+(p[9]*X4)+(p[10]*X5))/(p[4]))*

(1-exp((-p[3])*exp((y-(p[5]+(p[6]*X1)+(p[7]*X2)+

(p[8]*X3)+(p[9]*X4)+(p[10]*X5))/(p[4]))))))))))+

((p[1]-1)*sum(log((((p[3]+1)/(p[3]))*(1-exp(-exp((y-(p[5]+

(p[6]*X1)+(p[7]*X2)+(p[8]*X3)+(p[9]*X4)+

(p[10]*X5))/(p[4]))))))-(1/(p[3]+1))*(1-exp(-(1+p[3])*
```

```
exp(((y-(p[5]+(p[6]*X1)+(p[7]*X2)+(p[8]*X3)+(p[9]*X4)+
(p[10]*X5))/(p[4]))))))+((p[2]-1)*sum(1-log(((p[3]+1)/
(p[4]))*(1-exp(-exp(((y-(p[5]+(p[6]*X1)+(p[7]*X2)+
(p[8]*X3)+(p[9]*X4)+(p[10]*X5))/(p[4])))))))-
(1/(p[3]+1))*(1-exp(-(1+p[3])*exp((y-(p[5]+(p[6]*X1)+
(p[7]*X2)+(p[8]*X3)+(p[9]*X4)+(p[10]*X5))/(p[4]))))))))
c<- maxLik(Loglik, start=c(4.15,10.25,5.12,3.21,2.55,3.25,1.25,4.15,1.31,
 5.15))
summary(c)
Maximum Likelihood estimation
Newton-Raphson maximisation, 1 iterations
Return code 2: successive function values within tolerance limit
Log-Likelihood: 4556387
10 free parameters
Estimates:
Estimate Std. error   t value Pr(> t)
[1,] 4.150e+00  1.291e-02 3.215e+02  <2e-16 ***
[2,] 1.025e+01  1.900e-02 5.395e+02  <2e-16 ***
[3,] 5.120e+00  4.303e-03 1.190e+03  <2e-16 ***
[4,] 3.210e+00  1.006e-07 3.190e+07  <2e-16 ***
[5,] 2.550e+00  9.270e-06 2.751e+05  <2e-16 ***
[6,] 3.250e+00  9.259e-06 3.510e+05  <2e-16 ***
[7,] 1.250e+00  1.065e-05 1.174e+05  <2e-16 ***
[8,] 4.150e+00  1.080e-05 3.841e+05  <2e-16 ***
```

```
[9,] 1.310e+00  1.752e-06 7.476e+05  <2e-16 ***

[10,] 5.150e+00  7.978e-06 6.455e+05  <2e-16 ***

---

Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

vcov(c)

[,1]           [,2]           [,3]           [,4]           [,5]

[1,]  1.666177e-04  7.404324e-05  5.553426e-05  3.255832e-12  2.129952e-10

[2,]  7.404324e-05  3.610240e-04  2.036535e-04 -1.027013e-11 -7.131404e-10

[3,]  5.553426e-05  2.036535e-04  1.851940e-05 -5.256120e-12 -3.710969e-10

[4,]  3.255832e-12 -1.027013e-11 -5.256120e-12  1.012731e-14 -5.625766e-13

[5,]  2.129952e-10 -7.131404e-10 -3.710969e-10 -5.625766e-13  8.592538e-11

[6,] -7.371182e-11  2.430940e-10  1.274341e-10  9.782269e-14 -4.352246e-11

[7,]  2.476065e-13  1.202153e-10  5.513336e-11  2.060787e-13 -3.645181e-11

[8,] -6.987984e-11  1.493104e-10  8.234421e-11  1.739832e-13 -1.973864e-11

[9,] -7.857745e-11  2.414947e-10  1.255075e-10 -7.524011e-15 -8.369452e-12

[10,]  2.511449e-10 -7.735585e-10 -4.034079e-10 -8.384183e-13  3.888311e-11

[,6]                     [,7]                          [,8]                     [,9]

[1,] -7.371182e-11  2.476065e-13 -6.987984e-11 -7.857745e-11  2.511449e-10

[2,]  2.430940e-10  1.202153e-10  1.493104e-10  2.414947e-10 -7.735585e-10

[3,]  1.274341e-10  5.513336e-11  8.234421e-11  1.255075e-10 -4.034079e-10

[4,]  9.782269e-14  2.060787e-13  1.739832e-13 -7.524011e-15 -8.384183e-13

[5,] -4.352246e-11 -3.645181e-11 -1.973864e-11 -8.369452e-12  3.888311e-11

[6,]  8.573706e-11 -3.140609e-12 -3.248456e-11  1.513004e-12 -7.192508e-12

[7,] -3.140609e-12  1.134017e-10 -8.469966e-11  1.920290e-12 -9.219874e-12
```

```
[8,] -3.248456e-11 -8.469966e-11  1.167104e-10  3.376751e-12 -1.531676e-11
[9,]  1.513004e-12  1.920290e-12  3.376751e-12  3.070150e-12 -1.367144e-11
[10,] -7.192508e-12 -9.219874e-12 -1.531676e-11 -1.367144e-11  6.365451e-11
AIC=-9112754, BIC=-9112751, CAIC=-9112741
```

Log-Exponentiated Weighted Weibull Regression Model. When b=1

```
Loglik<-function(p)-n*log((p[1]))+sum(log(((p[2]+1)/
(p[3]*p[2]))*exp((y-(p[4]+(p[5]*X1)+(p[6]*X2)+
(p[7]*X3)+(p[8]*X4)+(p[9]*X5))/(p[3]))*
exp(-exp((y-(p[4]+(p[5]*X1)+(p[6]*X2)+(p[7]*X3)+
(p[8]*X4)+(p[9]*X5))/(p[3]))*(1-exp((-p[2])*exp((y-(p[4]+
(p[5]*X1)+(p[6]*X2)+(p[7]*X3)+(p[8]*X4)+
(p[9]*X5))/(p[3]))))))))))+
((p[1]-1)*sum(log(((p[2]+1)/(p[2]))*
(1-exp(-exp((y-(p[4]+(p[5]*X1)+(p[6]*X2)+
(p[7]*X3)+(p[8]*X4) +(p[9]*X5))/(p[3])))))))-
(1/(p[2]+1))*(1-exp(-(1+p[2])*
exp(((y-(p[4]+(p[5]*X1)+(p[6]*X2)+(p[7]*X3)+(p[8]*X4)
+(p[9]*X5))/(p[3])))))))
c<- maxLik(Loglik, start=c(100.25,2.10,4.50,2.25,
2.55,3.55,4.20,1.21,5.05))
Log-Likelihood: 795888.7
```

Log-Lehmann Type II Weighted WeibullRegression (when a=1)

```
Loglik<-function(p)-n*log((p[1])+sum(log(((p[2]+1)/(p[3]*p[2]))*
```

```
exp((y-(p[4]+(p[5]*X1)+(p[6]*X2)+(p[7]*X3)+(p[8]*X4)+
(p[9]*X5))/(p[3]))*exp(-exp((y-(p[4]+(p[5]*X1)+
(p[6]*X2)+(p[7]*X3)+(p[8]*X4)+(p[9]*X5))/(p[3]))*
(1-exp((-p[2])*exp((y-(p[4]+(p[5]*X1)+(p[6]*X2)+
(p[7]*X3)+(p[8]*X4)+(p[9]*X5))/(p[3]))))))))+
((p[1]-1)*sum(1-log(((p[2]+1)/(p[3]))*
(1-exp(-exp(((y-(p[4]+(p[5]*X1)+(p[6]*X2)+
(p[7]*X3)+(p[8]*X4)+(p[9]*X5))/(p[3])))))))-
(1/(p[2]+1))*(1-exp(-(1+p[2])*exp((y-(p[4]+(p[5]*X1)+
(p[6]*X2)+(p[7]*X3)+(p[8]*X4)+(p[9]*X5))/(p[3]))))))))
c<- maxLik(Loglik, start=c(100.25,2.10,4.50,2.25,
2.55,3.55,4.20,1.21,5.05))
```

Log-Weighted Weibull Regression Model (when $a = b = 1$)

```
Loglik<-function(p)-n*log((p[1]+1/(p[2]*p[1])*exp((y-(p[3]+
(p[4]*X1)+(p[5]*X2)+(p[6]*X3)+(p[7]*X4)+(p[8]*X5))/(p[2]))*
exp(-exp((y-(p[3]+(p[4]*X1)+(p[5]*X2)+(p[6]*X3)+(p[7]*X4)+
(p[8]*X5))/(p[2]))*(1-exp((-p[1])*exp((y-(p[3]+(p[4]*X1)+
(p[5]*X2)+(p[6]*X3)+(p[7]*X4)+(p[8]*X5))/(p[2]))))))))
c<- maxLik(Loglik, start=c(14.25,4.50,3.55,2.55,
3.35,4.25,1.11,5.10))
```

Log-Weighted Regression Model; (when $a = b = 1, \lambda = \sigma$)

```
Loglik<-function(p)-n*log((p[1]+1/(p[2]*p[1])*
exp(-exp((y-(p[3]+(p[4]*X1)+(p[5]*X2)+(p[6]*X3)+
```

```
(p[7]*X4)+(p[8]*X5))/(p[2]))*(1-exp((-p[1])*
exp((y-(p[3]+(p[4]*X1)+(p[5]*X2)+(p[6]*X3)+(p[7]*X4)+
(p[8]*X5))/(p[2])))))))))
c<- maxLik(Loglik, start=c(15.25,4.50,3.55,2.55,
3.35,4.35,1.11,5.01))
```

Log-WeibullRegression Model.

```
Loglik<-function(p)-n*log((1/(p[1])*exp((y-(p[2]+
(p[3]*X1)+(p[4]*X2)+(p[5]*X3)+(p[6]*X4)+(p[7]*X5))/
(p[1]))*exp(-exp((y-(p[2]+(p[3]*X1)+(p[4]*X2)+
(p[5]*X3)+(p[6]*X4)+(p[7]*X5))/(p[1])))))))
c<- maxLik(Loglik, start=c(4.25,1.50,1.45,
3.10,3.50,1.02,0.51))
```

Log-Beta Normal Regression Model

```
Loglik<-function(p)-(n)/(2)*(beta(p[1],p[2]))*
log(2*3.142*(p[3]^2))-sum(exp((y-(p[4]+(p[5]*X1)+
(p[6]*X2)+(p[7]*X3)+(p[8]*X4)+(p[9]*X5))/(p[3]^2))))+
((p[1]-1)*sum(log(exp((y-(p[4]+(p[5]*X1)+(p[6]*X2)+
(p[7]*X3)+(p[8]*X4)+(p[9]*X5))/(p[3]^2))))))+
((p[2]-1)*sum(1-log(exp((y-(p[4]+(p[5]*X1)+(p[6]*X2)+
(p[7]*X3)+(p[8]*X4)+(p[9]*X5))/(p[3]^2))))))
c<- maxLik(Loglik, start=c(6.05,5.05,5.10,3.21,
2.45,4.36,0.25,4.15,0.25))real
```

Log-Normal Regression Model

```
Loglik<-function(p)-(n)/(2)*log(2*3.142*(p[1]^2))-
sum(exp((y-(p[2]+(p[3]*X1)+(p[4]*X2)+(p[5]*X3)+
(p[6]*X4)+(p[7]*X5))/(p[1]^2))))
c<- maxLik(Loglik, start=c(6.45,5.55,0.11,2.25,4.55,3.25,5.25))
Log-Beta Normal Regrn Model
Loglik<-function(p)-(n)/(2)*(beta(p[1],p[2]))*
log(2*3.142*(p[3]^2))-sum(exp((y-(p[4]+(p[5]*X1)+
(p[6]*X2)+(p[7]*X3)+(p[8]*X4)+(p[9]*X5))/(p[3]^2))))+
((p[1]-1)*sum(log(exp((y-(p[4]+(p[5]*X1)+(p[6]*X2)+
(p[7]*X3)+(p[8]*X4)+(p[9]*X5))/(p[3]^2))))))+
((p[2]-1)*sum(1-log(exp((y-(p[4]+(p[5]*X1)+
(p[6]*X2)+(p[7]*X3)+(p[8]*X4)+(p[9]*X5))/(p[3]^2))))))
c<- maxLik(Loglik, start=c(6.05,5.05,5.10,3.21,2.45,
4.36,0.25,4.15,0.25))real
```

Log-Beta Weibull Regression Model

```
Loglik<-function(p)-n*(log(p[3])+(beta(p[1],p[2])))+
sum(log(exp((y-(p[4]+(p[5]*X1)+(p[6]*X2)+(p[7]*X3)+
(p[8]*X4)+(p[9]*X5))/(p[3])))))-((p[2])*
sum(log(exp((y-(p[4]+(p[5]*X1)+(p[6]*X2)+
(p[7]*X3)+(p[8]*X4)+(p[9]*X5))/(p[3]))))))+
((p[1]-1)*sum(log(1-exp(-exp((y-(p[4]+(p[5]*X1)+
(p[6]*X2)+(p[7]*X3)+(p[8]*X4)+(p[9]*X5))/(p[3])))))))
c<- maxLik(Loglik, start=c(0.05,0.01,100.10,3.21,
2.45,4.36,0.25,4.15,2.25))
```

```
Loglik<-function(p)-n*log(log(p[3])+log(beta(p[1],p[2])))+
sum(log(exp((y-(p[4]+(p[5]*X1)+(p[6]*X2)+(p[7]*X3)+
(p[8]*X4)+(p[9]*X5))/(p[3])))))-((p[2])*
sum(log(exp((y-(p[4]+(p[5]*X1)+(p[6]*X2)+
(p[7]*X3)+(p[8]*X4)+(p[9]*X5))/(p[3]))))))+
((p[1]-1)*sum(log(1-exp(-exp((y-(p[4]+(p[5]*X1)+
(p[6]*X2)+(p[7]*X3)+(p[8]*X4)+(p[9]*X5))/(p[3])))))))
c<- maxLik(Loglik, start=c(0.05,0.05,100.10,3.11,
2.45,4.36,1.25,4.15,0.25))
Log-Beta LogLogistic Regrn Model
Loglik<-function(p)-n*log(p[3])-n*(log(beta(p[1],p[2])))+
(p[1])*sum(log(exp((y-(p[4]+(p[5]*X1)+
(p[6]*X2)+(p[7]*X3)+(p[8]*X4)+(p[9]*X5))/
(p[3])))))-(p[1]+p[2])*sum(log(1+exp((y-(p[4]+(p[5]*X1)+
(p[6]*X2)+(p[7]*X3)+(p[8]*X4)+(p[9]*X5))/(p[3])))))
c<- maxLik(Loglik, start=c(1.05,0.05,10.10,3.11,
2.45,4.36,1.25,4.15,0.15))
Loglik<-function(p)+n*(log(p[3])*(log(beta(p[1],p[2]))))+
(p[1])*sum(log(exp((y-(p[4]+(p[5]*X1)+(p[6]*X2)+
(p[7]*X3)+(p[8]*X4)+(p[9]*X5))/(p[3])))))-(p[1]+
p[2])*sum(log(1+exp((y-(p[4]+(p[5]*X1)+(p[6]*X2)+
(p[7]*X3)+(p[8]*X4)+(p[9]*X5))/(p[3])))))
c<- maxLik(Loglik, start=c(2.05,5.25,10.05,3.11,
0.25,4.36,2.15,4.45,4.25))
```

Log-Logistic Regression Model

```
Loglik<-function(p)-n*(log(p[1]))+sum(log(exp((y-(p[2]+
(p[3]*X1)+(p[4]*X2)+(p[5]*X3)+(p[6]*X4)+(p[7]*X5))/
(p[1])))))-sum(log(1+exp((y-(p[2]+(p[3]*X1)+
(p[4]*X2)+(p[5]*X3)+(p[6]*X4)+(p[7]*X5))/(p[1])))))
c<- maxLik(Loglik, start=c(2.05,5.25,10.05,
3.11,4.36,2.15,5.25))
```

Calculation of skewness

```
a=0.5
b=0.5
c=2
d=10
k1=c+1
k2=c+1
k3=(1/d)
k4=gamma(1+k3)
k5=beta(a,b)
k6=c*k5
k7=(k2)^(-k3)
k8=(k1-k7)*(k4)
k9=k8/k6
k9


a=0.5
```

```
b=0.5

c=2

d=10

k1=c+1

k2=c+1

k3=(2/d)

k4=gamma(1+k3)

k5=beta(a,b)

k6=c*k5

k7=(k2)^(-k3)

k8=(k1-k7)*(k4)

k9=k8/k6

k9

a=0.5

b=0.5

c=2

d=10

k1=c+1

k2=c+1

k3=(3/d)

k4=gamma(1+k3)

k5=beta(a,b)

k6=c*k5

k7=(k2)^(-k3)
```

```
k8=(k1-k7)*(k4)

k9=k8/k6

k9


a=0.5

b=0.5

c=2

d=10

k1=c+1

k2=c+1

k3=(4/d)

k4=gamma(1+k3)

k5=beta(a,b)

k6=c*k5

k7=(k2)^(-k3)

k8=(k1-k7)*(k4)

k9=k8/k6

k9

a=2

b=0.5

c=2

d=10

k1=c+1

k2=c+1
```

```
k3=(1/d)

k4=gamma(1+k3)

k5=beta(a,b)

k6=c*k5

k7=(k2)^(-k3)

k8=(k1-k7)*(k4)

k9=k8/k6

k9


a=2

b=0.5

c=2

d=10

k1=c+1

k2=c+1

k3=(2/d)

k4=gamma(1+k3)

k5=beta(a,b)

k6=c*k5

k7=(k2)^(-k3)

k8=(k1-k7)*(k4)

k9=k8/k6

k9
```

```
a=2

b=0.5

c=2

d=10

k1=c+1

k2=c+1

k3=(3/d)

k4=gamma(1+k3)

k5=beta(a,b)

k6=c*k5

k7=(k2)^(-k3)

k8=(k1-k7)*(k4)

k9=k8/k6

k9
```

```
a=2

b=0.5

c=2

d=10

k1=c+1

k2=c+1

k3=(4/d)

k4=gamma(1+k3)

k5=beta(a,b)
```

```
k6=c*k5

k7=(k2)^(-k3)

k8=(k1-k7)*(k4)

k9=k8/k6

k9

AGWW.

a=0.5

b=0.5

c=0.5

d=3

e=1

k1=c+1

k2=c+1

k3=(1/d)

k4=gamma(1+k3)

k5=beta(a,b)

k6=(c*e)*k5

k7=e*((k2)^(-k3))

k8=(k1-k7)*(k4)

k9=k8/k6

k9


a=0.5

b=0.5
```

```
c=0.5

d=3

e=1

k1=c+1

k2=c+1

k3=(2/d)

k4=gamma(1+k3)

k5=beta(a,b)

k6=(c*e)*k5

k7=e*((k2)^(-k3))

k8=(k1-k7)*(k4)

k9=k8/k6

k9


a=0.5

b=0.5

c=0.5

d=3

e=1

k1=c+1

k2=c+1

k3=(3/d)

k4=gamma(1+k3)

k5=beta(a,b)
```

```
k6=(c*e)*k5

k7=e*((k2)^(-k3))

k8=(k1-k7)*(k4)

k9=k8/k6

k9



a=0.5

b=0.5

c=0.5

d=3

e=1

k1=c+1

k2=c+1

k3=(4/d)

k4=gamma(1+k3)

k5=beta(a,b)

k6=(c*e)*k5

k7=e*((k2)^(-k3))

k8=(k1-k7)*(k4)

k9=k8/k6

k9


a=2
```

```
b=0.5

c=0.5

d=3

e=1

k1=c+1

k2=c+1

k3=(1/d)

k4=gamma(1+k3)

k5=beta(a,b)

k6=(c*e)*k5

k7=e*((k2)^(-k3))

k8=(k1-k7)*(k4)

k9=k8/k6

k9


a=2

b=0.5

c=0.5

d=3

e=1

k1=c+1

k2=c+1

k3=(2/d)

k4=gamma(1+k3)
```

```
k5=beta(a,b)

k6=(c*e)*k5

k7=e*((k2)^(-k3))

k8=(k1-k7)*(k4)

k9=k8/k6

k9


a=2

b=0.5

c=0.5

d=3

e=1

k1=c+1

k2=c+1

k3=(3/d)

k4=gamma(1+k3)

k5=beta(a,b)

k6=(c*e)*k5

k7=e*((k2)^(-k3))

k8=(k1-k7)*(k4)

k9=k8/k6

k9


a=2
```

```
b=0.5

c=0.5

d=3

e=1

k1=c+1

k2=c+1

k3=(4/d)

k4=gamma(1+k3)

k5=beta(a,b)

k6=(c*e)*k5

k7=e*((k2)^(-k3))

k8=(k1-k7)*(k4)

k9=k8/k6

k9


a=1

b=0.5

c=0.5

d=3

e=1

k1=c+1

k2=c+1

k3=(1/d)

k4=gamma(1+k3)
```

```
k5=beta(a,b)

k6=(c*e)*k5

k7=e*((k2)^(-k3))

k8=(k1-k7)*(k4)

k9=k8/k6

k9


a=1

b=0.5

c=0.5

d=3

e=1

k1=c+1

k2=c+1

k3=(2/d)

k4=gamma(1+k3)

k5=beta(a,b)

k6=(c*e)*k5

k7=e*((k2)^(-k3))

k8=(k1-k7)*(k4)

k9=k8/k6

k9


a=1
```

```
b=0.5

c=0.5

d=3

e=1

k1=c+1

k2=c+1

k3=(3/d)

k4=gamma(1+k3)

k5=beta(a,b)

k6=(c*e)*k5

k7=e*((k2)^(-k3))

k8=(k1-k7)*(k4)

k9=k8/k6

k9


a=1

b=0.5

c=0.5

d=3

e=1

k1=c+1

k2=c+1

k3=(4/d)

k4=gamma(1+k3)
```

```
k5=beta(a,b)

k6=(c*e)*k5

k7=e*((k2)^(-k3))

k8=(k1-k7)*(k4)

k9=k8/k6

k9
```

LBWD & LBND

```
a=2

b=2

c=1

d=3

e=1

k1=c+1

k2=c+1

k3=(1/d)

k4=gamma(1+k3)

k5=beta(a,b)

k6=(c*e)*k5

k7=e*((k2)^(-k3))

k8=(k1-k7)*(k4)

k9=(k8/k6)/6

k9


a=2
```

```
b=2

c=1

d=3

e=1

k1=c+1

k2=c+1

k3=(2/d)

k4=gamma(1+k3)

k5=beta(a,b)

k6=(c*e)*k5

k7=e*((k2)^(-k3))

k8=(k1-k7)*(k4)

k9=(k8/k6)/6

k9


a=2

b=2

c=1

d=3

e=1

k1=c+1

k2=c+1

k3=(3/d)

k4=gamma(1+k3)
```

```
k5=beta(a,b)

k6=(c*e)*k5

k7=e*((k2)^(-k3))

k8=(k1-k7)*(k4)

k9=(k8/k6)/6

k9


a=2

b=2

c=1

d=3

e=1

k1=c+1

k2=c+1

k3=(4/d)

k4=gamma(1+k3)

k5=beta(a,b)

k6=(c*e)*k5

k7=e*((k2)^(-k3))

k8=(k1-k7)*(k4)

k9=(k8/k6)/6

k9
```

AGWWD

```
a=2
```

```
b=2

c=1.5

d=3

e=1

k1=c+1

k2=c+1

k3=(1/d)

k4=gamma(1+k3)

k5=beta(a,b)

k6=(c*e)*k5

k7=e*((k2)^(-k3))

k8=(k1-k7)*(k4)

k9=(k8/k6)/6

k9


a=2

b=2

c=1.5

d=3

e=1

k1=c+1

k2=c+1

k3=(2/d)

k4=gamma(1+k3)
```

```
k5=beta(a,b)

k6=(c*e)*k5

k7=e*((k2)^(-k3))

k8=(k1-k7)*(k4)

k9=(k8/k6)/6

k9


a=2

b=2

c=1.5

d=3

e=1

k1=c+1

k2=c+1

k3=(3/d)

k4=gamma(1+k3)

k5=beta(a,b)

k6=(c*e)*k5

k7=e*((k2)^(-k3))

k8=(k1-k7)*(k4)

k9=(k8/k6)/6

k9

a=2

b=2
```

```
c=1.5

d=3

e=1

k1=c+1

k2=c+1

k3=(4/d)

k4=gamma(1+k3)

k5=beta(a,b)

k6=(c*e)*k5

k7=e*((k2)^(-k3))

k8=(k1-k7)*(k4)

k9=(k8/k6)/6

k9
```

LWWD

```
a=1

b=1

c=1.5

d=3

e=1

k1=c+1

k2=c+1

k3=(1/d)

k4=gamma(1+k3)

k5=beta(a,b)
```

```
k6=(c*e)*k5

k7=e*((k2)^(-k3))

k8=(k1-k7)*(k4)

k9=(k8/k6)/6

k9


a=1

b=1

c=1.5

d=3

e=1

k1=c+1

k2=c+1

k3=(2/d)

k4=gamma(1+k3)

k5=beta(a,b)

k6=(c*e)*k5

k7=e*((k2)^(-k3))

k8=(k1-k7)*(k4)

k9=(k8/k6)/6

k9


a=1

b=1
```

```
c=1.5

d=3

e=1

k1=c+1

k2=c+1

k3=(3/d)

k4=gamma(1+k3)

k5=beta(a,b)

k6=(c*e)*k5

k7=e*((k2)^(-k3))

k8=(k1-k7)*(k4)

k9=(k8/k6)/6

k9


a=1

b=1

c=1.5

d=3

e=1

k1=c+1

k2=c+1

k3=(4/d)

k4=gamma(1+k3)

k5=beta(a,b)
```

```
k6=(c*e)*k5

k7=e*((k2)^(-k3))

k8=(k1-k7)*(k4)

k9=(k8/k6)/6

k9
```

LWD, LND & LLogD

```
a=1

b=1

c=1

d=3

e=1

k1=c+1

k2=c+1

k3=(1/d)

k4=gamma(1+k3)

k5=beta(a,b)

k6=(c*e)*k5

k7=e*((k2)^(-k3))

k8=(k1-k7)*(k4)

k9=(k8/k6)/6

k9


a=1

b=1
```

```
c=1

d=3

e=1

k1=c+1

k2=c+1

k3=(2/d)

k4=gamma(1+k3)

k5=beta(a,b)

k6=(c*e)*k5

k7=e*((k2)^(-k3))

k8=(k1-k7)*(k4)

k9=(k8/k6)/6

k9


a=1

b=1

c=1

d=3

e=1

k1=c+1

k2=c+1

k3=(3/d)

k4=gamma(1+k3)

k5=beta(a,b)
```

```
k6=(c*e)*k5

k7=e*((k2)^(-k3))

k8=(k1-k7)*(k4)

k9=(k8/k6)/6

k9

a=1

b=1

c=1

d=3

e=1

k1=c+1

k2=c+1

k3=(4/d)

k4=gamma(1+k3)

k5=beta(a,b)

k6=(c*e)*k5

k7=e*((k2)^(-k3))

k8=(k1-k7)*(k4)

k9=(k8/k6)/6

k9


{\bf LWeib, LND \& LLog}

k=((0.25000)-(3*(0.17953*0.20613))+(2*(0.17953)^3))

kk=(0.17390)^1.5
```

```
kkk=k/kk

kkk

k=((0.31813)-(4*(0.17953*0.25000))+(6*((0.17953)^2)*0.20613)-(3*(0.17953)^4))

kk=(0.17390)^2

kkk=(k/kk)-3

kkk
```

LWWD

```
k=((0.23333)-(3*(0.17494*0.19631))+(2*(0.17494)^3))

kk=(0.16570)^1.5

kkk=k/kk

kkk

k=((0.29174)-(4*(0.17494*0.23333))+(6*((0.17494)^2)*0.19631)-(3*(0.17494)^4))

kk=(0.16570)^2

kkk=(k/kk)-3

kkk
```

AGWWD

```
k=((1.40000)-(3*(1.04967*1.17785))+(2*(1.04967)^3))

kk=(0.07606)^1.5

kkk=k/kk

kkk

k=((1.75046)-(4*(1.04967*1.40000))+(6*((1.04967)^2)*

1.17785)-(3*(1.04967)^4))

kk=(0.07606)^2
```

```
kkk=(k/kk)-3

kkk
```

LBWD & LBND

```
k=((1.50000)-(3*(1.08720*1.23690))+(2*(1.08720)^3))

kk=(0.054896)^1.5

kkk=k/kk

kkk

k=((1.90877)-(4*(1.07720*1.50000))+(6*((1.07720)^2)*

1.23690)-(3*(1.07720)^4))

kk=(0.07644)^2

kkk=(k/kk)-3

kkk
```

LWeighted D

```
k=((1.62400)-(3*(0.23333*0.52000))+(2*(0.23333)^3))

kk=(0.46556)^1.5

kkk=k/kk

kkk

k=((6.5984)-(4*(0.23333*1.62400))+(6*((0.23333)^2)*

0.52000)-(3*(0.23333)^4))

kk=(0.46556)^2

kkk=(k/kk)-3

kkk
```

The histogram and theoretical densities of Weibull, Lognormal, Gamma and BWW distribution

```
fw <- fitdist(Duration, "weibull")

summary(fw)

fg <- fitdist(Duration, "gamma")

fln <- fitdist(Duration, "lnorm")

plot.legend <- c("Weibull", "lognormal","gamma")

denscomp(list(fw, fln,fg), legendtext = plot.legend)

a=7

b=7

c=3

d=3.5

Duration=x

x=seq(0,1.2,0.01)

bww1.pdf<-function(x,a,b,c,d){

k1=(1-exp(-x^d))

k2=(1-exp(-(1+c)*x^d))/(c+1)

k3=((c+1)/c)*k1

k6=(1-exp(-c*x^d))

k7=((c+1)/c)*d*x^(d-1)*exp(-x^d)

k8=(k3-k2)^(a-1)

k9=(1-(k3-k2))^(b-1)

k4=beta(a,b)

k5=1/k4

bww1.pdf=k5*k8*k9*k7*k6

}
```

```
lines(x,bww1.pdf(x,7,7,3,3.5),col="black",
main="(i)",ylim=c(0,5),lty=2)
legend("topleft",col=c("black"),legend=c("i"),lty=1:2:4)
legend("right",inset=0.10,col=c("black"),legend=c("BWW"),lty=1:2:4)


fw <- fitdist(Duration, "weibull")
summary(fw)
fg <- fitdist(Duration, "gamma")
fln <- fitdist(Duration, "lnorm")
plot.legend <- c("Weibull", "lognormal","gamma")
denscomp(list(fw, fln,fg), legendtext = plot.legend)
a=5
b=5
c=3.5
d=3
Duration=x
x=seq(0,1.2,0.01)
bww1.pdf<-function(x,a,b,c,d){
k1=(1-exp(-x^d))
k2=(1-exp(-(1+c)*x^d))/(c+1)
k3=((c+1)/c)*k1
k6=(1-exp(-c*x^d))
k7=((c+1)/c)*d*x^(d-1)*exp(-x^d)
k8=(k3-k2)^(a-1)
```

```
k9=(1-(k3-k2))^(b-1)

k4=beta(a,b)

k5=1/k4

bww1.pdf=k5*k8*k9*k7*k6

}

lines(x,bww1.pdf(x,5,5,3,3.5),col="black",main="(ii)",ylim=c(0,5),lty=4)

legend("topleft",col=c("black"),legend=c("ii"),lty=1:2:4)

legend("right",inset=0.10,col=c("black"),legend=c("BWW"),lty=1:2:4)
```

```
fw <- fitdist(Duration, "weibull")

summary(fw)

fg <- fitdist(Duration, "gamma")

fln <- fitdist(Duration, "lnorm")

plot.legend <- c("Weibull", "lognormal","gamma")

denscomp(list(fw, fln,fg), legendtext = plot.legend)

a=3.5

b=3.5

c=3

d=3.5

Duration=x

x=seq(0,1.2,0.01)

bww1.pdf<-function(x,a,b,c,d){

k1=(1-exp(-x^d))
```

```
k2=(1-exp(-(1+c)*x^d))/(c+1)

k3=((c+1)/c)*k1

k6=(1-exp(-c*x^d))

k7=((c+1)/c)*d*x^(d-1)*exp(-x^d)

k8=(k3-k2)^(a-1)

k9=(1-(k3-k2))^(b-1)

k4=beta(a,b)

k5=1/k4

bww1.pdf=k5*k8*k9*k7*k6

}

lines(x,bww1.pdf(x,3.5,3.5,3,3.5),col="black",

main="(iii)",ylim=c(0,5),lty=6)

legend("topleft",col=c("black"),legend=c("iii"),lty=1:2:4)

legend("right",inset=0.10,col=c("black"),legend=c("BWW"),lty=1:2:4)


fw <- fitdist(Duration, "weibull")

summary(fw)

fg <- fitdist(Duration, "gamma")

fln <- fitdist(Duration, "lnorm")

plot.legend <- c("Weibull", "lognormal","gamma")

denscomp(list(fw, fln,fg), legendtext = plot.legend)

a=2.5

b=2.5

c=3.5
```

```
d=3
Duration=x
x=seq(0,1.2,0.01)
bww1.pdf<-function(x,a,b,c,d){
k1=(1-exp(-x^d))
k2=(1-exp(-(1+c)*x^d))/(c+1)
k3=((c+1)/c)*k1
k6=(1-exp(-c*x^d))
k7=((c+1)/c)*d*x^(d-1)*exp(-x^d)
k8=(k3-k2)^(a-1)
k9=(1-(k3-k2))^(b-1)
k4=beta(a,b)
k5=1/k4
bww1.pdf=k5*k8*k9*k7*k6
}
lines(x,bww1.pdf(x,2.5,2.5,3,3.5),col="black",
main="(iv)",ylim=c(0,5),lty=8)
legend("topleft",col=c("black"),legend=c("iv"),lty=1:2:4)
legend("right",inset=0.10,col=c("black"),legend=c("BWW"),lty=1:2:4)
```

The dummy trapping (coding) Analysis

```
de<-read.csv("pm.csv",header=1)
de
AR X0 X0.1 X0.2 X0.3 X0.4 X0.5 X0.6 X0.7 X0.8
```

```
1 AG 1    0    0    0    0    0    0    0    0

2 FB 0    1    0    0    0    0    0    0    0

3 IN 0    0    1    0    0    0    0    0    0

4 PH 0    0    0    1    0    0    0    0    0

5 PU 0    0    0    0    1    0    0    0    0

6 SC 0    0    0    0    0    1    0    0    0

7 SS 0    0    0    0    0    0    1    0    0

8 TH 0    0    0    0    0    0    0    1    0

9 VM 0    0    0    0    0    0    0    0    1

X<-as.factor(sample(LETTERS[1:10],50,replace=TRUE))

y<-rnorm(50)

lm(y~X)

lm(formula=y~X)

Call:

lm(formula = y ~ X)

summary(lm(y~X))

Call:

lm(formula = y ~ X)

Residuals:

Min       1Q   Median       3Q      Max

-2.76123 -0.61166  0.04582  0.50666  2.01215

Coefficients:

Estimate Std. Error t value Pr(>|t|)

(Intercept)  -0.5641     0.5170  -1.091   0.2818
```

```
XB            0.8974     0.7898   1.136   0.2626

XC            0.8118     0.6675   1.216   0.2310

XD            0.5816     0.6937   0.838   0.4067

XE            1.2616     0.7898   1.597   0.1180

XF            0.8476     0.7898   1.073   0.2896

XG            0.8201     0.6937   1.182   0.2441

XH            1.9266     0.8955   2.151   0.0375 *

XI            0.7904     0.5970   1.324   0.1930

XJ            0.2395     0.6481   0.370   0.7136

Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 1.034 on 40 degrees of freedom

Multiple R-squared:  0.1556,    Adjusted R-squared:  -0.03441

F-statistic: 0.8189 on 9 and 40 DF,  p-value: 0.6023
```