SENSITIVITY ANALYSIS OF INTEREST RATE DERIVATIVES IN SOME LÉVY MARKETS

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ABSTRACT

Interest Rate Derivatives (IRDs) are generally jump-diffusion processes which are usually modelled with Lévy processes. Brownian motion has been used extensively for modelling IRDs, however, this does not capture the jumps inherent in the IRDs. To hedge risks in a Lévy market, it is important to consider the presence of jumps. This work was therefore designed to model IRDs driven by some subordinated Lévy processes that consider jumps.

The classical Vasicek short rate model $dr_t = a(b - r_t)dt + \sigma dW_t$ (where r_t, a, b, σ and W_t denote interest rate, speed of mean-reversion, long-term mean rate, volatility of the short rate model and Brownian motion, respectively) was extended to a model driven by subordinated Lévy processes using Itô formula for semimartingales. Using the extended Vasicek model, expressions for the price of IRDs: zero-coupon bond, with Variance Gamma (VG) and Normal Inverse Gaussian (NIG) as the underlying sources of uncertainties, were derived. Expressions for the greeks were derived by means of Skorohod integral, Ornstein-Uhlenbeck operator and the Malliavin calculus. Consequently, the greeks obtained were used to determine the sensitivities of the parameters of the model. Monthly dataset of the Nigerian Interbank Offer Rate from 2007 to 2017 was obtained from the Central Bank of Nigeria website and used to validate the model.

The greek expressions that measure the price sensitivities to interest rate, namely, the delta $\triangle^{VG}$ associated with the VG process and the delta $\triangle^{NIG}$ associated with the NIG process were obtained as

$$\triangle^{VG} = e^{-r_0 T}\Bigg(-T\mathbb{E}[\Phi(P)] + \mathbb{E}\Bigg[\Phi(P)\Bigg(\frac{\sigma^2}{a}(e^{-aT} - e^{-at})\mathcal{K}^{-2}\Big(\sum_{t\leq u\leq T}(\widetilde{\sigma}\Delta\sqrt{G(u)})^2\Big) - \frac{1}{a}(e^{-aT} - e^{-at})\mathcal{K}^{-1}Z\Bigg)\Bigg]\Bigg),$$

$$\triangle^{NIG} = e^{-r_0 T}\Bigg(-T\mathbb{E}[\Phi(P)] + \mathbb{E}\Bigg[\Phi(P)\Bigg(\frac{1}{a}(e^{-aT} - e^{-at})\sigma^2\widetilde{K}^{-2}\Big(\sum_{t\leq u\leq T}(\delta\Delta\sqrt{IG(u)})^2\Big) - Z\frac{1}{a}(e^{-aT} - e^{-at})\widetilde{K}^{-1}\Bigg)\Bigg]\Bigg);$$

where $e, r_0, T, P, \Phi(P), \mathbb{E}[\Phi(P)], a, \sigma, \mathcal{K}, \widetilde{K}, \widetilde{\sigma}, \theta, \delta$ and β denote exponential, initial interest rate, expiration time, zero-coupon bond price, payoff function, expectation of the payoff function, mean-reversion of the extended Vasicek model,

volatility of the extended Vasicek model,

$$\sigma\widetilde{\sigma}\sum_{t\leq u\leq T}\sum_{0\leq s\leq t}\Delta\sqrt{G(s)}e^{-a(u-s)}+\sigma\widetilde{\sigma}\sum_{t\leq u\leq T}((\Delta\sqrt{G(u)})$$

$$-\sigma^2\widetilde{\sigma}\sum_{t\leq u\leq T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)},$$

$$\sigma\delta\sum_{t\leq u\leq T}\sum_{0\leq s\leq t}(\Delta\sqrt{IG(s)}e^{-a(u-s)})+\sigma\delta\sum_{t\leq u\leq T}(\Delta\sqrt{IG(u)})$$

$$-\sigma^2\delta^2\sum_{t\leq u\leq T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)},$$

volatility of the VG process, skewness of the VG process, volatility of the NIG process and skewness of the NIG process, respectively. The Z, G and IG are certain random variables, and $\Delta G(t) = G(t) - G(t_-)$. Other greek expressions derived include gamma^{VG}, $\text{gamma}^{\text{NIG}}$, vega^{VG}, vega^{NIG}, $\text{vega}_2^{\text{VG}}$, $\text{vega}_2^{\text{NIG}}$, $\text{vega}_3^{\text{VG}}$, $\text{vega}_3^{\text{NIG}}$, and $\text{vega}_4^{\text{NIG}}$. The zero-coupon bond prices were found to be suitable for both skewed and heavily-tailed IRDs markets. The greek delta indicates the sensitivity of the zero-coupon bond price to changes in the interest rate. The dynamics of the extended Vasicek model and zero-coupon bond price for the Nigeria market revealed that the distribution of the IRDs was skewed to the left and heavily-tailed. This was an indication of high risk in the Nigerian market.

The newly extended Vasicek model captured the jumps in the interest rate Lévy market. This model should be applied in the interest rate derivative market in order to monitor and minimise risks.

Keywords: Subordinated Lévy processes, Extended Vasicek model, Variance gamma, Normal inverse Gaussian, Greek expressions

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DEDICATION

To my loved ones.

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All the glory, honour and adoration be to the Almighty God for everything He has done and He will continue to do in my life. He is a faithful God.

CERTIFICATION

I certify that this work was carried out by **Miss Adaobi Mmachukwu Udoye** with matriculation number 128347 in the Department of Mathematics, Faculty of Science, University of Ibadan under my supervision

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Notations

$(\Omega, \mathcal{F}, \mathbb{P})$	Probability space
$(\mathbf{b}, \sigma^2, \nu)$	Lévy characteristic triplet, where $\mathbf{b}$, σ and ν denote drift coefficient, difussion coefficient and Lévy measure, respectively
$\mathcal{S}$	subordinator
G	Gamma random variable
Z	Gaussian random variable
IG	inverse Gaussian random variable
$\text{Gamma}(c, \lambda)$	Gamma distribution where λ and c denote scale parameter and shape parameter, respectively
$\mathbf{X} = \{X_i\}_{i=1}^n$	sequence of random variables
$\mathbb{R}$	set of real numbers
$C^\infty(\mathbb{R}^n)$	$\widetilde{F} : \mathbb{R}^n \to \mathbb{R}$ that are infinitely continuously differentiable
$S_{n,p}$	set of simple functionals in $C^p(\mathbb{R}^n)$
$D(\cdot)$	Malliavin derivative of $(\cdot)$
$U = (U_i)_{i \leq n}$	simple process of length n: $U_1 = u_1(X_1, ..., X_n)$
$P_{n,p}$	set of simple processes of length n in $C^p(\mathbb{R}^n)$
$\langle \cdot, \cdot \rangle$	inner product
$\phi(\cdot)$	characteristic function of $\cdot$
$\mathbb{F}$	the filtration $(\mathcal{F}_t, 0 \leq t \leq T)$
$\delta(u)$	Skorohod integral of u

$\psi(\cdot)$	characteristic exponent
$f(\cdot)$	density function of $\cdot$
$\varphi(x)$	$\partial_x \ln[f(x)]$
$\mathbf{w}$	cumulant generating function
$\mathcal{N}$	normally distributed
$\mathrm{NIG}(\alpha, \beta, \delta, \mu)$	NIG process with parameters α for tail heaviness, β for skewness, δ for scale and μ for location
$\mathrm{VG}(\theta, \sigma, \kappa)$	VG process with parameters θ being the drift of the arithmetic Brownian motion, σ for volatility, κ is variance of the subordinator
$M_X(\cdot)$	moment generation function of $(\cdot)$
$\mathbb{E}[\cdot]$	Expectation of '$\cdot$'
$f(t, T)$	forward rate at time t for maturity T
L	Ornstein-Uhlenbeck operator
$\Phi(\cdot)$	payoff function
$\mathcal{B}$	Borel σ-algebra
$\mathcal{M}(\cdot)$	$\langle D(\cdot), D(\cdot)\rangle$
$\mathbf{s}$	standard deviation
B_t	Bank account
$\mathbb{V}$	value of a zero-coupon bond
$\Delta G(t)$	$G(t) - G(t_-)$

$\Delta IG(t)$	$IG(t) - IG(t_{-})$
a	mean reversion speed
b	long-term mean rate

Abbreviations

CIR	Cox-Ingersoll-Ross
iid	independent identically distributed
IG	inverse Gaussian
VG	variance gamma
NIG	normal inverse Gaussian
EMM	Equivalent Martingale Measure
a.s.	almost surely
LIBOR	London interbank offer rate
NIBOR	Nigerian Inter-Bank Offered Rate
EURIBOR	Euro interbank offer rate
repo	repurchasement agreement
HJM	Heath, Jarrow and Morton
càdlàg	right continuous with left limit
O-U	Ornstein-Uhlenbeck

Contents

Chapter 1

General Introduction

1.1 Background

Lévy processes, a vital class of stochastic processes containing both Brownian motion and Poisson types, are standard examples of semimartingales and Markov processes. As a class of stochastic processes with discontinuous paths, they are applicable in finance and insurance, as well as the physical and biological sciences. They are used to model jumps. A Lévy-type market is characterised by semimartingale price processes in which the martingale part is represented as a sum of stochastic integrals with respect to a Brownian motion and a compensated Poisson random measure. Improved descriptions of price processes of financial assets are found by substituting Brownian motion with suitably chosen alternative Lévy process. Lévy-based models give better representations of financial markets than Brownian motion models. Thus, Lévy processes provide us with the appropriate frameworks to describe observations in real and risk-neutral worlds.

An interest rate derivative is a derivative whose underlying asset is the right to disburse or receive a certain sum at a particular interest rate. According to Investopedia (2017), an interest rate derivative refers to a financial instrument with a value that increases or decreases, based on movements in interest rates. In other words, an interest rate derivative is a financial instrument affected by movements in interest rates. Interest rate derivatives are used by corporate investors, for example, banks and insurance firms, as hedges to the movements in market interest rates.

Sensitivity analysis focuses on how variations or errors in parameter values impinge on model outputs (Rappaport, 1967). Bayazit and Nolder (2009) and

Bayazit (2010) computed some options' sensitivities with exponential Lévy processes as underlying. They also showed the application of Malliavin calculus to calculation of the sensitivities.

The few available works, which include Grandet (2011), considered sensitivity analysis of interest rate derivatives but did not consider the availability of excess kurtosis and leptokurticity. This thesis will extend the work of Bayazit and Nolder (2009) and Grandet (2011) to the interest rate derivatives market driven by Lévy processes. Two major types of subordinated Lévy processes namely, variance gamma (VG) and normal inverse Gaussian (NIG) processes, are considered in the model for the interest rate derivative. To carry out sensitivity analysis, greeks of the derived prices of the interest rate derivative are computed and necessary comparison is made.

1.2 Research Questions

The following questions are to be considered in the study:

(i) How can one inculcate jump in the pricing of interest rate derivatives?

(ii) What model will solve the problem of skewness; tail heaviness or excess kurtosis in the interest rate derivative market?

(iii) Which method is suitable to measure the effects of changes in the parameters of an interest rate derivative in a Lévy market?

(iv) How can risk be reduced in the interest rate Lévy market?

1.3 Aim of the Study

The aim of the study is to contribute to the theory of interest rate derivatives in Lévy markets.

1.4 Objectives of the Study

The objectives are:

(1) to extend the existing Vasicek interest rate model to a Lévy market;

(2) to employ the improved Vasicek model in deriving an expression for an interest rate derivative in a Lévy market;

(3) to apply the Malliavin calculus in the sensitivity analysis of the interest rate derivative; and

(4) to compare the greeks of the prices of interest rate derivatives driven by subordinated Lévy processes.

1.5 Significance of the Study

This work extends the Vasicek interest rate model from a Brownian motion market to a Lévy market. It obtains expressions for interest rate derivative driven by subordinated Lévy processes, namely, variance gamma and normal inverse Gaussian processes. Using integration by parts formula of Malliavin calculus, we obtain the greeks involved in an interest rate derivative in a Lévy market. The greeks are useful in hedging risk in the interest rate derivative market.

1.6 Scope of Coverage

The study considers two types of Lévy processes, namely: variance gamma and normal inverse Gaussian processes in the interest rate Lévy market. The computation of the effects of the parameters of the models are also discussed.

1.7 Organisation of the Thesis

Following this chapter is Chapter 2 which discusses the contributions of some individuals in sensitivity analysis of financial instruments with emphasis on Lévy market. Chapter 3 discusses existing models and methodology to be employed in generating our results. Chapter 4 contains the main results, while Chapter 5 discusses the applications of our results. Chapter 6 contains summary of the work and future research.

Chapter 2

Literature Review

2.1 Background

This chapter discusses the contributions of different individuals in the theory of interest rate derivatives. Section 2.2 focuses on some contributions related to the Lévy market while section 2.3 discusses contributions of some researchers in the interest rate derivatives market, in order to identify gaps and establish the need for this research. Section 2.4 lay emphasis on types of interest rate derivatives whereas section 2.5 focuses on some existing interest rate models.

The term 'Lévy processes' is in honour of the French mathematician, Paul Lévy (1886-1971). He contributed to the study of Gaussian processes, law of large numbers, stable and infinitely divisible laws, the central limit theorem and pioneered processes of independent and stationary increments. Between 1930s and 1940s, the major contributors of Lévy processes were Paul Lévy, a Russian mathematician, Aleksandr Khintchine (1894-1959) in the fields of probability theory and number theory, and a Japanese mathematician, Kiyoshi Itô (1915-2008), whose work advanced the understanding of random events. Lévy processes are a popular tool in engineering, physics, mathematical finance, crude oil options, etc. (Kyprianou, 2006; Papapantoleon, 2008).

2.2 Applications of Lévy processes

Mandelbrot (1963) introduced α-stable Lévy distributions for modelling asset prices in order to overcome the deficiency that the Gaussian distribution does not address, tail-heaviness and asymmetry of financial return series. (Kim et al. 2008; Raible, 2000). Adopting the α-stable distributions, Rachev et al.

(2005) gave financial models for credit and market risk control, option pricing, and portfolio choosing; they concluded that empirical evidence does not support the use of Gaussian distributions and α-stable distributions. Moreover, Kim et al. (2008) found that the distribution of returns for assets has weightier tails than the Gaussian distribution but thinner tails when compared to the α-stable distributions. A number of extensions of the α-stable distributions were suggested in the literature, namely, the 'classical tempered stable'(CTS) distribution (Boyarchenko and Levendorskiĭ, 2000; Carr et al., 2002) and the 'modified tempered stable'(MTS) distribution (Kim et al., 2007). Furthermore, Kim et al. (2007) also introduced an extended version of the CTS distribution, namely, KR-distribution. Later, Kim et al. (2008) proposed subclasses of the tempered distribution (KR-distribution) as a model for describing return distribution. However, Salminen and Yor (2007) developed a Tanaka formula for local times of symmetric α-stable Lévy processes for $\alpha \in (1, 2]$ and determined which powers of such processes are semimartingales.

Studies of some specific types of Lévy processes are also carried out by researchers. The normal inverse Gaussian (NIG) process was pioneered by Barndorf-Nielsen in 1995 to generate better models for log-return price processes and exchange rates (Barndorff-Nielsen, 1998). The processes allow jumps and important empirical properties of skewness with fat tails to be modelled in a better way. Rydberg (1997) studied NIG process in connection with German and Danish securities. He provided an approximation of the process that allows for an equivalent martingale measure. Núñez et al. (2018) suggested the NIG process to replace normality assumption of underlying asset returns since it can model heavy tails, a fact mainly found in returns data series. Benth et al. (2018) modelled the logarithm of spot price of electricity driven by an NIG process by replacing the small jumps of the process with a Brownian term.

Madan and Seneta (1990) introduced the symmetric variance gamma (VG) process as a Lévy process for modelling of stock market returns, while, Madan et al. (1998) utilised the asymmetry aspect of the VG process to obtain a closed-form solution for return densities as well as prices of European options. Hamza et al. (2015) considered option pricing when assumption of normality is replaced with that of symmetry of the underlying distribution, and obtained Black-Scholes (B-S) type option pricing formulae for symmetric VG and symmetric NIG processes.

Hainaut and MacGilchrist (2010) proposed an interest rate model driven by NIG where the stochastic differential equation chosen to govern the short term rate is Hull-White model. The Brownian motion is replaced by an NIG process since it provides a better fit of bond returns and captures the asymmetry and leptokurticity of short term rates distribution.

There are some related work on the application of Lévy processes and Malliavin calculus in sensitivity analysis as given below.
Petrou (2008) generalised results of Fournié et al. (1999) by extending the theory of Malliavin calculus to provide tools for sensitivity analysis in Lévy markets. The tools involve differentiability results for the solution of a stochastic differential equation. Bavouzet-Morel and Messaoud (2006) developed a Malliavin calculus for jump processes by working on functionals of a fixed set of random variables.

Bayazit (2010), in his thesis, applied Malliavin calculus in the sensitivity analysis of options under a Lévy market. Bavouzet et al. (2009) discussed its application to jump-market models, and provided numerical steps for sensitivity calculations of European options and American options pricing under compound Poisson process. Andersson and Lindiner (2017) introduced Hilbert space-valued Malliavin calculus for Poisson random measures. El-Kihatib and Hatemi-J (2018) applied the calculus to compute price sensitivities of stochastic volatility model.

2.3 Interest rate derivatives

There are ample publications (Khoshnevisan and Xiao, 2004; Chernov et al., 1999; Fajardo and Mordecki, 2003; Geiss and Laukkarinen, 2011; Klingler et al., 2013; Yang and Zhang, 2001; Vives, 2013) on the applications of Lévy processes in financial markets but little has been done on the interest rate derivative markets.

Chacko and Das (2000) discussed short rate modelling with emphasis on fixed income pricing, and applied jump-diffusion model to derive zero-coupon bond price. Zhou (2000) applied a multivariate weighted non-linear least square estimator for a set of jump-diffusion interest rate models that allow closed form solutions for bond prices under no-arbitrage situation. Kim and Wright (2014) derived a no-arbitrage term structure model involving jumps of random sizes and applied their model to term structure of the United States (US) treasury rates.

Hin and Dokuchaev (2015) proposed an approach to get information on investors expectation of forthcoming short date from zero-coupon bond prices in order to obtain a reasonable forecast based on inference from Cox-Ingersoll-Ross (CIR) model for extended yield curve dynamics. Swishchuk (2008) stated how to derive zero-coupon bond price for Gaussian Lévy one-factor and multi-factor models, using a change of time method. Furthermore, Sarais (2015), in his Ph.D thesis, developed a model to price inflation and interest rate derivatives using continuous-time dynamics associated to monetary macroeconomic models. Waldenberger (2017) proposed a model to price certain interest rate derivatives, namely: caps and floors. Pintoux and Privault (2017) computed zero-coupon bond price using the interest rate model of Dothan via integral representations of heat kernels.

Magnou (2017) proposed an approach for pricing fixed-income derivatives by introducing hedging derivatives in the Uruguayen market in order to minimise the risk of volatility threats. Yin et al. (2018) considered a corporate bond-pricing model of credit rating risk using Vasicek model, and observed possibility of jumps when there is credit-rating altering for the bond.

Brigo and Alfonsi (2005) introduced a shifted root diffusion model of two dimensions for interest rate derivatives and presented an analytical approximation for certain terms in credit derivatives involving CIR processes. Bormetti et al. (2018) presented an analysis of interest rate derivatives in credit risk and collateral modelling.

Epstein et al. (1999) described convertible bond as coupon-influenced bond where the holder gets coupon payments at predetermined periods; applied a non-probabilistic, non-linear interest rate and Vasicek model to derive its price, and discussed the sensitivity of the convertible bond price to changes in the parameters of the Vasicek model. Jiao et al. (2016) extended the CIR model by introducing a jump part driven by an α-stable Lévy process where $\alpha \in (1, 2]$, deduced an expression for bond price and concluded that the behaviour of bond price increases with respect to tail heaviness as observed in extended model with jumps in Duffie and Gârleanu (2001). Teneberg (2012) extended equity pricing model of standard geometric Brownian motion to jump-diffusion processes, and used the model to price convertible bonds.

Annaert et al. (2007) derived expressions for zero-coupon bond and coupon bond using Hull-White one-factor model calibrated to a class of cap prices and hedged a Belgian government bond by considering different values at risk measures. Park et al. (2014) proposed closed-form solutions on jump models of HJM and Hull-White for bond option pricing. Park and Kim (2015) derived solutions of Hull-White model with jumps using differential equations, and suggested that the connection between short rate and forward rate processes can be used to derive a formula for bond price. Ma (2003) extended HJM (1992) representation of term structure of interest rates in a jump-diffusion framework. Kluge (2005) presented interest rate model and credit risk model under time-inhomogeneous

Lévy processes.

Schönbucher (1996) derived an expression for zero-coupon bond price whose underlying short rate is driven by HJM model and concluded that their model permits jumps in defautable rates at defaultable times. Huotari (2016) presented an approach to model interest rate market when swap rates are normally distributed with jumps. Vullings (2016) proposed a type of contingent convertible bond with a market-based trigger and floating coupons, where the coupons rise near the trigger price to recompense holders for the possibility of bankruptcy before conversion.

Küchler and Naumann (2003) discussed Markovian short rates in a forward rate model of Lévy processes. Pirjol (2012) considered a class of interest rate models, and showed a relationship between the interest rate and lattice gases for attractive two-body interaction. Kooiman (2015) observed that there is a non-negligible possibility of floating rate dropping below zero so that bank pays twice (since receiving the floating rate will be equivalent to making a payment), and suggested using Hull-White model for interest rate modelling since it has the possibilities of negative interest rate. Kurman (2017) examined determinants of interest rate derivatives in some Indian commercial listed banks using simulation and market interest rate sensitivity, and noted that interest rate risks influence derivatives usage by banks. Sosa and Mordecki (2016) applied Gaussian model to derive a bond's price curve corresponding to sovereign Uruguayan debt.

Huang (2005) considered Lévy jump processes in a class of affine structure models of corporate bond pricing whose underlying asset return involves a high frequency jump component and a stochastic volatility model. Collin-Dufresne and Goldstein (2002) introduced an approach based on cumulant expansion for pricing of coupon bond options in affine structure.

Itkin and Lipton (2015) applied correlated jumps to model credit risk, while Itkin (2017) extended the work in pricing and hedging of exotic options using local stochastic volatility model modified to include stochastic interest rates.

Brigo et al. (2015) derived a model of quanto credit default swap based on a reduced form model for credit risk, introduced jump-at-default in foreign exchange dynamics and showed that it provides a better way to model credit or foreign exchange dependency.

Park et al. (2006) applied Monte-Carlo simulation method for bond option pricing with jumps by extending Vasicek and CIR models to include jumps. Lang et al. (2018) studied how different choices of interest rate models by banks affect financial stability and observed that good interest rate models do not entail aggregate financial stability.

Kou (1999) extended the work of Glasserman and Kou (1999) on the pricing of interest rate caps and floors with jump risk, and developed a solution for prices of caps and floors in a double exponential jump model that produces a volatility smile. Das (2002) observed that information surprises lead to discontinuity of interest rates in bond markets and derived a set of Poisson-Gaussian models in order to capture the consequences.

Coke (2016) estimated quarterly government of Jamaica zero-coupon bond yield curve, and fitted it into interest rates stress testing structure to measure the effect on portfolio holdings. Grandet (2011) considered sensitivity analysis and stress testing in an interest rate market, and discussed sensitivity analysis of zero-coupon bond price in a Brownian motion market.

2.4 Some Interest Rate Derivatives

In this section, we give description of interest rate derivatives and lay emphasis on bonds. However, our work will concentrate on a type of bond called the zero-coupon bond.

An *interest rate derivative* is a financial instrument whose underlying asset is an interest rate. A forward contract (known as futures contract if it is on exchange) is an agreement flanked by two parties where one buys an asset from the counterparty on a given future date for a predetermined price.

In what follows, we discuss some interest rate derivatives.

2.4.1 Swap products and swaptions

A *swap* is a financial agreement amid two parties to interchange cash flows later, based on certain predetermined plan. In a cross-currency swap, the payments of the two legs depend on the floating rates of interest in two separate currencies. Swap contracts have been in existence since 1981 (Carmona and Tehranchi, 2006). A swaption is an option whose underlying asset must be a swap. It gives the investor the privilege to go into a particular interest rate swap at particular time in the future.

An *interest rate swap* is a type of swap that include asset swap, basis swap, currency swap and forward rate agreement. Interest rate swap markets have experienced great increase since what is commonly regarded as first swap was executed in 1981 (Corb, 2012). A fixed-for-floating swap is an interest rate swap where a sequence of payments obtained by applying a fixed interest rate to a principal amount, are interchanged for a sequence of payments obtained using a floating rate of interest. Cash flows are exchanged in net sum on selected swap dates all through the swap contract's life. All payments can be made in similar currency. The principal amount is termed *notional* as no exchange occurs on the principal, and is employed only to calculate actual amount to be swapped at regular intervals on the swap dates. One can take the floating rate to be any money market rate, e.g., London interbank offer rate (LIBOR), federal fund rate and treasury bill rate.

LIBOR is the interest rate that leading banks propose to pay on Eurodollar deposits accessible to other leading banks for a predetermined maturity. A Eurodollar is a US dollar deposited in and outside US banks. LIBOR comes with diverse maturities, e.g., one-month, three-month and six-month LIBOR. A pair of distinct reference floating rates are employed to compute the exchange payments in the floating-for-floating interest rate swaps.

A *margin swap* is a type of swap where the parties prefer the rates for both sides to be boosted by a margin, for adequate accounting.

Based on the pattern of LIBOR, Nigeria Inter-Bank Offered Rate (NIBOR) was created on 6th April, 1998 in UK. A quote is the rate naira is offered in the inter-bank markets.

A *basis swap* allows exchange of floating-rate cash flows between parties to an agreement.

A *currency swap* is used to exchange loans in different currencies. There is a spread between the market price and the plain full pricing 'swapwise' of the product. Its outlook is that exchanging interest rates from one currency to another adds some risks which come from the fact that the aim of a cross-currency swap is to get money in the foreign currency to invest it in assets quoted in the same currency. The exchange rate, spot or forward has no impact on the product's price. The spread in the price reflects the difference of liquidity available in each currency which may bring about a rise in the interest rate. It is quoted on the market as a *basis spread* with respect to a reference index. The basis swap are officially quoted against USD LIBOR.

A *plain vanilla swap* is a contract between two parties to trade a fixed rate against floating one (commonly EURIBOR).

2.4.2 Futures

A *future contract* is an arrangement connecting two parties to interchange certain goods at a certain rate at a given future date.

2.4.3 Repos

A *repo* (repurchasement agreement) is a kind of secured short-term lending, mostly between banks, where the counterpart gives a security as collateral. It is a two-way transaction where one party accepts to sell securities to another and accepts to buy back identical securities on a specified date at a specified price

(Grandet, 2011).

The most famous repos is the treasury repos where the security is a *bond*. The rates are quoted on the market and depend highly on the quality of the security and the counterpart. Sometimes banks prefer third party to direct transaction with another bank when arranging a repo; this is known as *tri-party repos* (Brown, 2006).

2.4.4 Caps and floors

A *caplet* is a contract in which the interest rate on a loan with floating rate, at any stipulated time, becomes the least of the existing LIBOR rate with predetermined cap rate.

A *floorlet* is a contract that allows the holder to get the highest of the current floating rate with the predetermined floor rate on a floating rate deposit. The holder is to have the least rate level for his floating rate deposit.

2.4.5 Interest rate options

An *interest rate call option* is an agreement between two parties which gives the holder right and not obligation to purchase an underlying asset at a given price and date called *strike price* and *expiry date*, respectively, while it gives the issuer the responsibility to sell the underlying asset at a given price and date.

An *interest rate put option* is an agreement between two parties which offers the holder the privilege and not duty to sell an underlying asset at a predetermined price and at a given time, while it offers the issuer the obligation to purchase the underlying asset at a given price and date.

European-style interest rate options are option contracts that can only be exercised on the expiry date. *An American-style interest rate option* can be exercised at any time during the life of the option contract. *A Bermudian-style interest rate options* are option contracts that can be exercised at some predetermined occasion during the lifetime of the option.

2.4.6 Bond

A *bond* is a loan from one party known as the holder to another known as the issuer. The issuer grants the investor an assurance of interest rate payment on the loan at specified intervals, then reimburse the loan at a given future time. The issuer can maintain or allow embedded option that he or the investor can exercise soon. Bonds can be grouped into *fixed-rate* such as *zero-coupon bonds* and *undated bonds*, *floating-rate* and *index-linked bonds* (Brown, 2006: 2).

Let $P(t,T)$ be a zero-coupon bond price at time t of a currency unit received for sure at time T. It can also be regarded as the *discounting factor* for cash-flows occurring at time T. The bond does not pay interest intermittently, but gives a face value that will be paid at maturity; the interest earned emerges as a discount to the face value at the point of starting, and depends on the maturity time. Under normal condition, the interest rate paid for a bond with many years to maturity is bigger than ones close to maturity. Given a set of zero-coupon bond prices $P(t,T), t < T \leq T^*$, the term structure of interest rate is the set of yields to maturity $r(t,T), t < T \leq T^*$ given (using continuous compounding) by

$$r(t,T) = -(T-t)^{-1} \log P(t,T), t < T \leq T^*,$$

known as the *yield to maturity.* The *yield curve* represents the plot of $r(t,T)$ against $T-t$, while reliance of the yield curve on the maturity time $T-t$ is called its *term structure.*

A *coupon bond* involves series of payments: $C_1, C_2, ..., C_n$, at times $T_1, T_2, ..., T_n$ and a terminal payment at maturity date T_n.

Treasury bills

They are securities issued by US government with maturity date of maximum of a year. They do not carry coupon payments. A treasury bill is an example of a zero-coupon bond. Yields, rates, spreads, etc., are commonly quoted in basis points. The treasury releases bills with maturity time: 13, 26 and 52 weeks, respectively called 3-month, 6-month and 1-year bills, although the names can be correct only at their beginning (Carmona and Tehranchi, 2006). 13 and 26 week bills are auctioned off on Mondays while 52 week bills are auctioned off monthly.

The short rate

The short rate r_t is the rate on instant borrowing and lending. It is stochastic. In some markets, the overnight interest rate is usually not considered as a good approximation for the short rate.

A unit sum invested in the rate at time 0 and instantaneously reinvested, is called the *money-market account*, and it is controlled by

$$\frac{dB_t}{dt} = r_t B_t, \; B_0 = 1.$$

If r is deterministic and constant, B_t reduces to the classical bank account:

$$B_t = \exp(rt).$$

2.4.7 The instantaneous forward rate

This is a rate at time t for date $T > t$ defined as

$$f(t,T) = -\frac{\partial}{\partial T} \ln P(t,T). \tag{2.4.1}$$

It is a contractable rate at time t on a risk-free loan that begins at time T and is returned an instant afterwards. From equation (2.4.1),

$$P(t,T) = \exp(-\int_t^T r_s ds).$$

Zero-coupon bond prices and forward rates represent equivalent information. The short interest rate (spot rate) at time t, is the instant forward rate at time t given by $r_t = f(t,t)$.

2.4.8 The Gaussian HJM model

Heath, Jarrow and Morton (HJM) suggested the use of the entire (forward) rate curve as (infinite-dimensional) state variable (Heath et al., 1992). Their model uses information available in the initial term structure.

The dynamics of HJM forward rate model is given by

$$df(t,T) = \alpha(t,T)dt + v(t,T)dW_t,\ f(0,T) > 0, \tag{2.4.2}$$

where W_t denote a standard Brownian motion, α and v are sufficiently smooth functions. The drift and volatility functions given by $\alpha(t,T)$ and $v(t,T)$ respectively, can be made path dependent.

The dynamics of forward rate for zero-coupon bond price is driven by

$$dP(t,T) = P(t,T)(m(t,T)dt + \sigma(t,T)dW_t), P(0,T) > 0$$

where

$$\begin{aligned} m(t,T) &= f(t,t) - \int_t^T \alpha(t,s)ds + \frac{1}{2}(\int_t^T v(t,s)ds)^2 \\ \sigma(t,T) &= -\int_t^T v(t,s)ds. \end{aligned}$$

From equation (2.4.2), the dynamics of the short rate is

$$r_t = f(0,t) - \int_0^t \alpha(s,t)ds + \int_0^t v(s,t)dW_s,\ r_0 > 0.$$

2.5 Interest Rate Models

In this section, we discuss some existing models of interest rates. The models assumed that bond market returns are normally distributed. Out of the existing models, we adopt the Vasicek model due to its properties.
Major existing interest rate models are given below.

Merton (1973) model

The model follows the dynamics

$$dr_t = \mu dt + \sigma dW_t.$$

The drift parameter $\mu > 0$ and the volatility σ are constants. The solution gives

$$r_t = r_0 + \mu t + \int_0^t \sigma dW_s.$$

Vasicek (1977) model

This model displays an analytic solution for a discount bond price. The model is given by

$$dr_t = a(b - r_t)dt + \sigma dW_t$$

where W_t is a Wiener process under the risk-neutral framework modelling random market risk factor; σ is the standard deviation that determines the volatility of the interest rate (higher σ occurs when there is much randomness); a is the reversion speed that gives the rate at which paths will restructure around b in time; b stands for long-term mean level. $(b - r_t)$ represents difference in return and $r_t = r(t)$. The solution is given by

$$r_t = r_0 e^{-at} + b(1 - e^{-at}) + \sigma e^{-at} \int_0^t e^{as} dW_s.$$

The long-term variance is $\dfrac{\sigma^2}{2a}$,

$$\mathbb{E}[r_t] = r_0 e^{-at} + b(1 - e^{-at}), \ \ \mathrm{Var}[r_t] = \frac{\sigma^2}{2a}(1 - e^{-2at})$$

$$\lim_{t\to\infty} \mathbb{E}[r_t] = b; \text{ while } \lim_{t\to\infty} \mathrm{Var}[r_t] = \frac{\sigma^2}{2a}.$$

Exponential Vasicek model

The short rate dynamics is given by

$$r_t = \exp(\lambda(t)) \text{ where } d\lambda(t) = a(b - \lambda(t))dt + \sigma dW_t.$$

It satisfies the stochastic differential equation

$$dr_t = (ab + \frac{\sigma^2}{2} - a\ln(r_t))r_t dt + \sigma r_t dW_t$$

where

$$r_t = \exp(\ln r_s e^{-a(t-s)} + \sigma(1 - e^{-a(t-s)}) + \sigma \int_s^t e^{-a(t-u)} dW_u), 0 \leq s \leq t \leq T.$$

This is log-normally distributed but cannot be calculated explicitly.

$$E[r_t \mid \mathcal{F}_s] = \exp\left(\ln r_s e^{-a(t-s)} + b(1 - e^{-a(t-s)}) + \frac{\sigma^2}{4a}(1 - e^{-2a(t-s)}) \right)$$

and

$$\begin{aligned}\text{Var}[r_t \mid \mathcal{F}_s] &= \exp\left(2 \ln r e^{-a(t-s)} + 2b(1 - e^{-a(t-s)})\right) \\ &\quad + \exp\left(\frac{\sigma^2}{2a}(1 - e^{-2a(t-s)})\right)\left(\exp(\frac{\sigma^2}{2a}(1 - e^{-2a(t-s)})\right) - 1\Big).\end{aligned}$$

The model is mean reverting with

$$\lim_{t\to\infty} E[r_t \mid \mathcal{F}_s] = \exp\left(b + \frac{\sigma^2}{4a}\right)$$

and

$$\lim_{t\to\infty} \text{Var}[r_t \mid \mathcal{F}_s] = \exp\left(2b + \frac{\sigma^2}{2a}\right)\left[\exp(\frac{\sigma^2}{2a}) - 1\right].$$

Dothan (1978) model

The model is given by

$$dr_t = \mu r_t + \sigma r_t dW_t$$

where σ and μ are constant parameters for volatility and drift, respectively. The short rate is log-normally distributed. The solution to the dynamics is given by

$$r_t = r_0 \exp(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t.$$

Furthermore,

$$\mathbb{E}[r_t] = r_0 e^{\mu t} \qquad \text{and} \qquad \text{Var}[r_t] = r_0^2 e^{2\mu t}(e^{\sigma^2 t} - 1).$$

The model's disadvantage is that r is not mean-reverting except when $\mu < 0$ and the mean-reversion level has the value 0.

Brennan-Schwartz (1980) model

The model is used to analyse convertible bonds and is given by

$$dr_t = (b + ar_t)dt + \sigma r_t dW_t$$

where b, a and σ are non-negative constants. The solution is given by

$$r_t = e^{(a-\frac{1}{2}\sigma^2)(t-u)+\sigma(W_t - W_u)} r_u + \int_u^t e^{(a-\frac{1}{2}\sigma)(t-s)+\sigma(W_t-W_s)} b ds.$$

Hull-White (1990) model

The short rate follows the dynamics

$$dr_t = (b - ar_t)dt + \sigma dW(t)$$

where b and W_t denote a positive deterministic function of time and a Wiener process, respectively. The solution is given by

$$r_t = r_0 e^{-at} + \frac{b}{a}(1 - e^{-at}) + \sigma e^{-at} \int_0^t e^{au} dW(u).$$

Its mean satisfies

$$\mathbb{E}[r_t] = e^{-at} r_0 + \frac{b}{a}(1 - e^{-at}),$$

when $t \to \infty$, the mean tends to $\frac{b}{a}$; while the variance is

$$\text{Var}(r_t) = \frac{\sigma^2}{2a}(1 - e^{-2at}).$$

As $t \to \infty$, $\text{Var}(r_t) \to \frac{\sigma^2}{2a}$ and the distribution of r_t tends to $N(\frac{b}{a}, \frac{\sigma^2}{2a})$.

As the value of b (the mean reversion) gets bigger, r_t tends faster to its limit distribution. Bigger mean reversion implies lower variance of r_t.

The Cox-Ingersoll-Ross (1985) model

The dynamics of the Cox-Ingersol-Ross (CIR) model is given by

$$dr_t = a(b - r_t)dt + \sigma \sqrt{r_t} dW_t$$

where b is the mean reversion. The drift $a(b - r_t)$ guarantees mean reversion of interest rate. Standard deviation factor $\sigma\sqrt{r_t}$ prevents the occurrence of non-positive interest rate for all non-negative values of a and b. The standard deviation reduces if the rate is near zero. The distribution is not normal or log-normal, but has the property of a noncentral chi-squared distribution.

The solution of the dynamics is given by

$$r_t = r_0 e^{-at} + b(1 - e^{-at}) + \sigma \int_0^t e^{-a(t-u)} \sqrt{r_u} dW_u$$

where r_0, a, b, σ are constants and

$$\mathbb{E}[r_t] = r_0 e^{-at} + b(1 - e^{-at})$$

$$\text{Var}(r_t) = r_0 \frac{\sigma^2}{a}(e^{-at} - e^{-2at}) + \frac{b\sigma^2}{2a}(1 - e^{-at})^2.$$

Orlando et al. (2018) observed that CIR model is not suitable for modelling current market environment with negative short interest rates; the diffusion term in the rate dynamics gets to zero when short rates are small while volatility and long-run mean do not alter with time; they do not fit with the asymmetric (fat tails) distribution of the interest rates. They suggested an extended CIR model, that will fit the term structure of short interest rates so that the market volatility structure is preserved with the analytical tractability of the original CIR model.

Ho and Lee (1986) model

The model is given by

$$dr_t = \sigma dW_t + \mu(t)dt$$

where W is a Brownian motion under the risk-neutral measure, μ is deterministic and σ is greater than zero. The solution is given by

$$r_t = r_s + \int_s^t \mu(u)du + \sigma(W(t) - W(s))$$

with

$$E[r_t \mid \mathcal{F}_s] = r_s + \int_s^t \mu(u)du \text{ and } \text{Var}[r_t \mid \mathcal{F}_s] = \sigma^2(t - s).$$

From the literature, interest rate derivatives have been studied with little emphasis on zero-coupon bond price with jumps. We shall adopt the Vasicek (1977) interest rate model in Chapter 4. Hence, in this work, we model zero-coupon bond price by extending the Vasicek model under certain Lévy processes and carry out sensitivity analysis using Maliavin calculus. The choice of Vasicek model is because interest rates are mean-reverting and can be negative, the model captures such properties.

In the next chapter, we discuss the methodology to be used in the work.

Chapter 3

Methodology

3.1 Background

This chapter discusses the methodology used in the work.

In section 3.2, we discuss some basic concepts and terminologies used in modelling a Lévy market.

3.2 Stochastic Processes

Definition 3.2.1. Let Ω, $\mathcal{F}$, $\mathbb{P}$ and T be a set of all possible outcomes, a σ-algebra containing subsets of Ω, probability that an event in $\mathcal{F}$ will occur and a fixed time, respectively. The triple $(\Omega, \mathcal{F}, \mathbb{P})$ is termed a *probability space.*

A *filtration* refers to a non-decreasing family $\mathbb{F} = (\mathcal{F}_t, 0 \leq t \leq T)$ of sub-σ-algebras: $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}_T \subset \mathcal{F}$ for $0 \leq s < t \leq T$.

Definition 3.2.2. A *stochastic process* is a family of random variables $X = (X_t)_{t\geq 0}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in a measurable space $(E, \mathcal{E})$, where $\mathbb{P}$, E and $\mathcal{E}$ denote a positive measure on $(\Omega, \mathcal{F})$, the state space and a σ-algebra, respectively.

Definition 3.2.3. A stochastic process $X = (X_t)_{t\geq 0}$ is a *Brownian motion* on $(\Omega, \mathcal{F}, \mathbb{P})$ if the following conditions hold:

(i) Paths of X_t is $\mathbb{P}$-a.s. continuous,

(ii) $\mathbb{P}(X_0 = 0) = 1$,

(iii) $X_t - X_s$ has the same distribution as X_{t-s}, for $s \in [0, t]$,

(iv) $X_t - X_s$ is independent of $\{X_u : u \leq s\}$, for $s \in [0, t]$,

(v) For any $t > 0$, X_t is a Gaussian random variable of mean 0 and variance t.

Definition 3.2.4. The *characteristic function* ϕ of a random variable X, with a distribution function $F(x) = \mathbb{P}(X \leq x)$, is given by

$$\phi(u) := \mathbb{E}[\exp{(iuX)}] = \int_{-\infty}^{\infty} \exp{(iux)} dF(x).$$

It follows that $\phi(0) = 1$ and $|\phi(u)| \leq 1$, $u \in \mathbb{R}$.

Definition 3.2.5. The *moment generating function* M of a continuous random variable X whose distribution function has a density $f(\cdot)$ is defined as

$$M(u) := \mathbb{E}[e^{uX}] = \int_{\mathbb{R}} e^{ux} f(x) dx = \phi(-iu).$$

Definition 3.2.6. A real-valued nonanticipating stochastic process $X = (X_t)_{t \geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be *càdlàg* or right continuous with left limits if

(i) $\lim\limits_{s \to t, s > t} X_s = X_{t^+}$ (ii) $\lim\limits_{s \to t, s < t} X_s = X_{t^-}$ (iii) $X_{t^+} = X_t$.

Definition 3.2.7. Let X_t be a stochastic process. The *jump* of X_t at time t is given by

$$\Delta X_t = X_t - X_{t^-}.$$

Definition 3.2.8. Let $X_1, X_2, ..., X_n$ be a univariate data, then *skewness* and *kurtosis* are defined by

$$\text{skewness} = \sum_{i=1}^{n} (X_i - \overline{X})^3((n-1)\mathbf{s}^3)^{-1}; \text{ kurtosis} = \sum_{i=1}^{n} (X_i - \overline{X})^4((n-1)\mathbf{s}^4)^{-1}$$

where $\overline{X}$, $\mathbf{s}$ and n denote mean, standard deviation and number of data points, respectively.

Skewness is the asymmetry of a dataset distribution, it describes the degree of distortion or deformation from the Gaussian distribution. Skewness for a Gaussian distribution is 0. A positive skewness occurs when the distribution has fatter or longer tail on its right side, while a negative skewness means that the fatter or longer tail is on its left side.

Jain (2018) explained that

(i) fairly symmetrical data have skewness between -0.5 and 0.5.

(ii) moderately negative skewed data have skewness between -1 and -0.5, while moderately positive ones have skewness between 0.5 and 1.

(iii) highly negative skewed data have skewness less than -1 while highly positive skewed data have skewness above 1.

Skewness is important to investors when taking a decision on return distribution because it does not focus only on the average but considers the extremes of the data.

Kurtosis describes the tails of the distribution, it measures the outliers present in a distribution. Kurtosis for a standard Gaussian distribution is 3. When the kurtosis is greater than 3, it is called *leptokurtic*; this means the presence of heavily-tailed data or profuseness of the outlier. When the kurtosis is less than 3, it is called *platykurtic*; this means that the tails are thinner than the tail of the Gaussian distribution and that the extreme values are not as great in amount as that of Gaussian distribution.

High kurtosis of the return distribution means that the investors will experience either positive or negative extreme returns from time to time, and this will lead to kurtosis risk.

Definition 3.2.9. Let $\mathbb{R}^d$ be the d-dimensional Euclidean space, $E \subseteq \mathbb{R}^d$, and let A be measurable subsets of E. Then a *measure* on E is a positive number $\mu(A)$, where $0 \leq \mu(A) \leq \infty$.

Let Ω be a sample space and $\mathcal{E}$ be a collection of its subsets. Then, $\mathcal{E}$ is called a σ-*algebra* if and only if it satisfies the conditions:

(i) $\{\} \in \mathcal{E}$ where $\{\}$ is an empty set.

(ii) If $A_1 \in \mathcal{E}$, then, A_1^c (the complement of A_1) also belongs to $\mathcal{E}$

(iii) If $A_i, i = 1, 2, ...$ belongs to $\mathcal{E}$, then their union $\bigcup A_i \in \mathcal{E}$.

A *measurable space* is a set in conjunction with a non-empty collection of the subsets of the set.

Definition 3.2.10. Let $(\Omega, \mathcal{E})$ be a measurable space. Then, a *measure* on the measurable space is the function $\mu : \mathcal{E} \to [0, \infty]$, that satisfies the properties:

(i) For every $A \in \mathcal{E}$, $0 \leq \mu(A) \leq \infty$.

(ii) $\mu(\{\}) = 0$.

(iii) If $A_i \in \mathcal{E}, i = 1, 2, ...,$ are disjoint members, then $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$.

The space $(\Omega, \mathcal{E}, \mu)$ is called a *measure space.* If $\mu(\Omega) = 1$, then, the measure is called a *probability measure.*

Definition 3.2.11. A *Lebesgue measure* on d-dimensional space is defined on the Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)$ and can be viewed as the d-dimensional volume $v(A) = \int_A dx$, where dx is the Lebesgue measure.

Definition 3.2.12. Let $X = (X_t)_{t \geq 0}$ be a stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$. Then, X is a *martingale* if

i. $\mathbb{E}[|X_t|] < \infty \ \forall \, t \geq 0$

ii. $E[X_t | \mathcal{F}_s] = X_s$, $\mathbb{P}$-a.s., $0 \leq s \leq t$.

Definition 3.2.13. Probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ is an *equivalent martingale measure (EMM)* if

(i) $\mathbb{Q}$ is equivalent to $\mathbb{P}$.

(ii) the discounted stock-price process $\widetilde{S}_t = \exp(-rt) S_t$, $t \geq 0$, where r and S_t denote interest rate and the stock price at time t, respectively, is a martingale under $\mathbb{Q}$. Bank account $B_t = \exp(-rt)$ is a *numeraire*, that is, it is a strictly increasing positive price process for all time t, where $0 \leq t \leq T$.

Definition 3.2.14. Let $[a, b] \subset \mathbb{R}$ be a partition given by $\mathcal{P} = \{a = t_1 < t_2 < \cdots < t_{n+1} = b\}$. The *variation of a real-valued stochastic process* X_t on

$(\Omega, \mathcal{F}, \mathbb{P})$ over $\mathcal{P}$ is defined as

$$\mathcal{V}_{\mathcal{P}}(X) = \sum_{i \in [1,n]} |X(t_{i+1}) - X(t_i)|.$$

X has a *finite variation* on $[a, b]$ if the supremum over all the partitions is finite, otherwise it is of *infinite variation.*

Every non-decreasing stochastic process X_t is of finite variation, and can be written as the difference between two non-decreasing functions; for example, Poisson process is of finite variation whereas Brownian motion is of infinite variation (Schoutens, 2003: 14).

3.3 Lévy Processes

This section discusses definitions and types of Lévy processes needed for the success of this work.

Definition 3.3.1. Given $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ a filtered probability space, whose filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}$, let $X = (X_t)_{0 \le t \le T}$ with $X_0 = 0$ a.s., be a càdlàg, adapted, real-valued stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$. X is called a *Lévy process* if it

(i) has independent increments, that is, the random variables X_{t_0}, $X_{t_1} - X_{t_0}, ..., X_{t_n} - X_{t_{n-1}}$ are independent, where $0 \le t_0 < t_1 < \cdots < t_n$.

(ii) has stationary increments, that is, the distribution of $X_{t+s} - X_t$ does not depend on t for all $0 \le s, t \le T$.

(iii) is stochastically continuous, that is, for every $0 \le t \le T$ and $\epsilon > 0$, $\lim_{t \to s} \mathbb{P}(| X_t - X_s | > \epsilon) = 0$.

Definition 3.3.2. Let $X = (X_t)_{t \ge 0}$ be a Lévy process on $\mathbb{R}$. For $A \in \mathcal{B}(\mathbb{R})$, the *measure* ν on $\mathbb{R}$ is the expected number of jumps for each unit time, with jump sizes belonging to A. Mathematically, it is denoted

$$\nu(A) = \mathbb{E}[\chi\{t \in [0, 1] : \Delta X_t \in A, \Delta X_t \neq 0\}].$$

ν is called the *Lévy measure* of X.

Types of Lévy processes. These include

(i) Brownian motion (ii) Brownian motion with drift (iii) Poisson process

(iv) Compound Poisson process (v) Gamma process

(vi) Inverse Gaussian process (vii) Stable processes and Subordinators.

The Poisson process

Definition 3.3.3. A stochastic process $X_t, t \geq 0$, is called a *Poisson process* with intensity $\lambda \in (0, \infty)$ if its distribution and characteristic function satisfy

$$\mathbb{P}(X_t = k) = (\lambda t)^k (k! e^{\lambda t})^{-1}, \; k \geq 0, \; t \geq 0$$

and

$$\mathbb{E}[e^{iuX_t}] = \exp(-t\lambda(1 - e^{iu})), \; u \in \mathbb{R}, \;\; \text{respectively.}$$

A *compensated Poisson process* $\widetilde{X}$ is a process of the form $X_t - \lambda t$ where X_t is a Poisson process and λt is the compensator, and its characteristic function is

$$\mathbb{E}[e^{iu\widetilde{X}_t}] = \exp(-t\lambda(1 - e^{iu} + iu)).$$

Remark 3.3.1. A compensated Poisson process is a martingale. Mathematically, $E[\widetilde{X}_t \mid \mathcal{F}_s] = \widetilde{X}_s \; \forall \; s \leq t$:

$$\begin{aligned} E[\widetilde{X}_t \mid \mathcal{F}_s] &= E[X_t - \lambda t \mid \mathcal{F}_s] = E[(X_t - X_s) + X_s - \lambda t \mid \mathcal{F}_s] \\ &= E[X_t - X_s] + X_s - \lambda t = \lambda(t-s) + X_s - \lambda t = X_s - \lambda s = \widetilde{X}_s. \end{aligned}$$

Definition 3.3.4. Given that $N = (N_t)_{t \geq 0}$ is a Poisson process of intensity λ, let $Y_1, Y_2, ...$ be independent identically distributed (iid) random variables with common distribution, and which are independent of N. Then, the process $X_t, t \geq 0$ given by $X_t = \displaystyle\sum_{k=1}^{N_t} Y_k$ is a *compound Poisson process* of intensity λ and step distribution ν.

Its characteristic function is given by

$$\mathbb{E}[e^{iuX_t}] = \exp(-t\lambda \int_{-\infty}^{\infty} (1 - e^{ius}) \nu(ds)), \; u \in \mathbb{R}.$$

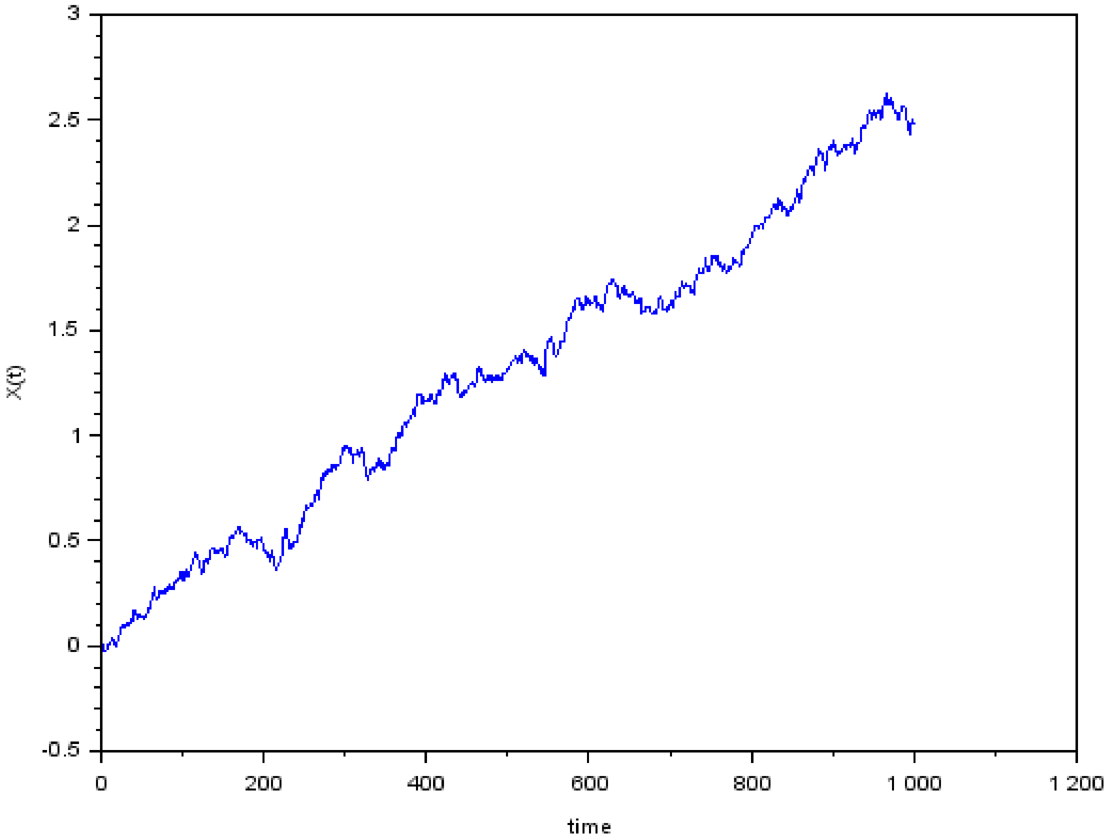

Figure 3.1: Brownian motion with $\mu = 0.1$, $\sigma = 0.8$.

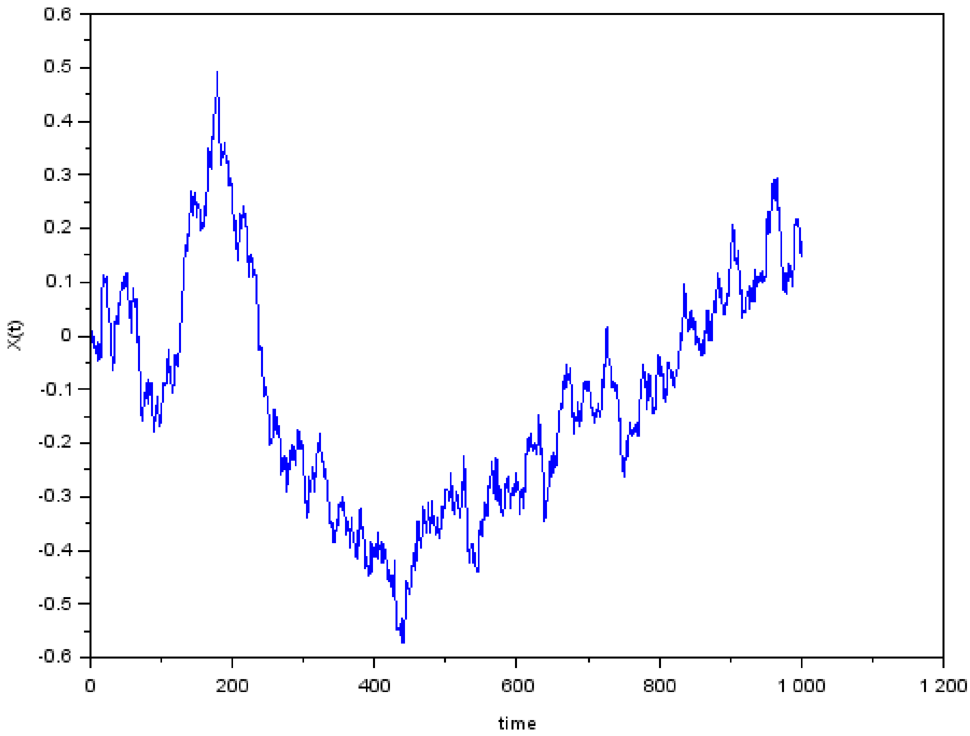

Figure 3.2: Standard Brownian motion with $\mu = 0$, $\sigma = 1$.

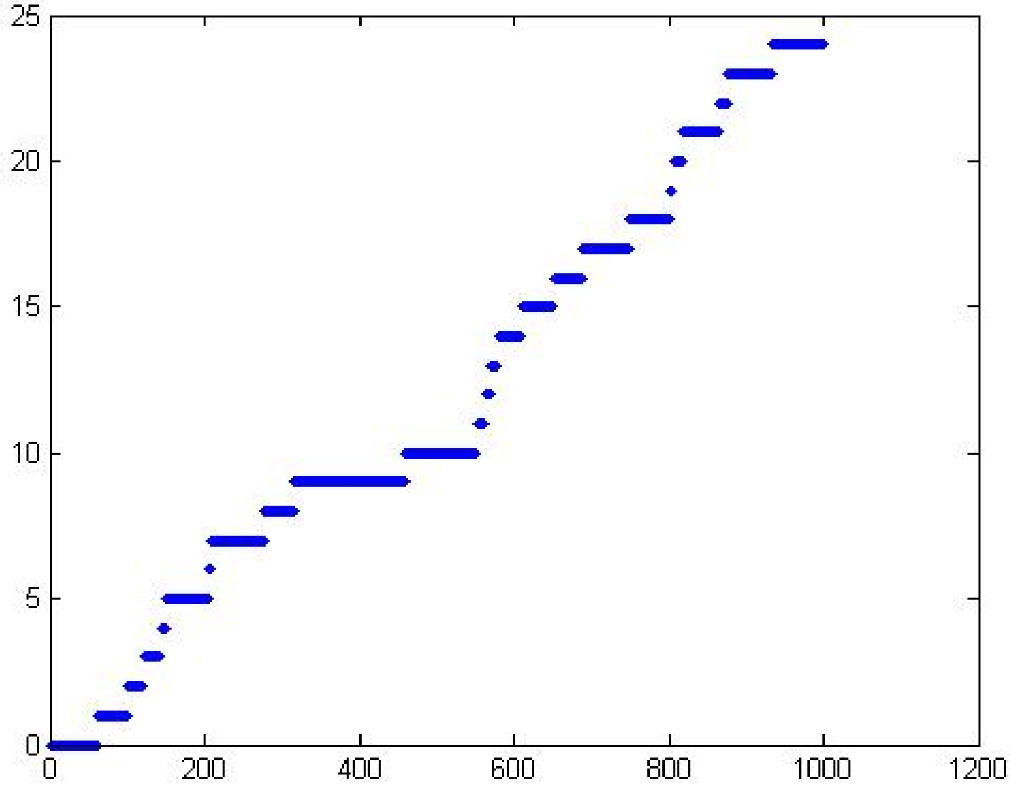

Figure 3.3: Paths of Poisson process with $\lambda = 20$.

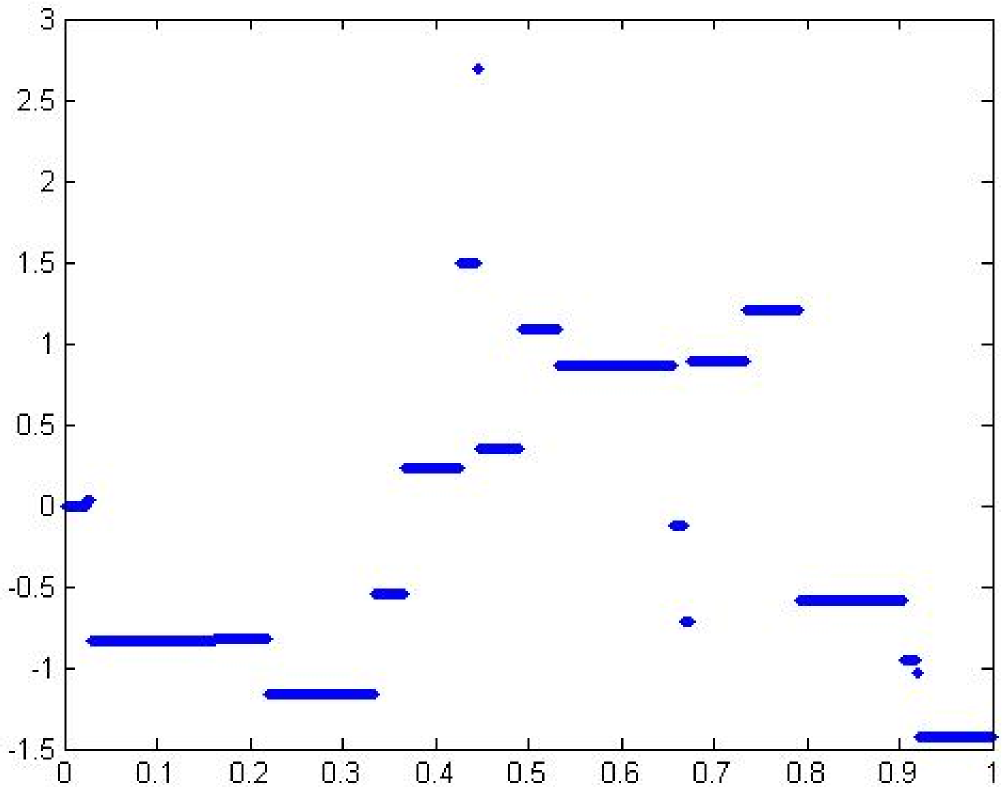

Figure 3.4: Compound Poisson process of $\lambda = 20$ with Gaussian distribution of jump sizes.

Definition 3.3.5. The density function of a gamma distribution with parameter $c > 0$ and $\lambda > 0$ is given by

$$f(x; c, \lambda) = \frac{\lambda^c}{\Gamma(c)} x^{c-1} \exp(-c\lambda),\ x > 0;$$

where c and λ denote shape and scale parameters, respectively.
Its characteristic function is $\phi(u; c, \lambda) = (1 - iu/\lambda)^{-c}$.

In what follows, the jump of a Lévy process X_t at time t is written as

$$\Delta X_t = X(t) - X(t^-).$$

Lévy jump-diffusion process

Definition 3.3.6. A Lévy jump-diffusion process has the form

$$X_t = \mathbf{b}t + \sigma W_t + \sum_{k=1}^{N_t} Y_k$$

where $(N_t)_{t\geq 0}$ is the Poisson process that counts the iid jump sizes of X and Y_k.

Definition 3.3.7. An *infinitely divisible distribution (i.d.d.)* is a distribution of a random variable that can be written as a sum of n iid random variables, where n is a positive integer (Mainardi and Rogosin (2006)).

Theorem 3.3.1. The Lévy-Khintchine (Winkel, 2010).
A real-valued random variable X is i.d.d. provided there are parameters $\mathbf{b} \in \mathbb{R}$, $\sigma^2 \geq 0$ and a measure ν on $\mathbb{R} \setminus \{0\}$ with $\int_{-\infty}^{\infty} (1 \wedge x^2)\nu(dx) < \infty$ and $\mathbb{E}(e^{iuX}) = e^{-\psi(u)}$, where

$$\psi(u) = -i\mathbf{b}u + 0.5\sigma^2 u^2 - \int_{-\infty}^{\infty} (e^{iux} - 1 - iux1_{\{|x|\leq 1\}})\nu(dx),\ u \in \mathbb{R}.$$

Definition 3.3.8. The *Lévy-Khintchine triplet* $(\mathbf{b}, \sigma^2, \nu)$ of an i.d.d. is made up of the constants $\mathbf{b} \in \mathbb{R}$, $\sigma^2 \geq 0$ and the measure $\nu(dx)$, that appear in Theorem 3.3.1 (Mainardi and Rogosin, 2006).

Subordinator

A *subordinator* is defined as a non-negative increasing Lévy process with characteristic exponent

$$\psi(u) = i\mathbf{b}u + \int_{-\infty}^{\infty} (e^{iux} - 1)\nu(dx)$$

where $\mathbf{b}$ is the drift coefficient.

Song (2012) described a subordinator as a nonnegative and necessarily increasing Lévy process starting from 0. It has no Brownian part which implies $\sigma^2 = 0$, hence, it does not decrease.

Theorem 3.3.2. (Winkel, 2010). A Lévy process $X = (X_t)_{t\geq 0}$ is a subordinator provided that its characteristic triplet satisfies the properties:

$$\sigma = 0;\ \nu((-\infty, 0]) = 0;\ \int_{0}^{\infty} \min\{x, 1\}\nu(dx) < \infty \text{ and } 0 \leq \mathbf{b} - \int_{|x|\leq 1} x\nu(dx).$$

Proof. See Winkel (2010).

Theorem 3.3.3. (Rhee and Kim, 2004). Let X_t be a Lévy process.

If $g : \mathbb{R} \to \mathbb{C}$ is complex-valued and left continuous with right limit, then

$$\mathbb{E}\big[\exp\big(\int_{0}^{t} g(s)dX\big)\big] = \exp\big(\int_{0}^{t} \psi(g(s))ds\big),$$

where the log of its moment generating function is

$$\psi(u) = \mathbf{b}u + \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}} (e^{ux} - 1 - ux)\nu(dx).$$

Proof. See Kluge (2005), pg. 12, Proposition 1.9.

Definition 3.3.9. Let X_t be a Lévy process with characteristic triplet $(\mathbf{b}, \sigma^2, \nu)$. Then the *quadratic variation process* of X_t is

$$[X, X]_t = \sigma^2 t + \sum_{\substack{0\leq s\leq t \\ \Delta X_0 \neq 0}} \mid \Delta X_s \mid^2 = \sigma^2 t + \int_{0}^{T}\int_{\mathbb{R}} x^2 J_X(ds\ dx)$$

where J_X is a Poisson random measure of intensity $\nu(dx)$.

The quadratic variation is a subordinator.

Definition 3.3.10. A *semimartingale* is a stochastic process $X = (X_t)_{t \geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, whose trajectories are càdlàg, and can be represented as the sum $X_t = M_t + V_t$ where M_t and V_t denote local martingale and locally bounded variation process, respectively.

Let X and X^* be semimartingales. The *quadratic covariation process* of X and X^* is the semimartingale given by

$$[X, X^*]_t = X_t X_t^* - X_0 X_0^* - \int_0^t X_{s-} dX_s^* - \int_0^t X_{s-}^* dX_s.$$

The quadratic variation of a semimartingale X_t is given by

$$[X, X]_t = X_t^2 - 2\int_0^t X_{s-} dX_s.$$

Itô formula for Lévy processes

Let $X = X_t, t \geq 0$ be an n-dimensional Lévy process with characteristic triplet $(\mathbf{b}, \sigma^2, \nu)$ and a function $f \in C^{1,2}$ being a map $[0, T] \times \mathbb{R}^n \to \mathbb{R}$. Then

$$\begin{aligned} f(t, X_t) &= f(0,0) + \int_0^t \frac{\partial f}{\partial s}(s, X_s) ds + \int_0^t \sum_{1 \leq i \leq n} \frac{\partial f}{\partial x_i}(s, X_{s-}) \mathbf{b}_i(t) dX_s^i \\ & +0.5 \int_0^t \sum_{1 \leq i,j \leq n} \sigma_{ij}^2 \frac{\partial^2 f}{\partial x_i \partial x_j}(s, X_s) ds \\ & + \sum_{0 \leq s \leq t}^{\Delta X_s \neq 0} \left[f(s, X_{s-} + \Delta X_s) - f(s, X_{s-}) - \sum_{1 \leq i \leq n} \Delta X_s^i \frac{\partial f}{\partial x_i}(s, X_{s-}) \right]. \end{aligned}$$

Theorem 3.3.4. Itô formula for semi-martingale.

Let $Y = (Y_t)_{0 \leq t \leq T}$ be a semi-martingale. If f maps $[0, T] \times \mathbb{R} \to \mathbb{R}$ is a $C^{1,2}$ function, then

$$f(t, Y_t) = f(0, Y_0) + \int_0^t \frac{\partial f}{\partial s}(s, Y_s) ds + \int_0^t \frac{\partial f}{\partial x}(s, Y_{s\_}) dY_s$$

$$+0.5 \int_0^t \frac{\partial^2 f}{\partial x^2}(s, Y_s) d[Y, Y]_s^c + \sum_{0 \leq s \leq t, \Delta Y_s \neq 0} [f(s, Y_s) - f(s, Y_{s\_}) - \Delta Y_s \frac{\partial f}{\partial x}(s, Y_{s\_})]$$

where $[Y, Y]_s^c$ is the continuous segment of the quadratic variation of Y (Cont and Tankov, 2004: Proposition 8.13, pg. 285).

The Itô formula for semi-martingale is very useful in deriving expressions in a Lévy market.

In the next four sections, we discuss the subordinators, namely, gamma process and inverse Gaussian process. For each subordinator, we discuss the corresponding subordinated Lévy process, to be employed in deriving expressions for an interest rate derivative in a Lévy market.

3.4 The Gamma Process

A *gamma process*, denoted by $\gamma(t; c, \lambda)$, is a random process having gamma distributed increments. It is a pure-jump non-decreasing process having intensity measure $\nu(x) = cx^{-1}\exp(-\lambda x)$, for positive x. Jump sizes in the interval $[x, x + dx]$ arise as a Poisson process with intensity $\nu(x)dx$. Parameter c manages the intensity of jump arrival and scaling parameter λ inversely influences the jump size. The process starts from 0 at time $t = 0$.

A gamma process is sometimes parameterised in terms of the mean μ and variance κ of the increase at each time, e.g., $c = \mu^2/\kappa$ and $\lambda = \mu/\kappa$.

The gamma process $X \sim \gamma(c, \lambda)$ with parameters $c, \lambda > 0$, is an infinite activity Lévy process whose density function and characteristic triplet are given by

$$f(x; c, \lambda) = \frac{\lambda^c}{\Gamma(c)} x^{c-1} e^{-\lambda x}, \qquad x > 0$$

and

$$(\mathbf{b}, \sigma^2, \nu) = (0, c(1 - e^{(-\lambda)})/\lambda, ce^{-\lambda x} x^{-1} \mathbf{1}_{x>0})$$

respectively, where $c > 0$ and λ control skewness and scale, respectively.

The characteristic function of the process is

$$\begin{aligned}\phi(u) &= \exp\left(c\int_0^\infty (e^{iux} - 1)\frac{e^{-\lambda x}}{x}dx\right) \\ &= \exp\left(-c\log\left(1 + \frac{iu}{\lambda}\right)\right), \qquad \text{for all } u \in \mathbb{R}, c \in \mathbb{R}^+.\end{aligned}$$

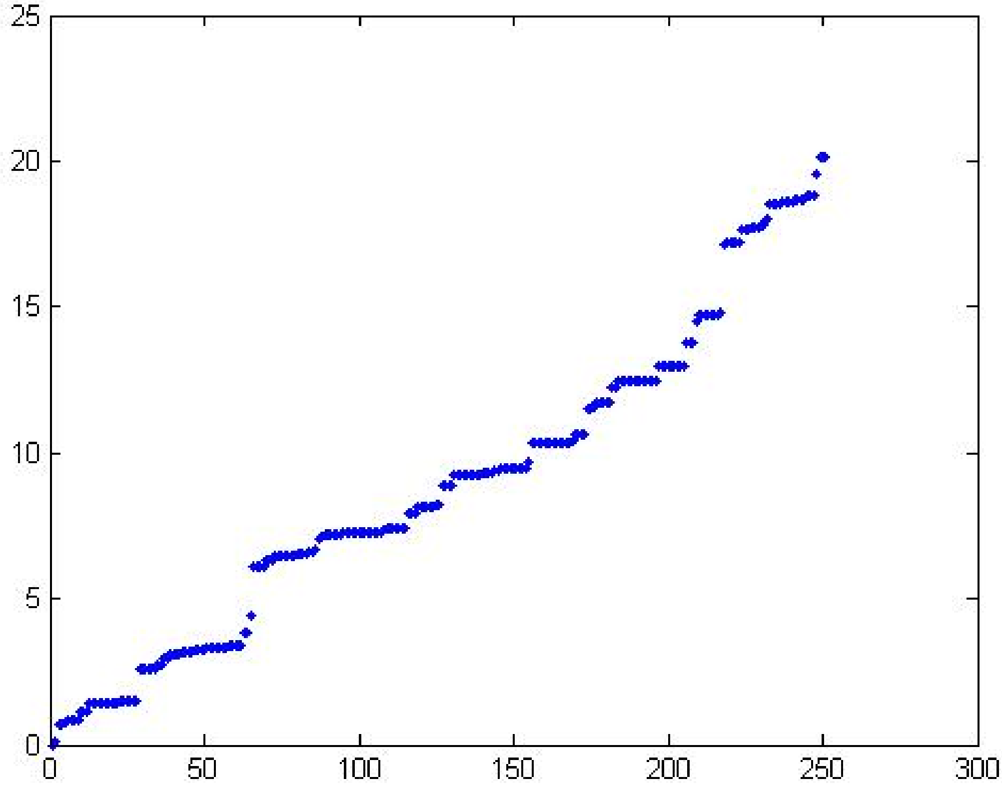

Figure 3.5: Path of gamma process with parameters c=25 and lambda=1.

Its Laplace exponent is given by

$$l(u) = -c \log\left(1 + \frac{u}{\lambda}\right).$$

For the density of a gamma process given by $f(x; c/n, \lambda)$, its characteristic function is $\phi_n(u) = \left(1 + \frac{iu}{\lambda}\right)^{-c/n}$. The gamma distribution is infinitely divisible, and is used for random time-change of a variance gamma (VG) process.

Table: Moments of the gamma distribution

	$\gamma(c, \lambda)$	$\gamma(ct, \lambda)$
mean	$c\lambda^{-1}$	$(ct)\lambda^{-1}$
variance	$c\lambda^{-2}$	$(ct)\lambda^{-2}$
skewness	$2c^{-1}$	$2(ct)^{-1/2}$
kurtosis	$3(1 + 2c^{-1/2})$	$3(1 + 2(ct)^{-1/2})$

In what follows, the density function and characteristic function of a gamma process with the parametrisation $\gamma(t; c, \lambda) = \gamma(t; \frac{\mu^2}{\kappa}, \frac{\mu}{\kappa})$ are given by

$$f(x; \mu, \kappa) = \frac{\left(\frac{\mu x}{\kappa}\right)^{\frac{\mu^2 t}{\kappa}} \exp\left(-\frac{\mu}{\kappa}x\right)}{\Gamma\left(\frac{\mu^2}{\kappa}t\right)} \tag{3.4.1}$$

and

$$\phi(u) = \left(1 - iu\frac{\kappa}{\mu}\right)^{-\frac{\mu^2}{\kappa}t} \quad \text{respectively.}$$

We proceed to the next section and discuss the subordinated Lévy process obtained by time-changing the time in arithmetic Brownian motion with a gamma process.

3.5 The Variance Gamma Process

A variance gamma (VG) process is a pure jump-type Lévy process, derived as a result of random time change with inter-arrival time of a gamma process. It is a three-parameter process, developed by Madan et al. (1998) for the dynamics of log stock prices; and is derived by assessing arithmetic Brownian motion with drift θ and volatility $\widetilde{\sigma}$ at a random time of a gamma process with a mean rate at each time and a variance rate of κ. The ensuing process $X_t(\widetilde{\sigma}, \theta, \kappa)$ has two additional parameters θ and κ offering control over skewness and kurtosis, respectively. Madan and Seneta (1990) initially introduced a symmetric VG process for the modelling of the underlying uncertainty driving stock market returns, while Madan et al. (1998) obtained a closed form result for return densities and European option prices by extending the symmetric VG to allow asymmetric form.

Definition 3.5.1. The characteristic function of a VG process $X_t(\widetilde{\sigma}, \kappa, \theta)$ is given by

$$\phi(u; \widetilde{\sigma}, \kappa, \theta) = (1 - iu\theta\kappa + 0.5\widetilde{\sigma}^2\kappa u^2)^{-1/\kappa}$$

where $u \in \mathbb{R}$.

A VG process has finite variation and infinitely many jumps in any given interval of time, and replaces Brownian motion in option pricing to solve its weakness. Its major controls include volatility and drift of the arithmetic Brownian motion in addition to variance of the gamma process. Its parameters allow for the control of skewness and kurtosis of return distribution with mean and variance.

VG as time-changed Brownian motion

Let arithmetic Brownian motion with drift θ and volatility $\widetilde{\sigma}$ be defined as

$$X(t; \theta, \widetilde{\sigma}) = \theta t + \widetilde{\sigma} W(t)$$

and let $\gamma(t; \mu, \kappa)$ be a gamma process of independent increments over separate intervals of time having mean rate μ and variance rate κ. Then, the increment of the gamma process over a time is distributed with gamma density function having mean μ and variance κ; and is denoted by $\gamma(t + \Delta; \mu, \kappa) - \gamma(t; \mu, \kappa) > 0$. The gamma density function is given by equation (3.4.1) and its characteristic function is given by

$$\phi(u) = \left(\frac{1}{1 - iu\kappa\mu^{-1}}\right)^{(\mu^2\kappa^{-1})t}.$$

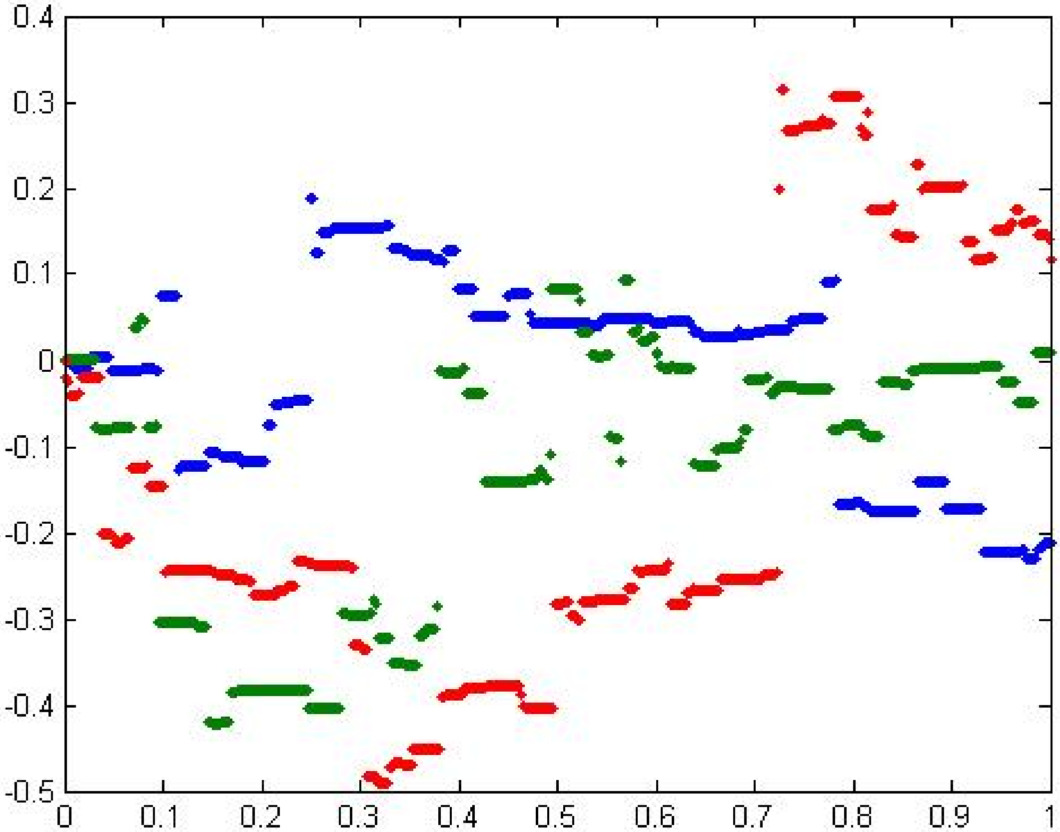

Figure 3.6: Paths of VG process.

The VG process $X = X_t(\widetilde{\sigma}, \kappa, \theta)$ is a Brownian motion whose stochastic time is a gamma process with mean rate 1, that is, $\gamma(t; 1, \kappa)$; and its density function and characteristic function are given by

$$f(x) = \int_0^\infty \frac{1}{\sqrt{2\pi\widetilde{\sigma}^2 y}} \exp\Big(- \frac{(x-\theta y)^2}{2\widetilde{\sigma}^2 y}\Big) \frac{y^{(\frac{t}{\kappa}-1)}\kappa^{(-\frac{t}{\kappa})}e^{-\frac{y}{\kappa}}}{\Gamma(\frac{t}{\kappa})} dy$$

and

$$\phi(u) = (1 - i\theta\kappa u + \frac{1}{2}\widetilde{\sigma}^2\kappa u^2)^{-t/\kappa}, \text{ respectively.}$$

The VG process has moments $\mathbb{E}[X] = \theta t$; Variance$(X) = (\kappa\theta^2 + \widetilde{\sigma}^2)t$,

$$\text{Skewness} = (2\theta^3\kappa^2 + 3\widetilde{\sigma}^2\kappa\theta)t$$

and

$$\text{Kurtosis} = 3(\widetilde{\sigma}^4\kappa + 4\widetilde{\sigma}^2\theta^2\kappa^2)t + 3(\widetilde{\sigma}^4 + 2\widetilde{\sigma}^2\theta^2\kappa + \theta^4\kappa^2)t^2.$$

Its Lévy measure is given by

$$\nu(x)dx = \begin{cases} \frac{\mu^2 e^{-\frac{\mu}{\kappa}x}}{\kappa x} dx, & x > 0 \\ 0, & x \leq 0. \end{cases}$$

The diffusion term in the Lévy Khintchine triplet for the process is zero.

Theorem 3.5.1 (Hirsa and Neftci, 2004). Let X_t be a VG process, then its characteristic function is given by

$$\phi(u) = \Big(1 - iu\kappa\theta + \frac{1}{2}\kappa\widetilde{\sigma}^2 u^2\Big)^{-t/\kappa}. \tag{3.5.1}$$

Proof. Let X_t^γ be a gamma process. Then

$$\begin{aligned} \phi(u) &= \mathbb{E}[e^{iuX_t}] = \mathbb{E}[e^{iu(\theta X_t^\gamma + \widetilde{\sigma} W_{X_t^\gamma})}] = \mathbb{E}[\mathbb{E}[e^{iu(\theta X_t^\gamma + \widetilde{\sigma} W_{X_t^\gamma})} \mid X_t^\gamma = x]] \\ &= \int_0^\infty \mathbb{E}[e^{iu(\theta X_t^\gamma + \widetilde{\sigma} W_{X_t^\gamma})} \mid X_t^\gamma = x]\mathbb{P}(X_t^\gamma \in dx) = \int_0^\infty e^{ix(u\theta + i\frac{u^2\widetilde{\sigma}^2}{2})} f_{X_t^\gamma}(x)dx \\ &= \mathbb{E}[e^{X_t^\gamma(iu\theta - \frac{u^2\widetilde{\sigma}^2}{2})}] = \Big(1 - \kappa(iu\theta - \frac{u^2\widetilde{\sigma}^2}{2})\Big)^{-t/\kappa} \\ &= \Big(\frac{1}{1 - iu\kappa\theta + \frac{1}{2}u^2\widetilde{\sigma}^2\kappa}\Big)^{t/\kappa} \end{aligned}$$

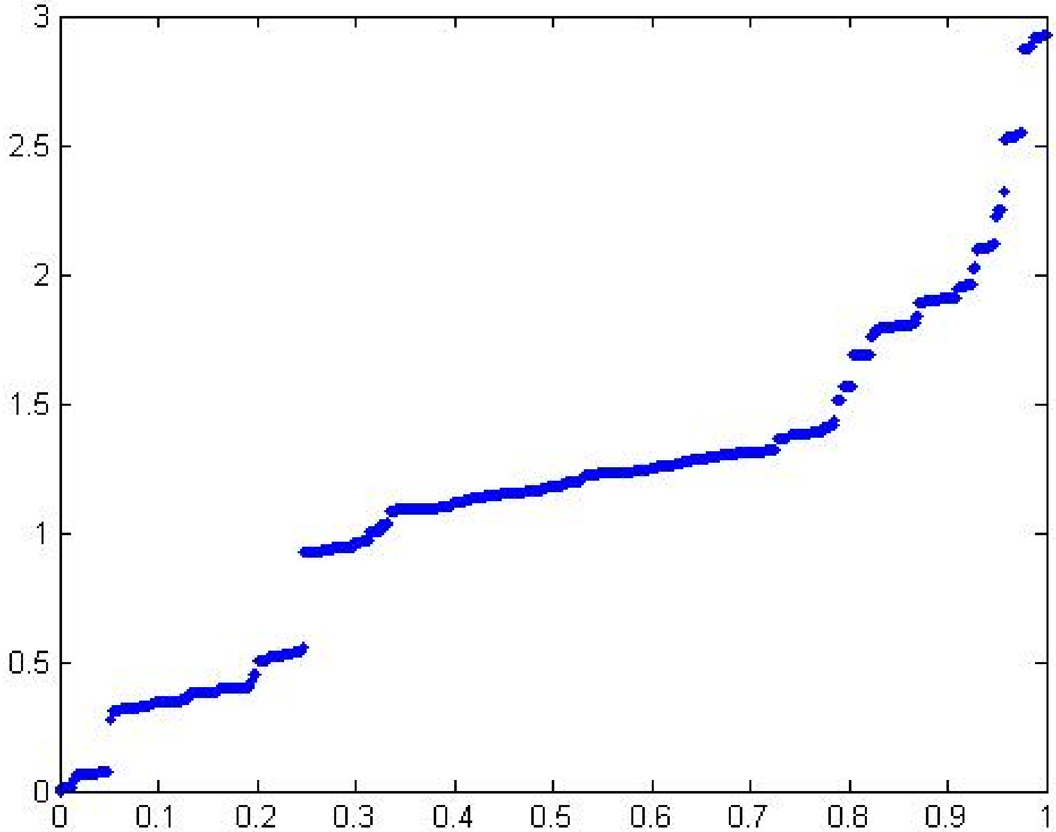

Figure 3.7: Path of IG process.

where the density function $f_{X_t^\gamma}(x)$ is given by

$$f_{X_t^\gamma}(x) = \frac{x^{\frac{t}{\kappa}-1} e^{-\frac{x}{\kappa}}}{\kappa^{t/\kappa}\Gamma(t/\kappa)}. \qquad \square$$

In the following section, we discuss the subordinator inverse Gaussian used to obtain the normal inverse Gaussian (NIG) process and proceed to discuss the NIG process.

3.6 The Inverse Gaussian Process

An inverse Gaussian (IG) process $X = X_t(\alpha, \beta)$, where $\alpha > 0, \beta \geq 0$ has density and characteristic function given by

$$f(x; \alpha, \beta) = \frac{\alpha}{\sqrt{2\pi}} e^{\alpha\beta} x^{-1.5} \exp\{-0.5(\alpha^2 x^{-1} + \beta^2 x)\} \mathbf{1}_{x>0}$$

and

$$\phi(u) = \exp(-\alpha(\sqrt{-2iu + \beta^2} - \beta)), \text{ respectively.}$$

It is the first time a standard Brownian motion $(\beta s + W_s, s \geq 0)$ with drift β reaches a positive level $\alpha > 0$ (Schoutens, 2003).

The process has infinite number of jumps in every finite time period (Barndorff-Nielsen and Shephard, 2012; Bayazit, 2010), and it is infinitely divisible, with mean, variance, skewness and kurtosis denoted by $\alpha t/\beta$, $\alpha t/\beta^3$, $3/\sqrt{\alpha t \beta}$ and $3(1 + 5/\sqrt{\alpha t \beta})$, respectively. The characteristic Lévy triplet of $X_t(\alpha, \beta)$ is given by

$$(\mathbf{b}, \sigma^2, \nu) = (\frac{\alpha}{\beta}(2\mathcal{N}(\beta) - 1), 0, \alpha(2\pi)^{-1/2} x^{-3/2} \exp(-\frac{1}{2}\beta^2 x) \mathbf{1}_{x>0})$$

where $\mathcal{N}(\beta)$ represents standard normal distribution.

We proceed to the next section and discuss the subordinated Lévy process called NIG process, obtained by time-changing the time in arithmetic Brownian motion with IG process.

3.7 The Normal Inverse Gaussian Process

The normal inverse Gaussian (NIG) process is a flexible four parameter distribution family with fat tails and skewness, originally initiated by Barndorff-Nielsen in 1995 as a model of grain-size distribution of wind-blown sand. The set is convolution stable under some conditions (Barndorff-Nielsen, 1998; Barndorff-Nielsen and Shephard, 2012).

The NIG distribution is made up of normal distribution $\mathcal{N}(\mu, \sigma^2)$ and IG distribution $IG(\alpha, \beta)$ having the probability density functions given by

$$f_{\mathcal{N}}(x) = (\sqrt{2\pi}\sigma^2)^{-1} e^{-0.5\frac{(x-\mu)^2}{\sigma^2}}, \ x \in \mathbb{R}$$

and

$$f_{X_t^{IG}}(x; \alpha, \beta) = \begin{cases} \frac{\alpha}{\sqrt{2\pi\beta}} \cdot \frac{1}{x^{3/2}} \exp\Bigg(- \frac{(\alpha-\beta x)^2}{2\beta x}\Bigg), & \text{if } x > 0 \\ 0, & \text{if } x \leq 0 \end{cases}, \text{ respectively.}$$

If a random variable X has an NIG distribution ($X \sim \text{NIG}(x; \alpha, \beta, \mu, \delta)$), then its probability density function is given by

$$f_{X_t^{NIG}}(x; \alpha, \beta, \mu, \delta) = \frac{\alpha\delta \exp(\delta(\alpha^2 - \beta^2)^{0.5} + \beta(x - \mu))}{\pi \cdot (\delta^2 + (x - \mu)^2)^{0.5}} K_1(\alpha(\delta^2 + (x - \mu)^2)^{0.5})$$

where $\alpha > 0$, $|\beta| < \alpha$, $\delta > 0$, and $K_1(x)$ represents modified Bessel function of the third kind with index λ given by

$$K_\lambda(x) = 0.5 \int_0^\infty t^{\lambda-1} \exp\Bigg(- 0.5x\big(t + \frac{1}{t}\big)\Bigg) dt, \ x > 0.$$

Alternative representation follows from Barndorff-Nielsen and Stelzer (2005), and is given by

$$K_\lambda(x) = \int_0^\infty \cosh(\lambda t) \cdot \exp(-x \cosh(t)) dt.$$

α, β, δ and μ are for tail heaviness, symmetry, scale and location, respectively. Its Lévy measure is

$$\nu(dx) = \frac{\alpha\delta}{\pi x} \exp(\beta x) \cdot K_1(\alpha x) dx, \ x > 0.$$

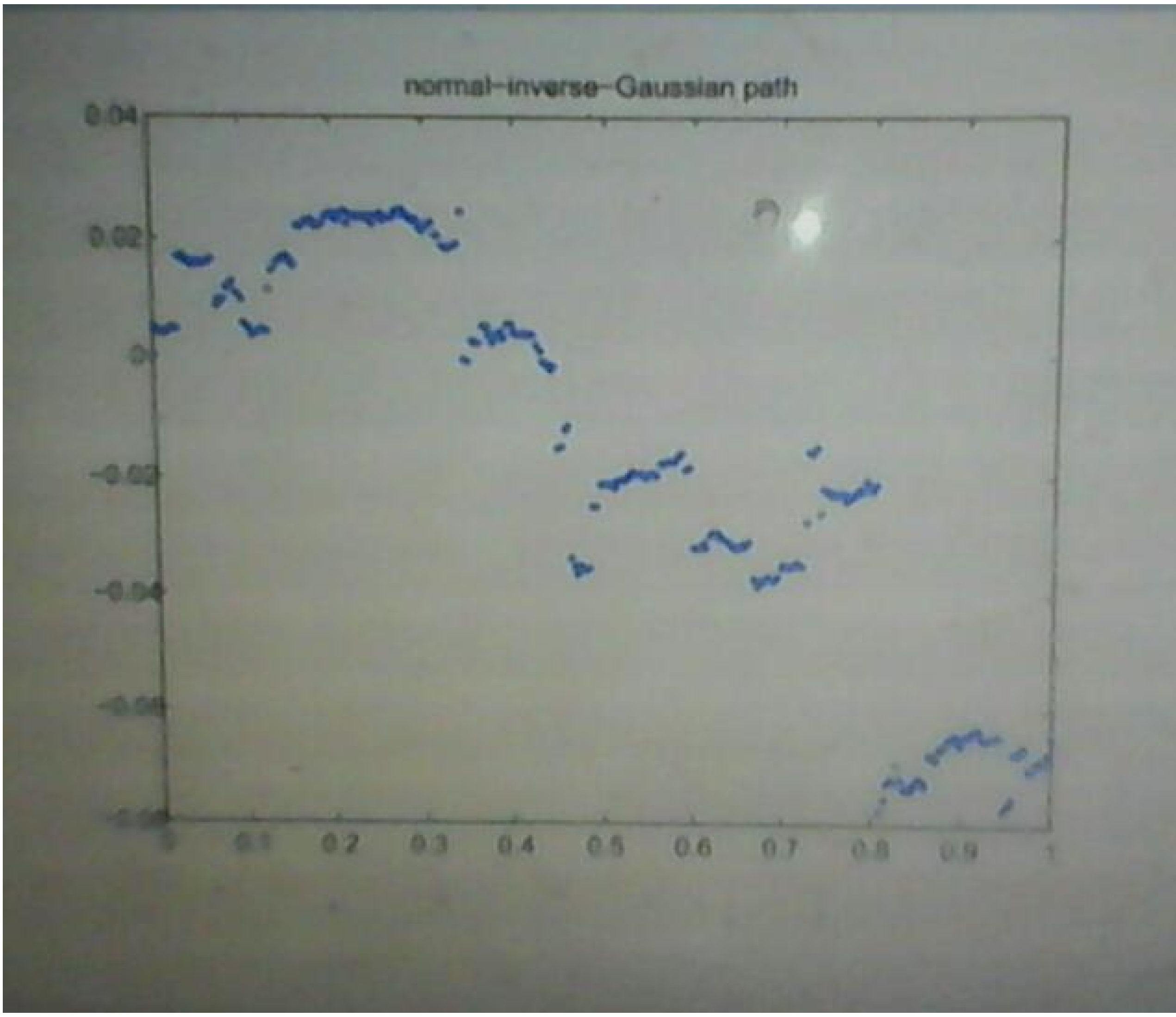

Figure 3.8: Path of NIG process.

Its moments are

$$\mathbb{E}[X] = \mu + \frac{\delta\beta}{\alpha}\Big(1 - \big(\frac{\beta}{\alpha}\big)^2\Big)^{-1/2} = \mu + \frac{\delta\beta}{\sqrt{\alpha^2 - \beta^2}}$$

$$\text{Var}(X) = \frac{\alpha^2\delta}{(\sqrt{\alpha^2 - \beta^2})^3} \qquad \text{Skewness} = 3\frac{\beta}{\alpha(\delta \cdot \sqrt{\alpha^2 - \beta^2})^{1/2}}$$

$$\text{and Kurtosis} = 3 + \frac{3}{\delta \cdot \sqrt{\alpha^2 - \beta^2}}\big(1 + 4\frac{\beta^2}{\alpha^2}\big).$$

Theorem 3.7.1 (Eriksson et al., 2009) Let X be an NIG$(\alpha, \beta, \mu, \delta)$-distributed random variable and let its mean, variance, skewness and excess kurtosis be denoted as Me, Va, Sk and Ku, respectively. Then the parameters are related to the moments by

$$\begin{aligned}
\alpha &= 3\big(\frac{4}{\varrho} + 1\big)\big(1 - \frac{1}{\varrho}\big)^{-1/2}(Ku)^{-1} \\
\beta &= sgn(Sk)\big\{3\big(\frac{4}{\varrho} + 1\big)\frac{1}{\sqrt{\varrho - 1}}(Ku)^{-1}\big\} \\
\mu &= (Me) - sgn(Sk)\sqrt{\frac{3}{\varrho}\big(\frac{4}{\varrho} + 1\big)\big(\frac{Va}{Ku}\big)} \\
\delta &= \sqrt{3\big(\frac{4}{\varrho} + 1\big)\big(1 - \frac{1}{\varrho}\big)\frac{Va}{Ku}}
\end{aligned}$$

where $\varrho = \dfrac{3Ku}{Sk^2} - 4 > 1$ and $sgn(\cdot)$ denotes the sign function.

Theorem 3.7.2. Let $X_t = \beta\delta^2 X_t^{IG} + \delta W_{X_t^{IG}}$ be an NIG process, then its characteristic function is given by

$$\phi(u) = \exp(-\delta t((-(\beta + iu)^2 + \alpha^2)^{0.5} - (-\beta^2 + \alpha^2)^{1/2})). \tag{3.7.1}$$

Proof. As in West (2012: 107), equation (3.7.1) is obtained as

$$\begin{aligned}
\phi &= \mathbb{E}[e^{iuX_t^{NIG}}] = \mathbb{E}[e^{iu(\mu X_t^{IG} + \tilde{\sigma} W_{X_t^{IG}})}] = \mathbb{E}[\mathbb{E}[e^{iu(\mu X_t^{IG} + \tilde{\sigma} W_{X_t^{IG}})} \mid X_t^{IG} = x]] \\
&= \int_0^\infty \mathbb{E}[e^{iu(\mu X_t^{IG} + \tilde{\sigma} W_{X_t^{IG}})} \mid X_t^{IG} = x]\mathbb{P}(X_t^{IG} \in dx) \\
&= \int_0^\infty e^{ix(u\mu + i\frac{u^2\tilde{\sigma}^2}{2})} f_{X_t^{IG}}(x)dx = \mathbb{E}[e^{X_t^{IG}(iu\mu - \frac{u^2\tilde{\sigma}^2}{2})}] \\
&= \exp(-\delta t((\alpha^2 - (\beta + iu)^2)^{0.5} - (\alpha^2 - \beta^2))^{0.5}
\end{aligned}$$

where

$$f_{X_t^{IG}}(x) = \frac{x^{\frac{t}{\kappa}-1}e^{-\frac{x}{\kappa}}}{\kappa^{t/\kappa}\Gamma(t/\kappa)}.$$

Next section discusses major options and pricing of options. We adopt call option in this work.

3.8 Options and Option Pricing

An option confers the holder the right to buy or sell an asset at a definite date for a predetermined price. A *call* (or *put*) option is a contract that confers its holder the right to buy (or sell) an underlying asset at a given date, called *expiration date*, for an agreed price, called *strike price*. *European option* is an option exercisable on the expiration date, while *American option* is an option exercisable at any time preceding the expiration date.

Pricing formula for European option

Let $S_t, 0 \leq t \leq T$ be a stock price at time t and $\Phi(S_T)$ be the payoff of the derivative at expiry time T. In the case of the European call with strike price K, we have

$$\Phi(S_T) = (S_T - K)^+ = \max(0, S_T - K).$$

The arbitrage-free price $\mathbb{V}_t$ of the derivative at time $0 \leq t \leq T$ is

$$\mathbb{V}_t = E^{\mathbb{Q}}[\exp(-r(T-t))\Phi(S_t)|\mathcal{F}_t],$$

where the expectation is taken with respect to an EMM measure $\mathbb{Q}$, while r and $\mathbb{F} = \{\mathcal{F}_t, 0 \leq t \leq T\}$ are the risk free rate and natural filtration, respectively. $\exp(-r(T-t))$ is called the *discounting factor*.

In the next section, we give description of Malliavin calculus to be applied in sensitivity analysis. We then proceed to Malliavin calculus for Lévy processes.

3.9 The Malliavin Calculus

The Malliavin calculus was introduced by Paul Malliavin in 1976, as an integration by paths technique, that has a lot of applications in many fields including finance, economics and Lie groups. It can be seen as a differential calculus in a Gaussian probability space. In the finite dimensional case, the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is $\Omega = \mathbb{R}^n$, $\mathcal{F} = \mathcal{B}(\mathbb{R}^n)$ and $\mathbb{P}$ is the standard Gaussian probability (Nualart, 2014; Nunno et al., 2009). Malliavin derivative of a given random variable $F = F(\omega)$, $\omega \in \Omega$, on the given probability space $(\Omega, \mathcal{F}, \mathbb{P})$, can be interpreted as a derivative with respect to the random parameter ω. The Malliavin derivative involves linear mapping from space of random variables to space of processes indexed by a Hilbert space.

In what follows, $C_p^{\infty}(\mathbb{R}^n)$ denote the set of all infinitely differentiable functions $\widetilde{F} : \mathbb{R}^n \to \mathbb{R}$ where $\widetilde{F}$ and all of its derivative have at most polynomial growth, while p denote partial derivatives with polynomial growth.

We proceed to the Malliavin calculus for Lévy processes, needed in the sensitivity analysis of interest rate derivatives in a Lévy market.

3.10 The Malliavin Calculus for Lévy Processes

In this section, we give description of some tools of the Malliavin calculus for Lévy processes. This will also involve the theorem on Malliavin integration by parts formula suitable for a Lévy market. The theorem will be needed in sensitivity analysis in Chapter 4.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $X_i, i = 1, ..., n$ be a sequence of random variables with absolutely continuous law $f_i(x)dx$ where $f_i,\ i = 1, ..., n$ are piecewise differentiable.

For $m \geq 1$ and $n \geq 1$, $\widetilde{F} \in C^m(\mathbb{R}^n)$ is the space of functions $\widetilde{F} : \mathbb{R}^n \to \mathbb{R}$.

Definition 3.10.1. Let $L^0(\Omega, \mathbb{R})$ be the linear space of all $\mathbb{R}$-valued random variables on $(\Omega, \mathcal{B}, \mathbb{P})$. A map $F : (L^0(\Omega, \mathbb{R}))^n \to L^0(\Omega, \mathbb{R}),\ n \in \mathbb{N}$ is said to be an (n,p)-*simple functional* of the n random variables if there exists an $\mathbb{R}$-valued function $\widetilde{F} \in C^p(\mathbb{R}^n)$ such that

$$F(X_1, ..., X_n)(\omega) = \widetilde{F}(X_1(\omega), ..., X_n(\omega)),\ \omega \in \Omega,\ X_1, ..., X_n \in L^0(\Omega, \mathbb{R}).$$

An (n,p)-*simple process* of length n is a sequence of random variables $U = (U_i)_{i\leq n}$: $U_i(\omega) = u_i(X_1(\omega), ..., X_n(\omega))$ where $u_i \in C^p(\mathbb{R}^n),\ X_1, ..., X_n \in L^0(\Omega, \mathbb{R})$ and $\omega \in \Omega$.

In what follows, $S_{n,p}$ and $P_{n,p}$ denote the space of all (n,p)-simple functionals and the space of all (n,p)-simple processes, respectively.

We proceed to give some definitions and lemmas needed in the Malliavin integration by parts theorem, to be applied in the computation of greeks in the next chapter.

Definition 3.10.2. Let $F \in S_{n,1}$ where $F(X_1, ..., X_n)(\omega) = \widetilde{F}(X_1(\omega), ..., X_n(\omega))$, $\omega \in \Omega,\ \widetilde{F} \in C^1(\mathbb{R}^n)$, and $X_1, ..., X_n \in L^0(\Omega, \mathbb{R})$. Define $D : S_{n,1} \to (P_{n,0})^n$ by $DF = (D_i F)_{i\leq n}$ where

$$D_i F(X_1, ..., X_n)(\omega) = (\partial_i \widetilde{F})(X_1(\omega), ..., X_n(\omega)) = \left(\frac{\partial \widetilde{F}}{\partial x_i}\right)(X_1(\omega), ..., X_n(\omega)),$$

$$X_1, ..., X_n \in L^0(\Omega, \mathbb{R}),\ \omega \in \Omega.$$

Then, the operator D is called the *Malliavin derivative operator.*

Definition 3.10.3. Let $F = (F_1, ..., F_d)$ be a d-dimensional vector of simple functionals where $F_i \in S_{n,1}$. The matrix $\mathcal{M} = \mathcal{M}(F)_{i,j}$ defined by

$$\mathcal{M}(F)_{i,j} = \langle DF_i, DF_j \rangle_n = \sum_{m=1}^{n} D_m F_i D_m F_j$$

is called the *Malliavin covariance matrix* of F (Bavouzet and Messaoud, 2006). Let α_i and β_i be $\mathcal{A}_i$-measurable random variables where $\alpha_i(\omega) < \beta_i(\omega)$. The weight function π_i is defined by

$$\pi_i(x)(\omega) = (x - \alpha_i(\omega))^{\iota}(\beta_i(\omega) - x)^{\iota} \text{ where } \iota > 0,\ x \in (\alpha_i(\omega), \beta_i(\omega))$$

and $\pi_i(x)(\omega) = 0$ for $x \notin (\alpha_i(\omega), \beta_i(\omega))$ (Bavouzet et al., 2009).

Definition 3.10.4. Let $\delta : P_{n,1} \to S_{n,0}$ be defined for a simple process $U \in P_{n,1}$ by

$$\delta(U) = \sum_{i=1}^{n} \delta_{i,\pi}(U),$$

where

$$\delta_{i,\pi}(U)(X_1, ..., X_n) = -[D_i(\pi_i u_i)(X_1, ..., X_n) + (\pi_i u_i)(X_1, ..., X_n)\varphi_i], \ U = (U_i)_{i=1,...,n},$$

$$\varphi_i(\mathbf{x}) = \frac{\partial \ln f(\mathbf{x})}{\partial x_i} = \begin{cases} \frac{f_i'(\mathbf{x})}{f(\mathbf{x})}, & \text{if } f(\mathbf{x}) \neq 0, \ 1 \leq i \leq n; \\ 0, & \text{otherwise.} \end{cases}$$

and f_i is the density function of the random variable $X_i, i = 1, ..., n$.

Then, the operator δ is called the *Skorohod integral operator.*

Remark 3.10.1. In literature (Bavouzet et al., 2009; Bavouzet-Morel and Messaoud, 2006; Bally et al., 2007)), the weight function is chosen in the following way:

1. If X_i is a set of Gaussian random variables with mean μ and variance σ^2, then the interval $(\tilde{\alpha}(w), \tilde{\beta}(w)) = \mathbb{R} = (-\infty, \infty)$. Since its density function is differentiable on the whole $\mathbb{R}$, the weight function $\pi(x) = 1$ for all $x \in \mathbb{R}$.

2. Moreover, If X_i is a sequence of standardised Gaussian random variables, the interval is given by $I = (-\infty, \infty)$ and its weight function is also $\pi(x) = 1$ for all $x \in \mathbb{R}$ since its density function is also differentiable on the whole $\mathbb{R}$.

In this work, the weight function $\pi(x) = 1$ since we are dealing with the standardised Gaussian random variables.

Definition 3.10.5. Let $L : S_{n,2} \to S_{n,0}$ be defined by

$$(LF)(X_1, ..., X_n) = -\sum_{i=1}^{n} [(\partial_{ii}^2 \widetilde{F})(X_1, ..., X_n) + \varphi_i(\partial_i \widetilde{F})(X_1, ..., X_n)],$$

where $X_1, ..., X_n \in L^0(\Omega, \mathbb{R})$ and $F \in S_{n,2}$. Then, the operator L is called the *Ornstein-Uhlenbeck (O-U) operator.*

Lemma 3.10.1 (The duality formula). Let $F \in S_{n,1}$ and $U \in P_{n,1}$. Then,

$$\mathbb{E}[\langle DF, U \rangle] = \mathbb{E}[F\delta(U)]$$

where $\langle \cdot, \cdot \rangle$ denote the scalar product in $\mathbb{R}^n$ (Bally and Clément, 2011; Bavouzet-Morel and Messaoud, 2006).

Lemma 3.10.2 (Bayazit, 2010; Bayazit and Nolder, 2009).

Let $F, Q \in S_{n,2}$. Then,

(i) $\mathbb{E}[FLQ] = \mathbb{E}[QLF]$.

(ii) $L(FQ) = FL(Q) + QL(F) - 2\langle DF, DQ \rangle$.

Integration by parts formula

Given $(\Omega, \mathcal{F}, \mathbb{P})$ as a probability space and let $F, Q : \Omega \to \mathbb{R}$ be integrable random variables. The integration by parts formula $\text{IP}(F, Q)$ is said to hold if there exists an integrable random variable $H(F, Q)$ where

$$\text{IP}(F, Q) : \mathbb{E}[\Phi'(F)Q] = \mathbb{E}[\Phi(F)H(F, Q)],\ \forall\ \Phi \in C_c^\infty(\mathbb{R})$$

where $C_c^\infty(\mathbb{R})$ denotes the space of the functions $\widetilde{F} : \mathbb{R}^d \to \mathbb{R}$ which are infinitely differentiable with compact support. If $\text{IP}(F, Q)$ and $\text{IP}(F, H(F, Q))$ hold, then $\text{IP}_2(F, Q)$ holds with $H^2(F, Q) = H(F, H(F, Q))$. Moreover, $\text{IP}_1(F, Q)$ implies $\text{IP}(F, Q)$.

Theorem 3.10.3.

"Malliavin Integration by Parts" (Bayazit and Nolder, 2009)

Let $X_i, i = 1, ..., n$, defined on $(\Omega, \mathcal{F}, \mathbb{P})$ be a sequence of random variables which are absolutely continuous in $\mathbb{R}^n$. Let $F = (F^1, ..., F^d) \in S_{n,2}^d$ and $Q \in S_{n,1}$, and let $\mathcal{M}_{ij}(F)$ be an invertible Malliavin covariance matrix and denote $\frac{1}{\mathcal{M}_{ij}(F)}$ by $(\mathcal{M}(F)_{ij})^{-1}$. Assume that $\mathbb{E}[\det(\mathcal{M}(F))^{-1}]^p < \infty,\ p \geq 1$ and $\Phi : \mathbb{R}^d \to \mathbb{R}$, with $i = 1, ..., d$, is a smooth bounded function with bounded derivative; then, it follows that

$$\mathbb{E}[\partial_i \Phi(F)Q] = \mathbb{E}[\Phi(F)H_i(F, Q)]$$

where

$$H_i(F, Q) = \sum_{j=1}^{d} Q(\mathcal{M}(F)_{ij})^{-1} LF_j - (\mathcal{M}(F)_{ij})^{-1}\langle DF_j, DQ\rangle - Q\langle DF_j, D(\mathcal{M}(F)_{ij})^{-1}\rangle$$

with $\mathbb{E}[H_i(F,Q)] < \infty$.

Proof.

$$\partial\Phi(F)Q = Q\partial\Phi(F) = Q\Phi'(F)DF_j = Q\langle D\Phi(F), DF_j\rangle$$

where

$$\begin{aligned}\langle D\Phi(F), DF_j\rangle &= \sum_{p=1}^{n} D_p\Phi(F)D_pF_j = \sum_{p=1}^{n}\sum_{i=1}^{d}\partial_i\Phi(F)\langle D_pF_i, D_pF_j\rangle \\ &= \sum_{i=1}^{d}\partial_i\Phi(F)\sum_{p=1}^{n}D_pF_iD_pF_j = \sum_{i=1}^{d}\partial_i\Phi(F)\mathcal{M}_{ij}(F)\end{aligned}$$

$$\Rightarrow \partial_i\Phi(F) = \sum_{j=1}^{d}\langle D\Phi(F), DF_j\rangle(\mathcal{M}(F)_{ij})^{-1}.$$

From Lemma 3.10.2,

$$L(FQ) = FL(Q) + QL(F) - 2\langle DF, DQ\rangle$$

which implies that

$$\langle D\Phi(F), DF_j\rangle = \frac{1}{2}[\phi(F)L(F_j) + F_jL\Phi(F) - L(\phi(F)F_j)]$$

$$\Rightarrow \partial_i\Phi(F)Q = \frac{1}{2}\Big([\Phi(F)L(F_j) + F_jL\Phi(F) - L(\Phi(F)F_j)](\mathcal{M}(F)_{ij})^{-1}\Big)Q.$$

Hence,

$\mathbb{E}[\partial_i\Phi(F)Q]$

$$= \frac{1}{2}\mathbb{E}\Big[\sum_{j=1}^{d}(-\Phi(F)F_jL(Q(\mathcal{M}(F))_{ij}^{-1}) + \Phi(F)Q(\mathcal{M}(F)_{ij})^{-1}LF_j + \Phi(F)L(F_j(\mathcal{M}(F)_{ij})^{-1}Q))\Big].$$

Also, from Lemma 3.10.2,

$$\mathbb{E}[FLQ] = \mathbb{E}[QLF].$$

Hence,

$\mathbb{E}[\partial_i\Phi(F)Q]$

$$= \frac{1}{2}\mathbb{E}\Big[\sum_{j=1}^{d}\phi(F)\Big(Q(\mathcal{M}(F)_{ij})^{-1}LF_j - F_jL(Q((\mathcal{M}(F)_{ij})^{-1}) + L(F_j(\mathcal{M}(F)_{ij})^{-1}Q)\Big)\Big]$$

$$= \frac{1}{2}\mathbb{E}\Bigg[\sum_{j=1}^{d}\Phi(F)\Bigg[Q(\mathcal{M}(F)_{ij})^{-1}LF_j - F_jL(Q(\mathcal{M}(F)_{ij})^{-1}) + F_jL(Q(\mathcal{M}(F)_{ij})^{-1})$$

$$+Q(\mathcal{M}(F)_{ij})^{-1}LF_j - 2\langle DF_j, D((\mathcal{M}(F)_{ij})^{-1}Q)\rangle\Bigg]\Bigg]$$

$$= \frac{1}{2}\mathbb{E}\Bigg[\Phi(F)\sum_{j=1}^{d}\Bigg[2Q(\mathcal{M}(F)_{ij})^{-1}LF_j - 2\langle DF_j, D((\mathcal{M}(F)_{ij})^{-1}Q)\rangle\Bigg]\Bigg]$$

$$= \mathbb{E}\Bigg[\phi(F)\sum_{j=1}^{d}\Bigg[Q(\mathcal{M}(F)_{ij})^{-1}LF_j - (\mathcal{M}(F)_{ij})^{-1}\langle DF_j, DQ\rangle - Q\langle DF_j, D((\mathcal{M}(F)_{ij})^{-1})\rangle\Bigg]\Bigg]. \quad \square$$

We proceed to the next chapter. The above theorem will be applied in sections 4.2 and 4.3 in order to compute the greeks.

Chapter 4

Results and Discussion

4.1 Background

Certain financial instruments experience jumps, that can occur due to monetary or fiscal policy, inflation, natural disaster, recession, scarce resources etc. Lévy processes give good models that consider the jumps (Rhee and Kim, 2004; Eberlein, 2007; Schoutens, 2003, pg. 43). An *interest rate derivative* is a financial instrument whose underlying asset is the right to purchase or receive a notional sum at a certain time and a given interest rate. In other words, it is a financial instrument whose value is affected by shifts in interest rate. The derivatives include swaps, bonds, options and money market. Interest rate derivatives experience jumps at some random time. The literature shows that much work has been done in the formulation and sensitivity analysis of interest rate derivatives in a Brownian motion market but not much has been done with respect to sensitivity analysis in a Lévy markets.

There are different kinds of Lévy processes.

(i) Brownian motion: This is a Lévy process with continuous sample paths. Its Lévy measure is 0.

(ii) Poisson process: It is a non-decreasing, pure jump Lévy process whose jump size is always 1.

(iii) Compound Poisson process: It is a type of Lévy process with a finite number of small and large jumps.

(iv) Jump-diffusion process: It is a process with Brownian motion in addition to compound Poission process. It is not a pure jump process.

(v) Subordinators: These are Lévy processes with increasing and non-continuous sample paths. Examples are gamma and inverse Gaussian processes, they are used for time-changing because of their non-decreasing feature.

(vi) Variance Gamma (VG) process: This is a pure jump Lévy process whose sample paths have a finite number of big jumps and an infinite number of small jumps in any finite time interval. It has parameters for the control of skewness and kurtosis of financial data. It is a subordinated Lévy process obtained by time-changing arithmetic Brownian motion with a gamma process.

(vii) Normal inverse Gaussian (NIG) process: This is a pure jump Lévy process with parameters for the control of skewness and tail-heaviness of distribution. It is obtained by time-changing arithmetic Brownian motion by inverse Gaussian process.

(viii) α-stable process: This is a Lévy process with parameters for the control of skewness and kurtosis less than 3.

Brownian motion has kurtosis 3. Interest rate markets generally exhibit jumps, excess kurtosis and skewness. Hence, they cannot be modelled with the Brownian motion. In this thesis we concentrate on VG and NIG processes since they have interesting properties and are most frequently encountered Lévy processes (Rhee and Kim, 2004; Hanssen and Øigard, 2001; Schoutens, 2003; Bayazit and Nolder, 2009).

Grandet (2011) studied sensitivity analysis in interest rate markets and stress testing but did not consider the presence of jumps. *Sensitivity analysis* involves determining the effects of changes on the parameters of a financial instrument. It therefore studies how possible changes or errors in parameter values affect model outputs (Rappaport, 1967).

Bavouzet-Morel and Messaoud (2006) developed a Malliavin calculus for jump processes by working on functionals of a finite set of random variables representing the source of randomness. Petrou (2008) extended the theory of Malliavin calculus adding some tools that are necessary for the computation of sensitivities, especially differentiability results for the solutions of stochastic differential equations. Bavouzet et al. (2009) applied Malliavin calculus to market models of jump-type. They provided numerical approach to the sensitivity analysis of European options and American options pricing in a compound Poisson market. Following Bavouzet-Morel and Messaoud (2006) and Petrou (2008), Bayazit and Nolder (2009) applied Malliavin calculus to compute sensitivities for exponential Lévy model involving VG and NIG processes.

Sensitivity analysis of an interest rate derivative called zero-coupon bond in a Lévy market has not been considered in the literature. A *bond* is a contract paid in advance, that yields a certain amount on a predetermined date in the future called the maturity date. A zero-coupon bond has no coupon payment. In this work, we focus on sensitivity analysis of zero-coupon bond price in a Lévy market. We extend Vasicek (1977) short rate model, that was formulated for a Brownian motion market, to a Lévy market by focusing on subordinated Lévy processes, namely, VG and NIG processes. We employ the extended Vasicek short rate model and derive zero-coupon bond price driven by the subordinated Lévy processes. We adopt the Malliavin calculus approach employed by Bavouzet-Morel and Messaoud (2006), Petrou (2008), Bayazit and Nolder (2009) to compute the greeks. A *greek* is the rate of change of the price of a financial instrument to any of its parameters. It measures the sensitivity of a financial instrument to a shift in its parameter.

In section 4.2, we derive an expression for a short rate model and zero-coupon bond price driven by VG process, and compute the greeks using Malliavin calculus; while in section 4.3, we derive an expression for a short rate model and zero-coupon bond price driven by an NIG process, and compute its greeks using Malliavin calculus. In section 4.4, we compare the greeks obtained from the zero-coupon bond price driven by VG and NIG processes.

4.2 Sensitivity analysis of zero-coupon bond under VG-driven Lévy market

In this section, we extend the Vasicek short rate model to an interest rate derivative market driven by VG process and derive an expression for the zero-coupon bond price. The price driven by a VG process will enable the asymmetry of the model to be captured. We derive expressions for the greeks of the price of the zero-coupon bond by means of Malliavin calculus.

4.2.1 Short rate model under VG process

We derive a modified Vasicek (1977) interest rate model driven by a VG process. Let the interest rate satisfy the stochastic differential equation

$$dr_t = a(b - r_t)dt + \sigma dX_t$$

where a, b, $\sigma \neq 0$ and X_t denote speed of mean-reversion, long-term mean rate, volatility of the short rate model and the Lévy process to be considered, respectively.

Since $dr = abdt - ardt + \sigma dX$, it follows that

$$e^{at}dr = abe^{at}dt - ae^{at}rdt + \sigma e^{at}dX_t,$$

whence $\quad d(re^{at}) = abe^{at}dt + \sigma e^{at}dX_t.$

Integrating,

$$\begin{aligned} r_t e^{at} - r_0 &= ab\int_0^t e^{as}ds + \sigma\int_0^t e^{as}dX_s = ab\cdot\frac{1}{a}e^{as}|_0^t + \sigma\int_0^t e^{as}dX_s \\ &= b(e^{at} - 1) + \sigma\int_0^t e^{as}dX_s, \end{aligned}$$

which implies that

$$r_t = r_0e^{-at} + b(1 - e^{-at}) + \sigma\int_0^t e^{-a(t-s)}dX_s. \qquad (4.2.1)$$

Definition 4.2.1. *Skewness* of a data set distribution is the degree of distortion of the distribution from the Gaussian distribution. It measures the asymmetry of the distribution. In data analysis, positive skewness means longer or fatter tail on the right side of the distribution while negative skewness means that the longer or fatter tail is on the left side.

Definition 4.2.2. *Kurtosis* defines an observed data set distribution around the mean. In data analysis, it measures the joint weight of the tails of a distribution in relation to the center of the distribution. High kurtosis indicates the presence of irregular extreme positive or negative returns, which leads to tail of the data distribution surpassing the tail of Gaussian distribution.

Definition 4.2.3. *Arithmetic Brownian motion* is a Lévy process given by

$$X_t = \theta t + \widetilde{\sigma} W_t$$

where θ and $\widetilde{\sigma} \neq 0$ denote drift and volatility of the arithmetic Brownian motion, respectively. W_t denote Wiener process.

The VG process is obtained by time-changing the time in the arithmetic Brownian motion by a gamma process.

We adopt the VG model given by $X_t = \mathbf{w}t + \theta G_t + \widetilde{\sigma} W(G_t)$ (Nicoletta (2011)) where $\mathbf{w}$ is the cumulant generating function given by

$$\mathbf{w} = -\ln(\phi(-i)) = \frac{1}{\kappa}\ln(1 - \theta\kappa - \frac{1}{2}\widetilde{\sigma}^2\kappa)$$

and ϕ is the characteristic function of the time-changed arithmetic Brownian motion $\theta G_t + \widetilde{\sigma} W(G_t)$. κ, which controls kurtosis, is the variance of the subordinator, G is for gamma random variable. θ, which controls skewness, is the drift of the arithmetic Brownian motion; $\widetilde{\sigma} \neq 0$ is the volatility of the arithmetic Brownian motion, while Z is a Gaussian random variable.

With $W(G(t)) = \sqrt{G(t)}Z$, we have

$$dX_t = \mathbf{w}dt + \theta \Delta G(t) + \widetilde{\sigma}\Delta(\sqrt{G(t)})Z$$

where $\Delta G(t) = G(t) - G(t_{-})$ and $\Delta(\sqrt{G(t)}) = \sqrt{G(t)} - \sqrt{G(t_{-})}$.

Then,

$$
\begin{aligned}
&\int_0^t e^{-a(t-s)} dX_s \\
&= \mathbf{w}\int_0^t e^{-a(t-s)}ds + \sum_{0\le s\le t}\theta\Delta G(s)e^{-a(t-s)} + \sum_{0\le s\le t}\widetilde{\sigma}\Delta\sqrt{G(s)}e^{-a(t-s)}Z \\
&= \mathbf{w}\int_0^t e^{-a(t-s)}ds + \theta\sum_{0\le s\le t}\Delta G(s)e^{-a(t-s)} + \widetilde{\sigma}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(t-s)}Z \\
&= \frac{\mathbf{w}}{a}(1-e^{-at}) + \theta\sum_{0\le s\le t}\Delta G(s)e^{-a(t-s)} + \widetilde{\sigma}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(t-s)}Z.
\end{aligned}
$$

Hence, equation (4.2.1) becomes

$$
\begin{aligned}
r_t &= r_0e^{-at} + b(1-e^{-at}) + \sigma\int_0^t e^{-a(t-s)}dX_s \\
&= r_0e^{-at} + b(1-e^{-at}) + \sigma\Bigg(\frac{\mathbf{w}}{a}(1-e^{-at}) + \theta\sum_{0\le s\le t}\Delta G(s)e^{-a(t-s)} \\
&\qquad + \widetilde{\sigma}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(t-s)}Z\Bigg). \qquad (4.2.2)
\end{aligned}
$$

We adopt the above expression to derive the price of a zero-coupon bond driven by a VG process.

4.2.2 Expression for the price of a zero-coupon bond with a Vasicek short rate model under VG process

In this subsection, we derive an expression for the price of a zero-coupon bond driven by a VG process by using the improved Vasicek short rate model obtained in the previous subsection.

Let $P = P(t,T)$ be the price of a zero-coupon bond. In the risk neutral world,

$$
dP(t,T) = r_tPdt + \sigma PdX_t \qquad (4.2.3)
$$

where $\sigma \neq 0$ is the same volatility of the short rate r_t.

Let $F(t,x) = \ln x$, then $\dfrac{\partial F}{\partial t} = 0$, $\dfrac{\partial F}{\partial x} = \dfrac{1}{x}$, $\dfrac{\partial^2 F}{\partial x^2} = -\dfrac{1}{x^2}$.

Applying Itô's formula, we have

$$\begin{aligned} d\ln P &= \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial P}dP + \frac{1}{2}\frac{\partial^2 F}{\partial P^2}\langle dP, dP\rangle = \frac{1}{P}dP - \frac{1}{2}\cdot\frac{1}{P^2}(dP)^2 \\ &= \frac{1}{P}(r_t dt + \sigma dX_t)P - \frac{1}{2P^2}(r_t dt + \sigma dX_t)^2 P^2 \\ &= (r_t dt + \sigma dX_t) - \frac{1}{2}(r_t dt + \sigma dX_t)^2 \\ &= r_t dt + \sigma dX_t - \frac{1}{2}(r_t^2 (dt)^2 + 2r_t\sigma dt\cdot dX_t + \sigma^2 (dX_t)^2). \end{aligned}$$

where $(dt)^2 = 0, \quad dt\cdot dX_t = 0,\ X = X_t$.

Hence,

$$d\ln P = (r_t dt + \sigma dX_t) - \frac{1}{2}\sigma^2 (dX_t)^2.$$

Moreover,

$$X_t = \mathbf{w}t + \theta G(t) + \widetilde{\sigma}\sqrt{G(t)}Z \Rightarrow dX_t = \mathbf{w}dt + \theta\Delta G(t) + \widetilde{\sigma}\Delta\sqrt{G(t)}Z.$$

Thus,

$$\begin{aligned} (dX)^2 &= d[X,X] = dX\cdot dX = (\mathbf{w}dt + \theta\Delta G(t) + \widetilde{\sigma}\Delta\sqrt{G(t)}Z)^2 \\ &= \mathbf{w}^2 (dt)^2 + 2(\mathbf{w}\theta dt\cdot\Delta G(t) + \mathbf{w}\widetilde{\sigma}dt\cdot\Delta\sqrt{G(t)}Z + \theta\widetilde{\sigma}\Delta G(t)\Delta\sqrt{G(t)}Z) \\ &\quad + \widetilde{\sigma}^2(\Delta\sqrt{G(t)})^2 Z^2 + \theta^2(\Delta G(t))^2 \\ &= (2\theta\widetilde{\sigma}\Delta G(t)\Delta\sqrt{G(t)}Z + \widetilde{\sigma}^2(\Delta\sqrt{G(t)})^2 Z^2 + \theta^2(\Delta G(t))^2 \\ &= (\theta\Delta G(t) + \widetilde{\sigma}\Delta\sqrt{G(t)}Z)^2. \end{aligned}$$

Hence,

$$d\ln P = r_t dt + \sigma(\mathbf{w}dt + \theta\Delta G(t) + \widetilde{\sigma}\Delta\sqrt{G(t)}Z) - \frac{1}{2}\sigma^2(\theta\Delta G(t) + \widetilde{\sigma}\Delta\sqrt{G(t)}Z)^2. \tag{4.2.4}$$

Integrating equation (4.2.4), we have

$$\int_t^T d\ln P(u,T) = \ln P(u,T)\mid_t^T = \ln 1 - \ln P(t,T),\ P(T,T) = 1,$$

$$\ln P(t,T) = -\Big(\int_t^T r_u du + \int_t^T \sigma\mathbf{w}du + \sum_{0\le u\le T}\sigma(\theta\Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z)$$

$$-\sum_{0\le u\le t}\sigma(\theta\Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z) - \frac{1}{2}\sigma^2\Big(\sum_{0\le u\le T}\sigma^2(\theta\Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z)^2$$

$$-\sum_{0\le u\le t}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)^2\Big)\Big)$$

which implies that

$$\begin{aligned}P(t,T) &= \exp\Big(-\Big(\int_t^T r_u du+\mathbf{w}\sigma[T-t]+\sigma\sum_{0\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\\ &-\sigma\sum_{0\le u\le t}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)-\frac{\sigma^2}{2}\Big(\sum_{0\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)^2\\ &-\sum_{0\le u\le t}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)^2\Big)\Big)\Big)\end{aligned}$$

where $P(T,T)=1$ and by equation (4.2.2),

$$r_t=r_0e^{-at}+b(1-e^{-at})+\sigma\Big(\frac{\mathbf{w}}{a}(1-e^{-at})$$

$$+\theta\sum_{0\le s\le t}\Delta G(s)e^{-a(t-s)}+\widetilde{\sigma}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(t-s)}Z\Big).$$

Thus,

$$\begin{aligned}\int_t^T r_u du &= r_0\int_t^T e^{-au}du+\int_t^T b(1-e^{-au})du+\sigma\int_t^T\frac{\mathbf{w}}{a}(1-e^{-at})du\\ &+\sigma\Big(\theta\sum_{0\le u\le T}\sum_{0\le s\le t}\Delta G(s)e^{-a(u-s)}+\widetilde{\sigma}\sum_{0\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)}Z\Big)\\ &-\sigma\Big(\theta\sum_{0\le u\le t}\sum_{0\le s\le t}\Delta G(s)e^{-a(u-s)}+\widetilde{\sigma}\sum_{0\le u\le t}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)}Z\Big)\\ &= -\frac{r_0}{a}\big(e^{-aT}-e^{-at}\big)+b\big(T-t+a^{-1}(e^{-aT}-e^{-at})\big)+\frac{\sigma\mathbf{w}}{a}\Big[T-t+a^{-1}(e^{-aT}-e^{-at})\Big]\\ &+\sigma\Big(\theta\sum_{0\le u\le T}\sum_{0\le s\le t}\Delta G(s)e^{-a(u-s)}+\widetilde{\sigma}\sum_{0\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)}Z\Big)\\ &-\sigma\Big(\theta\sum_{0\le u\le t}\sum_{0\le s\le t}\Delta G(s)e^{-a(u-s)}+\widetilde{\sigma}\sum_{0\le u\le t}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)}Z\Big).\end{aligned}$$

Hence, in a Lévy market driven by a VG process, the value of a zero-coupon bond is

$$P(t,T)=\exp\Big(-\Big(\Big[-\frac{r_0}{a}\big(e^{-aT}-e^{-at}\big)+b\big(T-t+a^{-1}(e^{-aT}-e^{-at})\big)$$

$$+\frac{\sigma\mathbf{w}}{a}\Big[T-t+a^{-1}(e^{-aT}-e^{-at})\Big]+\sigma\sum_{0\le u\le T}\sum_{0\le s\le t}\big(\theta\Delta G(s)e^{-a(u-s)}+\widetilde{\sigma}\Delta\sqrt{G(s)}e^{-a(u-s)}Z\big)$$

$$-\sigma \sum_{0\leq u\leq t}\sum_{0\leq s\leq t}\left(\theta\Delta G(s)e^{-a(u-s)}+\widetilde{\sigma}\Delta\sqrt{G(s)}e^{-a(u-s)}Z\right)\Bigg]+\mathbf{w}\sigma[T-t]$$

$$+\sigma\sum_{0\leq u\leq T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)-\sigma\sum_{0\leq u\leq t}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)$$

$$-\frac{\sigma^2}{2}\Bigg(\sum_{0\leq u\leq T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)^2-\sum_{0\leq u\leq t}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)^2\Bigg)\Bigg)\Bigg). \tag{4.2.5}$$

Besides being a function of t and T, the expression on the right hand side of the above equation also depends on r_0, σ, $\widetilde{\sigma}$, $\mathbf{w}$ and Z. Thus, in the sequel, we shall regard P driven by VG process as a function of t, T, r_0, σ, $\widetilde{\sigma}$, $\mathbf{w}$ and Z.

Remark 4.2.1. In the subsequent sections, we shall use the price of a call option, with P as the underlying, given by

$$\mathbb{V}=e^{-r_0T}\mathbb{E}[\Phi(P)] \tag{4.2.6}$$

where r_0, T and $\Phi(P)$ denote the initial interest rate, maturity time and the payoff, respectively.

4.2.3 The greeks of zero-coupon bonds driven by VG Lévy process

In this subsection, we compute the greeks of the price of a zero-coupon bond. Let $\mathbb{V}$ be given by equation (4.2.6). It is seen that $\mathbb{V}$ is sensitive to changes in several parameters. The following greeks will be computed:
(1) $\text{Delta}^{VG} := \triangle^{VG} = \frac{\partial\mathbb{V}}{\partial r_0}$, (2) $\text{Gamma}^{VG} := \frac{\partial^2\mathbb{V}}{\partial r_0^2}$,
(3) $\text{Vega}^{VG} := \mathcal{V}^{VG} = \frac{\partial\mathbb{V}}{\partial\sigma}$, (4) $\text{Drift} := \mathcal{D} = \frac{\partial\mathbb{V}}{\partial\theta}$, (5) $\text{Vega}_2^{VG} := \frac{\partial\mathbb{V}}{\partial\kappa}$,
(6) $\text{Vega}_3^{VG} := \frac{\partial\mathbb{V}}{\partial\widetilde{\sigma}}$, (7) $\text{Theta}^{VG} := \frac{\partial\mathbb{V}}{\partial T}$.

Greeks describe the sensitivity of a bond price to alterations in certain parameters and enable traders to hedge their risks. The greek 'delta^{VG}' represents the sensitivity of the interest rate derivative to changes in the interest rate. Option traders are interested in delta because movements in the underlying may alter the worth of their positions (Corb (2012, pg. 488)). The greek 'gamma^{VG}' of an interest rate derivative gives the sensitivity of delta to alterations in the underlying, that is, the interest rate. Vega^{VG} measures the sensitivity of the zero-coupon bond option price to alterations in the volatility of the short rate model; in other words, it measures the alteration in the option price for a unit alteration in volatility. The volatility of the underlying represents the uncertainty about future prices for the underlying contract (Carol (2008)), and high vega implies that an option's value is sensitive to little moves in volatility (Chorafas (2008)). The knowledge of vega assists the risk manager to reduce risk. The drift '$\mathcal{D}$' describes the sensitivity of zero-coupon bond option price to changes in the drift of the VG process, it measures the effect of changes in the skewness of the short rate model to the option price. Vega$_2^{VG}$ describes the sensitivity of the bond option price to changes in the variance of the gamma process. Theta^{VG} describes the sensitivity of the bond option price to maturity time, it measures the rate of depreciation of the option value with time. Vega$_3^{VG}$ describes the sensitivity of the bond option price to changes in the volatility of the arithmetic Brownian motion.

We derive the expressions for the above greeks in the case of a VG-driven interest rate derivative.

Remark 4.2.2. We shall use the following information in the sequel.

$$\sum_{t\leq u\leq T} f(u)\Delta(u) = \sum_{0\leq u\leq T} f(u)\Delta(u) - \sum_{0\leq u<t} f(u)\Delta(u).$$

This implies that at time $t = T$,

$$\sum_{t\leq u\leq T} f(u)\Delta(u) = 0.$$

By Remark 4.2.2, it follows that the price of the zero-coupon bond, expressed by equation (4.2.5), may be written as:

$$P(t,T) = \exp\Bigg(-\Bigg(\Bigg[-\frac{r_0}{a}\big(e^{-aT} - e^{-at}\big) + b\big(T - t + \frac{1}{a}(e^{-aT} - e^{-at})\big)$$

$$+\frac{\sigma\mathbf{w}}{a}\left[T - t + \frac{1}{a}(e^{-aT} - e^{-at})\right]$$

$$+\sigma\sum_{t\leq u\leq T}\sum_{0\leq s\leq t}\Big(\theta\Delta G(s)e^{-a(u-s)} + \widetilde{\sigma}\Delta\sqrt{G(s)}e^{-a(u-s)}Z\Big)\Bigg] + \mathbf{w}\sigma[T-t]$$

$$+\sigma \sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)-\frac{\sigma^2}{2}\bigg(\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)^2\bigg)\bigg)\bigg) \tag{4.2.7}$$

where

$$\mathbf{w}=\frac{1}{\kappa}\ln(1-\theta\kappa-\frac{1}{2}\widetilde{\sigma}^2\kappa).$$

Let Q be of the form $Q=\dfrac{\partial P}{\partial \eta}$ for some parameters η of the zero-coupon bond. By Definition 3.10.3 and Theorem 3.10.3, with $i=j=1$, we write $\mathcal{M}(P)=\langle DP, DP\rangle = DP\cdot DP$ for the Malliavin covariance matrix. Assume that the matrix is invertible, we write $\mathcal{M}(P)^{-1}=\dfrac{1}{\mathcal{M}(P)}$ provided $DP\neq 0$, and L for the Ornstein-Uhlenbeck (O-U) operator. For a smooth function $\Phi:\mathbb{R}\to\mathbb{R}$, the following equation holds:

$$\mathbb{E}[\partial\Phi(P)Q]=\mathbb{E}[\Phi(P)H(P,Q)]$$

where $H(P,Q)$ is the Malliavin weight given by

$$H(P,Q)=Q\mathcal{M}(P)^{-1}LP-\mathcal{M}(P)^{-1}\langle DP,DQ\rangle - Q\langle DP, D\mathcal{M}(P)^{-1}\rangle \tag{4.2.8}$$

with $\mathbb{E}[H(P,Q)]<\infty$ (Bally, V. and Clément, E., 2010; Bavouzet et al., 2009; Bayazit and Nolder, 2009).

Let $\Phi(P)=\max(P-K,0)$ denote the payoff of a call option on a zero-coupon bond $P=P(t,T)$. The price of a call option with P as the underlying is given by $\mathbb{V}=e^{-r_0T}\mathbb{E}[\Phi(P)]$ where K is the strike price. In the sequel, we obtain expressions for the following greeks:

(1) $\text{Delta}^{VG}:=\triangle^{VG}=\dfrac{\partial\mathbb{V}}{\partial r_0}$, (2) $\text{Gamma}^{VG}:=\Gamma^{VG}=\dfrac{\partial^2\mathbb{V}}{\partial r_0^2}$,

(3) $\text{Vega}^{VG}:=\mathcal{V}^{VG}=\dfrac{\partial\mathbb{V}}{\partial\sigma}$, (4) $\text{Drift}:=\mathcal{D}=\dfrac{\partial\mathbb{V}}{\partial\theta}$,

(5) $\text{Vega}_2^{VG}:=\mathcal{V}_2=\dfrac{\partial\mathbb{V}}{\partial\kappa}$, (6) $\text{Vega}_3^{VG}:=\mathcal{V}_3^{VG}=\dfrac{\partial\mathbb{V}}{\partial\widetilde{\sigma}}$,

(7) $\text{Theta}^{VG}:=\Theta_{VG}^{VG}=\dfrac{\partial\mathbb{V}}{\partial T}$.

Delta^{VG} denoted as $\triangle^{VG}$ describes the sensitivity to changes in the initial interest rate r_0. That is,

$$\triangle^{VG}:=\frac{\partial\mathbb{V}}{\partial r_0}=\frac{\partial}{\partial r_0}\Big(e^{-r_0T}\mathbb{E}(\Phi(P))\Big)$$

where $\Phi(P)$ is the payoff of the zero-coupon bond price.

In what follows, we adopt the zero-coupon bond price driven by a VG process as given by equation (4.2.7).

The following Lemmas will be needed for easier computation of the greeks.

Lemma 4.2.1. Let P be the price of a zero-coupon bond driven by a VG process. Then, the Malliavian derivative is given by

$$DP = -\Big[\sigma\widetilde{\sigma}\Big(\sum_{t\leq u\leq T}\sum_{0\leq s\leq t}\Delta\sqrt{G(s)}e^{-a(u-s)} + \sum_{t\leq u\leq T}(\Delta\sqrt{G(u)})\Big)$$

$$-\sigma^2\widetilde{\sigma}\Big(\sum_{t\leq u\leq T}(\theta\Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Big)\Big]P. \tag{4.2.9}$$

Proof. Let $P = P(t,T)$ be as given in equation (4.2.7). Then, the Malliavin derivative of P is given by

$$DP = D\exp\Big(-\Big(\Big[-\frac{r_0}{a}\big(e^{-aT} - e^{-at}\big) + b\big(T - t + \frac{1}{a}(e^{-aT} - e^{-at})\big)$$

$$+\frac{\sigma\mathbf{w}}{a}\big[T - t + \frac{1}{a}(e^{-aT} - e^{-at})\big] + \sigma\sum_{t\leq u\leq T}\sum_{0\leq s\leq t}\big(\theta\Delta G(s)e^{-a(u-s)} + \widetilde{\sigma}\Delta\sqrt{G(s)}e^{-a(u-s)}Z\big)\Big]$$

$$+\mathbf{w}\sigma[T-t] + \sigma\sum_{t\leq u\leq T}(\theta\Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z) - \frac{\sigma^2}{2}\sum_{t\leq u\leq T}(\theta\Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z)^2\Big)\Big)$$

$$= \Big(-\sigma\widetilde{\sigma}\sum_{t\leq u\leq T}\sum_{0\leq s\leq t}\Delta\sqrt{G(s)}e^{-a(u-s)} - \sigma\widetilde{\sigma}\sum_{t\leq u\leq T}(\Delta\sqrt{G(u)})$$

$$+\frac{\sigma^2\widetilde{\sigma}}{2}\big(2\sum_{t\leq u\leq T}(\theta\Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\big)\Big)$$

$$\times\exp\Big(-\Big(\Big[-\frac{r_0}{a}\big(e^{-aT} - e^{-at}\big) + b\big(T - t + \frac{1}{a}(e^{-aT} - e^{-at})\big)$$

$$+\frac{\sigma\mathbf{w}}{a}\big[T - t + \frac{1}{a}(e^{-aT} - e^{-at})\big] + \sigma\sum_{t\leq u\leq T}\sum_{0\leq s\leq t}\big(\theta\Delta G(s)e^{-a(u-s)} + \widetilde{\sigma}\Delta\sqrt{G(s)}e^{-a(u-s)}Z\big)\Big]$$

$$+\mathbf{w}\sigma[T-t] + \sigma\sum_{t\leq u\leq T}(\theta\Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z) - \frac{\sigma^2}{2}\sum_{t\leq u\leq T}(\theta\Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z)^2\Big)\Big)$$

$$= \Big(-\sigma\widetilde{\sigma}\sum_{t\leq u\leq T}\sum_{0\leq s\leq t}\Delta\sqrt{G(s)}e^{-a(u-s)} - \sigma\widetilde{\sigma}\sum_{t\leq u\leq T}(\Delta\sqrt{G(u)})$$

$$+\sigma^2\widetilde{\sigma}\big(\sum_{t\leq u\leq T}(\theta\Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\big)\Big)\times P.$$

Hence,

$$DP = -\Big[\sigma\widetilde{\sigma}\sum_{t\leq u\leq T}\sum_{0\leq s\leq t}\Delta\sqrt{G(s)}e^{-a(u-s)} + \sigma\widetilde{\sigma}\sum_{t\leq u\leq T}(\Delta\sqrt{G(u)})$$

$$-\sigma^2\widetilde{\sigma}(\sum_{t\leq u\leq T}(\theta\Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)})\Big]P,$$

which is equation (4.2.9). □

Lemma 4.2.2. Let P be the price of a zero-coupon bond driven by a VG process. Then the action of the Ornstein-Uhlenbeck operator L on P is given by

$$LP = -\Big[\sigma^2\widetilde{\sigma}^2\sum_{t\leq u\leq T}(\Delta\sqrt{G(u)})^2 + \Big(\sigma\widetilde{\sigma}\sum_{t\leq u\leq T}\sum_{0\leq s\leq t}\Delta\sqrt{G(s)}e^{-a(u-s)}$$

$$+\sigma\widetilde{\sigma}\sum_{t\leq u\leq T}(\Delta\sqrt{G(u)}) - \sigma^2\widetilde{\sigma}(\sum_{t\leq u\leq T}(\theta\Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Big)^2$$

$$+Z\Big(\sigma\widetilde{\sigma}(\sum_{t\leq u\leq T}\sum_{0\leq s\leq t}\Delta\sqrt{G(s)}e^{-a(u-s)} + \sum_{t\leq u\leq T}\Delta\sqrt{G(u)})$$

$$-\sigma^2\widetilde{\sigma}\Big(\sum_{t\leq u\leq T}(\theta\Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Big)\Big)\Big]P. \qquad (4.2.10)$$

Proof. From the Malliavin derivative of the price P of the zero-coupon bond, it follows from equation (4.2.9) that:

$$D(DP) = D\Big(-\Big[\sigma\widetilde{\sigma}\Big(\sum_{t\leq u\leq T}\sum_{0\leq s\leq t}\Delta\sqrt{G(s)}e^{-a(u-s)} + \sum_{t\leq u\leq T}(\Delta\sqrt{G(u)})\Big)$$

$$-\sigma^2\widetilde{\sigma}\Big(\sum_{t\leq u\leq T}(\theta\Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Big)\Big]P\Big)$$

$$= \Big(-\Big[\sigma\widetilde{\sigma}\Big(\sum_{t\leq u\leq T}\sum_{0\leq s\leq t}\Delta\sqrt{G(s)}e^{-a(u-s)} + \sum_{t\leq u\leq T}(\Delta\sqrt{G(u)})\Big)$$

$$-\sigma^2\widetilde{\sigma}\Big(\sum_{t\leq u\leq T}(\theta\Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Big)\Big]\Big)DP$$

$$+P\dot{D}\Big(-\Big[\sigma\widetilde{\sigma}\Big(\sum_{t\leq u\leq T}\sum_{0\leq s\leq t}\Delta\sqrt{G(s)}e^{-a(u-s)} + \sum_{t\leq u\leq T}(\Delta\sqrt{G(u)})\Big)$$

$$-\sigma^2\widetilde{\sigma}\Big(\sum_{t\leq u\leq T}(\theta\Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Big)\Big]\Big)$$

where P and DP are as given by equations (4.2.7) and (4.2.9), respectively.

Hence,

$$DDP = \Big(- \Big[\sigma\widetilde{\sigma}\big(\sum_{t\leq u\leq T} \sum_{0\leq s\leq t} \Delta\sqrt{G(s)}e^{-a(u-s)}\big) + \sigma\widetilde{\sigma} \sum_{t\leq u\leq T} (\Delta\sqrt{G(u)})$$

$$-\sigma^2\widetilde{\sigma} \sum_{t\leq u\leq T} (\theta\Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Big]\Big) DP - P\Big(- \sigma^2 \sum_{t\leq u\leq T} (\widetilde{\sigma}\Delta\sqrt{G(u)})^2\Big)$$

$$= \sigma^2\Big(\sum_{t\leq u\leq T} (\widetilde{\sigma}\Delta\sqrt{G(u)})^2\Big)P + \Big(\Big[\sigma\widetilde{\sigma} \sum_{t\leq u\leq T} \sum_{0\leq s\leq t} \Delta\sqrt{G(s)}e^{-a(u-s)}$$

$$+\sigma\widetilde{\sigma} \sum_{t\leq u\leq T} (\Delta\sqrt{G(u)}) - \sigma^2\widetilde{\sigma}\big(\sum_{t\leq u\leq T} (\theta\Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Big]^2\Big)P.$$

By Remark 3.10.1 and Definition (3.10.5), the Ornstein-Uhlenbeck operator on P becomes

$$LP(t, T) = -[DDP(t, T) + \varphi DP(t, T)] = -[DDP - ZDP] \qquad (4.2.11)$$

where

$\varphi(z) = \partial_z \ln f(z) = \dfrac{f'(z)}{f(z)}$, $f(z) \neq 0$, otherwise $\varphi(z) = 0$; $f(z) = \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}$ is the density function of the standardised Gaussian random variable Z.

Substituting DDP and equation (4.2.9) into equation (4.2.11), we obtain

$$LP = -\Big[\Big(\sigma^2 \sum_{t\leq u\leq T} (\widetilde{\sigma}\Delta\sqrt{G(u)})^2 + \Big[\sigma\widetilde{\sigma} \sum_{t\leq u\leq T} \sum_{0\leq s\leq t} \Delta\sqrt{G(s)}e^{-a(u-s)}$$

$$+\sigma\widetilde{\sigma} \sum_{t\leq u\leq T} (\Delta\sqrt{G(u)}) - \sigma^2\widetilde{\sigma} \sum_{t\leq u\leq T} (\theta\Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Big]^2\Big)P$$

$$+(-Z)\Big(- \Big[\sigma\widetilde{\sigma} \sum_{t\leq u\leq T} \sum_{0\leq s\leq t} \Delta\sqrt{G(s)}e^{-a(u-s)} + \sigma\widetilde{\sigma} \sum_{t\leq u\leq T} (\Delta\sqrt{G(u)})$$

$$-\sigma^2\widetilde{\sigma}\Big(\sum_{t\leq u\leq T} (\theta\Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Big)\Big]\Big)P\Big],$$

which is equation (4.2.10). □

Lemma 4.2.3. Let P be the price of a zero-coupon bond driven by a VG process. Then, the Malliavin covariance matrix of P is given by

$$\mathcal{M}(P) = \Big[\sigma\widetilde{\sigma} \sum_{t\leq u\leq T} \sum_{0\leq s\leq t} \Delta\sqrt{G(s)}e^{-a(u-s)} + \sigma\widetilde{\sigma} \sum_{t\leq u\leq T} (\Delta\sqrt{G(u)})$$

$$-\sigma^2\widetilde{\sigma}\sum_{t\leq u\leq T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Bigg]^2 P^2.$$

Furthermore,

$$\mathcal{M}(P)^{-1}=\Bigg(\Bigg[\sigma\widetilde{\sigma}\sum_{t\leq u\leq T}\sum_{0\leq s\leq t}\Delta\sqrt{G(s)}e^{-a(u-s)}+\sigma\widetilde{\sigma}\sum_{t\leq u\leq T}(\Delta\sqrt{G(u)})$$

$$-\sigma^2\widetilde{\sigma}\sum_{t\leq u\leq T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Bigg]P\Bigg)^{-2} \tag{4.2.12}$$

where equation (4.2.12) holds with the following conditions

$$\sigma\neq 0,\ \widetilde{\sigma}\neq 0 \text{ and } P\neq 0.$$

Proof. From equation (4.2.9), the Malliavin derivative is given by

$$DP=-\Bigg[\sigma\widetilde{\sigma}\sum_{t\leq u\leq T}\sum_{0\leq s\leq t}\Delta\sqrt{G(s)}e^{-a(u-s)}+\sigma\widetilde{\sigma}\sum_{t\leq u\leq T}(\Delta\sqrt{G(u)})$$

$$-\sigma^2\widetilde{\sigma}\sum_{t\leq u\leq T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Bigg]P.$$

The Malliavin covariance matrix $\mathcal{M}(P)$ is given by

$\mathcal{M}(P)=\langle DP,DP\rangle=(DP)^2$

$$=\Bigg(-\Bigg[\sigma\widetilde{\sigma}\sum_{t\leq u\leq T}\sum_{0\leq s\leq t}\Delta\sqrt{G(s)}e^{-a(u-s)}+\sigma\widetilde{\sigma}\sum_{t\leq u\leq T}(\Delta\sqrt{G(u)})$$

$$-\sigma^2\widetilde{\sigma}\sum_{t\leq u\leq T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Bigg]\Bigg)^2 P^2.$$

Hence, $\mathcal{M}(P)^{-1}=(DP)^{-2}$, which is equation (4.2.12). □

Lemma 4.2.4. Let $P=P(t,T)$ be the price of a zero-coupon bond driven by a VG process and $\mathcal{M}(P)^{-1}$ be the inverse Malliavin covariance matrix of $P(t,T)$. Then,

$$D(\mathcal{M}(P))^{-1}=2\Bigg[\Bigg(\sigma\widetilde{\sigma}\sum_{t\leq u\leq T}\sum_{0\leq s\leq t}\Delta\sqrt{G(s)}e^{-a(u-s)}+\sigma\widetilde{\sigma}\sum_{t\leq u\leq T}(\Delta\sqrt{G(u)})$$

$$-\sigma^2\widetilde{\sigma}\sum_{t\leq u\leq T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Bigg)^{-3}\Bigg]P^{-2}$$

$$\times\Bigg[\sigma^2\widetilde{\sigma}^2\sum_{t\le u\le T}(\Delta\sqrt{G(u)})^2+\Bigg[\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)}$$

$$+\sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)})-\sigma^2\widetilde{\sigma}\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Bigg]^2\Bigg]. \tag{4.2.13}$$

Proof. From equation (4.2.12),

$$\mathcal{M}(P)^{-1}=\Bigg(-\Bigg[\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)}+\sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)})$$

$$-\sigma^2\widetilde{\sigma}\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Bigg]P\Bigg)^{-2}.$$

Thus, the Malliavin derivative of $\mathcal{M}(P)^{-1}$ is

$$D(\mathcal{M}(P)^{-1})=-2\Bigg[\Bigg(\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)}+\sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)})$$

$$-\sigma^2\widetilde{\sigma}\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Bigg)P\Bigg]^{-3}$$

$$\times\Bigg(P\Bigg(-\sigma^2\sum_{t\le u\le T}(\widetilde{\sigma}\Delta\sqrt{G(u)})^2\Bigg)+\Bigg[\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)}$$

$$+\sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)})-\sigma^2\widetilde{\sigma}\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Bigg]$$

$$\cdot\Bigg(-\Bigg[\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)}+\sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)})$$

$$-\sigma^2\widetilde{\sigma}\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Bigg]P\Bigg)$$

$$=-2\Bigg[\Bigg(\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)}+\sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)})$$

$$-\sigma^2\widetilde{\sigma}\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Bigg)P\Bigg]^{-3}$$

$$\times\Bigg[-\sigma^2\sum_{t\le u\le T}(\widetilde{\sigma}\Delta\sqrt{G(u)})^2P-\Bigg[\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)}$$

$$+\sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)})-\sigma^2\widetilde{\sigma}\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Bigg]^2P\Bigg],$$

which is equation (4.2.13) $\square$

4.2.4 Computation of *delta* for VG-driven interest rate derivatives

In this subsection, we compute the greek '*delta*' for VG-driven interest rate derivative. Let $\Phi(P)$ be the payoff of the zero-coupon bond price P. Then,

$$\triangle^{VG} := \frac{\partial}{\partial r_0}[e^{-r_0 T}\mathbb{E}(\Phi(P))] = -Te^{-r_0 T}\mathbb{E}(\Phi(P)) + e^{-r_0 T}\mathbb{E}\left[\Phi'(P)\frac{\partial P}{\partial r_0}\right]$$
$$= -Te^{-r_0 T}\mathbb{E}(\Phi(P)) + e^{-r_0 T}\mathbb{E}\left[\Phi(P)H_{VG}\left(P, \frac{\partial P}{\partial r_0}\right)\right].$$

Lemma 4.2.5. Let $Q = \dfrac{\partial P}{\partial r_0}$, and let P be the price of the zero-coupon bond driven by a VG process. Then, the following holds:

$$Q = \frac{1}{a}(e^{-aT} - e^{-at})P \tag{4.2.14}$$

and

$$DQ = -\frac{1}{a}(e^{-aT} - e^{-at})\Bigg(\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)} + \sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)})$$
$$-\sigma^2\widetilde{\sigma}\sum_{t\le u\le T}(\theta\Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Bigg)P. \tag{4.2.15}$$

Proof. From equation (4.2.7), the partial derivative of P with respect to r_0 is given by

$$Q = \frac{\partial P}{\partial r_0} = \frac{1}{a}(e^{-aT} - e^{-at})P.$$

Also, the Malliavin derivative

$$DQ = \frac{1}{a}(e^{-aT} - e^{-at})DP.$$

Substituting equation (4.2.9) for DP in the above equation, we get the desired result. $\square$

By Lemmas 4.2.1-4.2.5, we state Lemmas 4.2.6 and 4.2.7 needed to obtain each term of the Malliavian weight for delta denoted $\triangle^{VG}$ in Theorem 4.2.1.

Lemma 4.2.6. Let P be the price of a zero-coupon bond driven by VG process and $Q = \dfrac{\partial P}{\partial r_0}$. Then

$$Q\mathcal{M}(P)^{-1}LP = -\frac{\sigma^2}{a}(e^{-aT} - e^{-at})\Big(\sum_{t\le u\le T}(\widetilde{\sigma}\Delta\sqrt{G(u)})^2\Big)\cdot\mathcal{K}^{-2}$$

$$-\frac{1}{a}(e^{-aT}-e^{-at})-\frac{\frac{1}{a}(e^{-aT}-e^{-at})Z}{\mathcal{K}} \tag{4.2.16}$$

where

$$\begin{aligned}\mathcal{K}=\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)}+\sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)})\\ -\sigma^2\widetilde{\sigma}\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}.\end{aligned} \tag{4.2.17}$$

Proof. Substituting equations (4.2.14), (4.2.12) and (4.2.10) for $Q\mathcal{M}(P)^{-1}LP$, we have

$$\begin{aligned}Q\mathcal{M}(P)^{-1}LP=\frac{1}{a}(e^{-aT}-e^{-at})P\cdot\Bigg(\Bigg[\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)}+\sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)})\\ -\sigma^2\widetilde{\sigma}\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Bigg]\Bigg)^{-2}P^{-2}\cdot\Bigg[-\Bigg[\sigma^2\sum_{t\le u\le T}(\widetilde{\sigma}\Delta\sqrt{G(u)})^2\\ +\Bigg[\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)}+\sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)})\\ -\sigma^2\widetilde{\sigma}\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Bigg]^2\\ +Z\Bigg[\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)}+\sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)})\\ -\sigma^2\widetilde{\sigma}\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Bigg]\Bigg]P\end{aligned}$$

$$\begin{aligned}=-\frac{1}{a}(e^{-aT}-e^{-at})\cdot\Bigg[\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)}+\sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)})\\ -\sigma^2\widetilde{\sigma}\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Bigg]^{-2}\\ \cdot\Bigg[\sigma^2\sum_{t\le u\le T}(\widetilde{\sigma}\Delta\sqrt{G(u)})^2+\Bigg[\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)}+\sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)})\\ -\sigma^2\widetilde{\sigma}\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Bigg]^2\\ +Z\Bigg[\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)}+\sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)})\end{aligned}$$

$$-\sigma^2\widetilde{\sigma}\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Big]\Big]$$

$$=-\frac{\sigma^2}{a}(e^{-aT}-e^{-at})\Big(\sum_{t\le u\le T}(\widetilde{\sigma}\Delta\sqrt{G(u)})^2\Big)\cdot\Big[\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)}$$

$$+\sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)})-\sigma^2\widetilde{\sigma}\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Big]^{-2}$$

$$-\frac{1}{a}(e^{-aT}-e^{-at})-\frac{1}{a}(e^{-aT}-e^{-at})Z\cdot\Big[\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)}$$

$$+\sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)})-\sigma^2\widetilde{\sigma}\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Big]^{-1}. \qquad \square$$

Lemma 4.2.7. Let P be the price of a zero-coupon bond driven by a VG process where $\mathcal{M}(P)^{-1}$ is the inverse Malliavin covariance matrix of P and $Q=\dfrac{\partial P}{\partial r_0}$. Then,

$$\mathcal{M}(P)^{-1}\langle DP, DQ\rangle=\frac{1}{a}(e^{-aT}-e^{-at}). \tag{4.2.18}$$

Proof. From equations (4.2.12), (4.2.9) and (4.2.15), it follows that

$$\mathcal{M}(P)^{-1}\langle DP, DQ\rangle=-\Big[\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)}+\sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)})$$

$$-\sigma^2\widetilde{\sigma}\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Big]P$$

$$\cdot-\frac{1}{a}(e^{-aT}-e^{-at})\Big[\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)}+\sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)})$$

$$-\sigma^2\widetilde{\sigma}\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Big]P\cdot\Big[\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)}$$

$$+\sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)})-\sigma^2\widetilde{\sigma}\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Big]^{-2}P^{-2}$$

$$=\frac{1}{a}(e^{-aT}-e^{-at}). \qquad \square$$

Lemma 4.2.8. Let P be the price of a zero-coupon bond driven by a VG process, $\mathcal{M}(P)^{-1}$ be the inverse Malliavin covariance matrix of P and $Q=\dfrac{\partial P}{\partial r_0}$. Then,

$$Q\langle DP, D\mathcal{M}(P)^{-1}\rangle=-\frac{2}{a}(e^{-aT}-e^{-at})-\frac{2\sigma^2}{a}(e^{-aT}-e^{-at})\Big(\widetilde{\sigma}^2\sum_{t\le u\le T}(\Delta\sqrt{G(u)})^2\Big)\mathcal{K}^{-2} \tag{4.2.19}$$

where $\mathcal{K}$ is given by equation (4.2.17).

Proof. From equations (4.2.14), (4.2.9) and (4.2.13), it follows that

$Q\langle DP, D\mathcal{M}(P)^{-1}\rangle$

$$= -\frac{1}{a}(e^{-aT} - e^{-at})P\Big[\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)} + \sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)})$$

$$-\sigma^2\widetilde{\sigma}\sum_{t\le u\le T}(\theta\Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Big]P\times 2\Big[\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)}$$

$$+\sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)}) - \sigma^2\widetilde{\sigma}\sum_{t\le u\le T}(\theta\Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Big]^{-3}P^{-2}$$

$$\cdot\Big[\sigma^2\sum_{t\le u\le T}(\widetilde{\sigma}\Delta\sqrt{G(u)})^2 + \Big[\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)}$$

$$+\sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)}) - \sigma^2\widetilde{\sigma}\sum_{t\le u\le T}(\theta\Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Big]^2\Big]$$

$$= -\frac{2}{a}(e^{-aT} - e^{-at})\Big[\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)} + \sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)})$$

$$-\sigma^2\widetilde{\sigma}\sum_{t\le u\le T}(\theta\Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Big]^{-2}$$

$$\cdot\Big[\sigma^2\sum_{t\le u\le T}(\widetilde{\sigma}\Delta\sqrt{G(u)})^2 + \Big[\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)} + \sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)})$$

$$-\sigma^2\widetilde{\sigma}\sum_{t\le u\le T}(\theta\Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Big]^2\Big]$$

$$= -\frac{2}{a}(e^{-aT} - e^{-at})\big(\sigma^2\widetilde{\sigma}^2\sum_{t\le u\le T}(\Delta\sqrt{G(u)})^2\cdot\Big[\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)}$$

$$+\sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)}) - \sigma^2\widetilde{\sigma}\sum_{t\le u\le T}(\theta\Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Big]^{-2} - \frac{2}{a}(e^{-aT} - e^{-at})$$

where

$$\mathcal{K} = \sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)} + \sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)})$$

$$-\sigma^2\widetilde{\sigma}\sum_{t\le u\le T}(\theta\Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}. \qquad \square$$

Theorem 4.2.1. Suppose that the price of a zero-coupon bond driven by VG process is given by equation (4.2.7) and $\Phi(P) = \max(P-K, 0)$ is the payoff with strike price K on the bond. Let the price of the call option be $e^{-r_0 T}\mathbb{E}[\Phi(P)]$, then, its sensitivity with respect to its initial underlying asset r_0 denoted by $\triangle^{VG}$ is given by

$$\triangle^{VG} = e^{-r_0 T}\Bigg(-T\mathbb{E}(\Phi(P))+\mathbb{E}\Bigg[\Phi(P)\frac{\sigma^2}{a}(e^{-aT} - e^{-at})\big(\widetilde{\sigma}^2 \sum_{t\le u\le T} (\Delta\sqrt{G(u)})^2\big)\cdot \mathcal{K}^{-2} - \frac{\frac{1}{a}(e^{-aT} - e^{-at})Z}{\mathcal{K}}\Bigg]\Bigg)$$

where the Malliavin weight is

$$H_{VG}(P,Q) = \frac{\sigma^2}{a}(e^{-aT} - e^{-at})\big(\widetilde{\sigma}^2 \sum_{t\le u\le T} (\Delta\sqrt{G(u)})^2\big)\cdot \mathcal{K}^{-2} - \frac{\frac{1}{a}(e^{-aT} - e^{-at})Z}{\mathcal{K}},$$

$Q = \dfrac{\partial P}{\partial r_0}$ and $\mathcal{K}$ is given by equation (4.2.17).

Proof. Recall that

$$\begin{aligned}\triangle^{VG} &= \frac{\partial}{\partial r_0}[e^{-r_0 T}E(\Phi(P))] = -Te^{-r_0 T}\mathbb{E}(\Phi(P)) + e^{-r_0 T}\mathbb{E}\left[\Phi'(P)\frac{\partial P}{\partial r_0}\right] \\ &= -Te^{-r_0 T}\mathbb{E}(\Phi(P)) + e^{-r_0 T}\mathbb{E}\left[\Phi(P)H_{VG}\left(P, \frac{\partial P}{\partial r_0}\right)\right].\end{aligned}$$

From equations (4.2.16), (4.2.18) and (4.2.19), the weight function (4.2.8) is given by

$$H_{VG}(P,Q) = H_{VG}\left(P, \frac{\partial P}{\partial r_0}\right) = Q\mathcal{M}(P)^{-1}LP - \mathcal{M}(P)^{-1}\langle DP, DQ\rangle - Q\langle DP, D\mathcal{M}(P)^{-1}\rangle$$

$$= -\frac{\sigma^2}{a}(e^{-aT} - e^{-at})\big(\widetilde{\sigma}^2 \sum_{t\le u\le T} (\Delta\sqrt{G(u)})^2\big)\cdot \mathcal{K}^{-2} - \frac{1}{a}(e^{-aT} - e^{-at}) - \frac{\frac{1}{a}(e^{-aT} - e^{-at})Z}{\mathcal{K}}$$

$$-\frac{1}{a}(e^{-aT} - e^{-at}) + \frac{2\sigma^2}{a}(e^{-aT} - e^{-at})\big(\widetilde{\sigma}^2 \sum_{t\le u\le T} (\Delta\sqrt{G(u)})^2\big)\mathcal{K}^{-2} + \frac{2}{a}(e^{-aT} - e^{-at}). \qquad \square$$

4.2.5 Computation of *gamma* for VG-driven interest rate derivatives

In this subsection, we compute the greek '*gamma*' for the VG-driven interest rate derivative from the second partial derivative. Suppose that $Q = \dfrac{\partial P}{\partial r_0}$, then

$$\begin{aligned}
\Gamma^{VG} &= \frac{\partial^2}{\partial r_0^2}(e^{-r_0T}\mathbb{E}[\Phi(P)]) = \frac{\partial}{\partial r_0}\bigg(- Te^{-r_0T}\mathbb{E}(\Phi(P)) + e^{-r_0T}\mathbb{E}\Big[\Phi'(P)\frac{\partial P}{\partial r_0}\Big]\bigg) \\
&= \frac{\partial}{\partial r_0}\bigg(- Te^{-r_0T}\mathbb{E}(\Phi(P)) + e^{-r_0T}\mathbb{E}\Big[\Phi(P)H_{VG}\Big(P,\frac{\partial P}{\partial r_0}\Big)\Big]\bigg) \\
&= T^2e^{-r_0T}\mathbb{E}[\Phi(P)] - Te^{-r_0T}\mathbb{E}\Big[\Phi(P)H_{VG}\Big(P,\frac{\partial P}{\partial r_0}\Big)\Big] \\
&\quad -Te^{-r_0T}\mathbb{E}\Big[\Phi(P)H_{VG}\Big(P,\frac{\partial P}{\partial r_0}\Big)\Big] \\
&\quad +e^{-r_0T}\mathbb{E}\Big[\Phi(P)H_{VG}\Big(P,\frac{\partial P}{\partial r_0}H_{VG}\Big(P,\frac{\partial P}{\partial r_0}\Big)\Big)\Big] \\
&= T^2e^{-r_0T}\mathbb{E}[\Phi(P)] - 2Te^{-r_0T}\mathbb{E}\Big[\Phi(P)H_{VG}\Big(P,\frac{\partial P}{\partial r_0}\Big)\Big] \\
&\quad +e^{-r_0T}\mathbb{E}\Big[\Phi(P)H_{VG}\Big(P,\frac{\partial P}{\partial r_0}H_{VG}\Big(P,\frac{\partial P}{\partial r_0}\Big)\Big)\Big]
\end{aligned}$$

where $H_{VG}\Big(P,\frac{\partial P}{\partial r_0}\Big)$ is given by Theorem 4.2.1.

In addition to Lemmas 4.2.1-4.2.4, we state Lemmas 4.2.9 and 4.2.10, required for Theorem 4.2.2.

Lemma 4.2.9. Let P be the price of a zero-coupon bond driven by a VG process, and let $Q = \frac{\partial P}{\partial r_0}$ and $Q_\Gamma = \frac{\partial P}{\partial r_0}H_{VG}\Big(P,\frac{\partial P}{\partial r_0}\Big)$ where $H_{VG}\Big(P,\frac{\partial P}{\partial r_0}\Big)$ is the Malliavin weight function of the greek '$\triangle^{VG}$'. Then,

$$Q_\Gamma = \frac{1}{a}(e^{-aT} - e^{-at})PH_{VG}\big(P,Q\big) \tag{4.2.20}$$

$$DQ_\Gamma = \Big[- \big(\frac{1}{a}(e^{-aT} - e^{-at})\big)\mathcal{K}H_{VG}(P,Q) - \frac{\big(\frac{1}{a}(e^{-aT} - e^{-at})\big)^2}{\mathcal{K}} + 2\big(\frac{1}{a}(e^{-aT} - e^{-at})\big)^2\mathcal{K}^{-3}$$
$$\times\big[\sigma^2\widetilde{\sigma}^2\sum_{t\le u\le T}(\Delta\sqrt{G(u)})^2\big]^2 + \frac{\big(\frac{1}{a}(e^{-aT} - e^{-at})\big)^2 Z}{\mathcal{K}^2}\big[- \sigma^2\widetilde{\sigma}^2\sum_{t\le u\le T}(\Delta\sqrt{G(u)})^2\big]\Big]P \tag{4.2.21}$$

where $\mathcal{K}$ is given by equation (4.2.17).

Proof. By Theorem 4.2.1, it follows that

$$Q_\Gamma = \frac{\partial P}{\partial r_0}H_{VG}\Big(P,\frac{\partial P}{\partial r_0}\Big) = \frac{1}{a}(e^{-aT} - e^{-at})PH_{VG}\big(P,Q\big)$$

where

$$H_{VG}(P,Q) = \frac{\sigma^2}{a}(e^{-aT} - e^{-at})\big(\widetilde{\sigma}^2 \sum_{t\leq u\leq T}(\Delta\sqrt{G(u)})^2\big)\cdot\mathcal{K}^{-2} - \frac{\frac{1}{a}(e^{-aT} - e^{-at})Z}{\mathcal{K}}$$

and

$$\mathcal{K} = \sigma\widetilde{\sigma}\sum_{t\leq u\leq T}\sum_{0\leq s\leq t}\Delta\sqrt{G(s)}e^{-a(u-s)} + \sigma\widetilde{\sigma}\sum_{t\leq u\leq T}(\Delta\sqrt{G(u)})$$
$$-\sigma^2\widetilde{\sigma}\sum_{t\leq u\leq T}(\theta\Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}$$

as given by equation (4.2.17). The Malliavin derivative of $\mathcal{K}$ is given by

$$D\mathcal{K} = -\sigma^2\widetilde{\sigma}^2\sum_{t\leq u\leq T}(\Delta\sqrt{G(u)})^2$$

and the Malliavin derivative of Q_Γ is given by

$$DQ_\Gamma = H_{VG}(P,Q)\big(\frac{1}{a}(e^{-aT} - e^{-at})\big)\Bigg(- \Bigg[\sigma\widetilde{\sigma}\sum_{t\leq u\leq T}\sum_{0\leq s\leq t}\Delta\sqrt{G(s)}e^{-a(u-s)}$$
$$+\sigma\widetilde{\sigma}\sum_{t\leq u\leq T}(\Delta\sqrt{G(u)}) - \sigma^2\widetilde{\sigma}\sum_{t\leq u\leq T}(\theta\Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Bigg]\Bigg)P$$
$$+\frac{1}{a}(e^{-aT} - e^{-at})PDH_{VG}(P,Q).$$

Also, the Malliavin derivative of the weight function is given by

$$DH_{VG}(P,Q) = -\frac{2\sigma^2}{a}(e^{-aT} - e^{-at})\big(\sum_{t\leq u\leq T}(\widetilde{\sigma}\Delta\sqrt{G(u)})^2\big)\mathcal{K}^{-3}\cdot D\mathcal{K}$$
$$-\Bigg(\mathcal{K}\big(\frac{1}{a}(e^{-aT} - e^{-at})\big) - \big(\frac{1}{a}(e^{-aT} - e^{-at})\big)ZD\mathcal{K}\Bigg)\cdot\mathcal{K}^{-2}$$
$$= -\frac{2}{a}(e^{-aT} - e^{-at})\big(\sum_{t\leq u\leq T}(\widetilde{\sigma}\Delta\sqrt{G(u)})^2\big)\sigma^2\mathcal{K}^{-3}\cdot\big[-\sigma^2\sum_{t\leq u\leq T}(\widetilde{\sigma}\Delta\sqrt{G(u)})^2\big]$$
$$-\frac{\frac{1}{a}(e^{-aT} - e^{-at})}{\mathcal{K}} + \frac{\big(\frac{1}{a}(e^{-aT} - e^{-at})\big)Z}{\mathcal{K}^2}\big[-\sigma^2\sum_{t\leq u\leq T}(\widetilde{\sigma}\Delta\sqrt{G(u)})^2\big]$$

$$= \frac{2}{a}(e^{-aT} - e^{-at})\mathcal{K}^{-3}\big[\sigma^2\widetilde{\sigma}^2\sum_{t\leq u\leq T}(\Delta\sqrt{G(u)})^2\big]^2 - \frac{\frac{1}{a}(e^{-aT} - e^{-at})}{\mathcal{K}}$$
$$+\frac{\big(\frac{1}{a}(e^{-aT} - e^{-at})\big)Z}{\mathcal{K}^2}\big[-\sigma^2\widetilde{\sigma}^2\sum_{t\leq u\leq T}(\Delta\sqrt{G(u)})^2\big].$$

Hence,

$$DQ_\Gamma = D\Big[\frac{\partial P}{\partial r_0}\cdot H_{VG}\Big(P,\frac{\partial P}{\partial r_0}\Big)\Big] = H_{VG}\Big(P,\frac{\partial P}{\partial r_0}\Big)D\Big(\frac{\partial P}{\partial r_0}\Big)+\frac{\partial P}{\partial r_0}D\Big(H_{VG}\Big(P,\frac{\partial P}{\partial r_0}\Big)\Big)$$

$$= -\big(\frac{1}{a}(e^{-aT}-e^{-at})\big)\mathcal{K}PH_{VG}(P,Q)+\frac{1}{a}(e^{-aT}-e^{-at})PDH_{VG}(P,Q)$$

$$= -\big(\frac{1}{a}(e^{-aT}-e^{-at})\big)\mathcal{K}PH_{VG}(P,Q)+2\big(\frac{1}{a}(e^{-aT}-e^{-at})\big)^2\cdot\mathcal{K}^{-3}\big[\sigma^2\sum_{t\le u\le T}(\widetilde{\sigma}\Delta\sqrt{G(u)})^2\big]^2P$$

$$-\frac{\big(\frac{1}{a}(e^{-aT}-e^{-at})\big)^2}{\mathcal{K}}P+\frac{\big(\frac{1}{a}(e^{-aT}-e^{-at})\big)^2Z}{\mathcal{K}^2}\big[-\sigma^2\sum_{t\le u\le T}(\widetilde{\sigma}\Delta\sqrt{G(u)})^2\big]P$$

$$= \Big[-\big(\frac{1}{a}(e^{-aT}-e^{-at})\big)\mathcal{K}H_{VG}(P,Q)+2\big(\frac{1}{a}(e^{-aT}-e^{-at})\big)^2\mathcal{K}^{-3}\big[\sigma^2\widetilde{\sigma}^2\sum_{t\le u\le T}(\Delta\sqrt{G(u)})^2\big]^2$$

$$-\frac{\big(\frac{1}{a}(e^{-aT}-e^{-at})\big)^2}{\mathcal{K}}+\frac{\big(\frac{1}{a}(e^{-aT}-e^{-at})\big)^2Z}{\mathcal{K}^2}\big[-\sigma^2\widetilde{\sigma}^2\sum_{t\le u\le T}(\Delta\sqrt{G(u)})^2\big]\Big]P. \qquad \square$$

Lemma 4.2.10. Let P be the price of a zero-coupon bond driven by the VG process, $Q=\frac{\partial P}{\partial r_0}$ and $Q_\Gamma=\frac{\partial P}{\partial r_0}H_{VG}\Big(P,\frac{\partial P}{\partial r_0}\Big)$. Then,

$$Q_\Gamma\mathcal{M}(P)^{-1}LP = -\big(\frac{1}{a}(e^{-aT}-e^{-at})\big)H_{VG}(P,Q)\Big[\Big(\sigma^2\widetilde{\sigma}^2\sum_{t\le u\le T}(\Delta\sqrt{G(u)})^2\Big)\mathcal{K}^{-2}+1+\frac{Z}{\mathcal{K}}\Big] \tag{4.2.22}$$

where $\mathcal{K}$ is given by equation (4.2.17).

Proof. From equations (4.2.20), (4.2.12) and (4.2.10), it follows that:

$Q_\Gamma\mathcal{M}(P)^{-1}LP$

$$= \frac{1}{a}(e^{-aT}-e^{-at})PH_{VG}\big(P,Q\big)\Big[\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)}+\sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)})$$

$$-\sigma^2\widetilde{\sigma}\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Big]^{-2}P^{-2}$$

$$\times\Big[-\Big[\sigma^2\sum_{t\le u\le T}(\widetilde{\sigma}\Delta\sqrt{G(u)})^2+\Big[\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)}+\sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)})$$

$$-\sigma^2\widetilde{\sigma}\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Big]^2+Z\Big[\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)}$$

$$+\sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)})-\sigma^2\widetilde{\sigma}\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Big]\Big]P$$

$$= -\frac{1}{a}(e^{-aT} - e^{-at})PH_{VG}\big(P,Q\big)\mathcal{K}^{-2}P^{-2}\cdot\Big[\sigma^2\widetilde{\sigma}^2\sum_{t\le u\le T}(\Delta\sqrt{G(u)})^2 + \mathcal{K}^2 + Z\mathcal{K}\Big]P$$

where

$$\mathcal{K} = \sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)} + \sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)})$$
$$-\sigma^2\widetilde{\sigma}\sum_{t\le u\le T}(\theta\Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}$$

as given by equation (4.2.17).

Hence,

$$Q_\Gamma\mathcal{M}(P)^{-1}LP = -\big(\frac{1}{a}(e^{-aT} - e^{-at})\big)H_{VG}(P,Q)\Big[\sigma^2\widetilde{\sigma}^2\sum_{t\le u\le T}(\Delta\sqrt{G(u)})^2\Big]\mathcal{K}^{-2}$$
$$-\frac{1}{a}(e^{-aT} - e^{-at})H_{VG}(P,Q) - H_{VG}(P,Q)\frac{\big(\frac{1}{a}(e^{-aT} - e^{-at})\big)Z}{\mathcal{K}}. \qquad \square$$

Lemma 4.2.11. Let P be the price of a zero-coupon bond driven by a VG process and $Q_\Gamma = \frac{\partial P}{\partial r_0}H_{VG}\Big(P, \frac{\partial P}{\partial r_0}\Big)$. Then,

$$\mathcal{M}(P)^{-1}\langle DP, DQ_\Gamma\rangle = \big(\frac{1}{a}(e^{-aT} - e^{-at})\big)\Big[H_{VG}(P,Q) + \frac{\big(\frac{1}{a}(e^{-aT} - e^{-at})\big)}{\mathcal{K}^2}$$
$$-2\big(\frac{1}{a}(e^{-aT} - e^{-at})\big)\cdot\mathcal{K}^{-4}\big[\sigma^2\widetilde{\sigma}^2\sum_{t\le u\le T}(\Delta\sqrt{G(u)})^2\big]^2$$
$$+\frac{\big(\frac{1}{a}(e^{-aT} - e^{-at})\big)^2 Z}{\mathcal{K}^3}\big[\sigma^2\widetilde{\sigma}^2\sum_{t\le u\le T}(\Delta\sqrt{G(u)})^2\big]\Big]. \qquad (4.2.23)$$

Proof. From equations (4.2.12), (4.2.9) and (4.2.21), it follows that

$\mathcal{M}(P)^{-1}\langle DP, DQ_\Gamma\rangle = \mathcal{M}(P)^{-1}(DP\cdot DQ_\Gamma)$

$$= -\Big[\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)} + \sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)})$$
$$-\sigma^2\widetilde{\sigma}\sum_{t\le u\le T}(\theta\Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Big]P\times\Big[\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)}$$
$$+\sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)}) - \sigma^2\widetilde{\sigma}\sum_{t\le u\le T}(\theta\Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Big]^{-2}P^{-2}$$
$$\cdot\Big[-\big(\frac{1}{a}(e^{-aT} - e^{-at})\big)\mathcal{K}H_{VG}(P,Q) - \frac{\big(\frac{1}{a}(e^{-aT} - e^{-at})\big)^2}{\mathcal{K}} + 2\big(\frac{1}{a}(e^{-aT} - e^{-at})\big)^2$$
$$\cdot\mathcal{K}^{-3}\big[\sigma^2\sum_{t\le u\le T}(\widetilde{\sigma}\Delta\sqrt{G(u)})^2\big]^2 + \frac{\big(\frac{1}{a}(e^{-aT} - e^{-at})\big)^2 Z}{\mathcal{K}^2}\big[-\sigma^2\sum_{t\le u\le T}(\widetilde{\sigma}\Delta\sqrt{G(u)})^2\big]\Big]P$$

where $\mathcal{K}$ satisfies equation (4.2.17).

Hence,

$$\mathcal{M}(P)^{-1}\langle DP, DQ_\Gamma\rangle = -\Big[\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)} + \sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)})$$

$$-\sigma^2\widetilde{\sigma}\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Big]^{-1}\times\Big[-\big(\frac{1}{a}(e^{-aT}-e^{-at})\big)\mathcal{K}H_{VG}(P,Q)$$

$$-\frac{\big(\frac{1}{a}(e^{-aT}-e^{-at})\big)^2}{\mathcal{K}}+2\big(\frac{1}{a}(e^{-aT}-e^{-at})\big)^2\mathcal{K}^{-3}\big[\sigma^2\sum_{t\le u\le T}(\widetilde{\sigma}\Delta\sqrt{G(u)})^2\big]^2$$

$$+\frac{\big(\frac{1}{a}(e^{-aT}-e^{-at})\big)^2 Z}{\mathcal{K}^2}\big[-\sigma^2\sum_{t\le u\le T}(\widetilde{\sigma}\Delta\sqrt{G(u)})^2\big]\Big]P$$

$$=\big(\frac{1}{a}(e^{-aT}-e^{-at})\big)H_{VG}(P,Q)+\frac{\big(\frac{1}{a}(e^{-aT}-e^{-at})\big)^2}{\mathcal{K}^2}-2\big(\frac{1}{a}(e^{-aT}-e^{-at})\big)^2\mathcal{K}^{-4}$$

$$\times\big[\sigma^2\widetilde{\sigma}^2\sum_{t\le u\le T}(\Delta\sqrt{G(u)})^2\big]^2-\frac{\big(\frac{1}{a}(e^{-aT}-e^{-at})\big)^2 Z}{\mathcal{K}^3}\big[-\sigma^2\widetilde{\sigma}^2\sum_{t\le u\le T}(\Delta\sqrt{G(u)})^2\big]\Big]. \qquad \square$$

Lemma 4.2.12. Let P be the price of the zero-coupon bond given by equation (4.2.7) and $Q_\Gamma = \frac{\partial P}{\partial r_0}H_{VG}\left(P, \frac{\partial P}{\partial r_0}\right)$. Then,

$Q_\Gamma\langle DP, D\mathcal{M}(P)^{-1}\rangle$

$$= -2\big(\frac{1}{a}(e^{-aT}-e^{-at})\big)H_{VG}(P,Q)\Big(\mathcal{K}^{-2}\sigma^2\widetilde{\sigma}^2\sum_{t\le u\le T}(\Delta\sqrt{G(u)})^2\big)-1\Big). \quad (4.2.24)$$

Proof. From equations (4.2.20), (4.2.9) and (4.2.13), it follows that

$Q_\Gamma\langle DP, D\mathcal{M}(P)^{-1}\rangle = Q_\Gamma\times(DP\cdot D\mathcal{M}(P)^{-1})$

$$=\frac{1}{a}(e^{-aT}-e^{-at})PH_{VG}(P,Q)\cdot\Big(-\Big[\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)}$$

$$+\sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)})-\sigma^2\widetilde{\sigma}\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Big]P\Big)$$

$$\times\frac{2}{\mathcal{K}^3P^2}\Big[\sigma^2\sum_{t\le u\le T}(\widetilde{\sigma}\Delta\sqrt{G(u)})^2+\mathcal{K}^2\Big]$$

$$=-2\big(\frac{1}{a}(e^{-aT}-e^{-at})\big)H_{VG}(P,Q)\mathcal{K}^{-2}\sigma^2\widetilde{\sigma}^2\sum_{t\le u\le T}(\Delta\sqrt{G(u)})^2$$

$$-2\big(\frac{1}{a}(e^{-aT}-e^{-at})\big)H_{VG}(P,Q). \qquad \square$$

Theorem 4.2.2

Let P be the price of the zero-coupon bond driven by a VG process. Then the greek *gamma* 'Γ^{VG}' is given by

$$\Gamma^{VG} = T^2 e^{-r_0 T}\mathbb{E}[\Phi(P)] - 2Te^{-r_0 T}\mathbb{E}\Big[\Phi(P)H(P,Q)\Big]$$
$$+e^{-r_0 T}\mathbb{E}\Big[\Phi(P)H_{VG}\Big(P, \frac{\partial P}{\partial r_0}H_{VG}\Big(P, \frac{\partial P}{\partial r_0}\Big)\Big)\Big]$$

where $H_{VG}(P,Q)$ is given by Theorem 4.2.1, $Q = \dfrac{\partial P}{\partial r_0}$,

$$H_{VG}\Big(P, \frac{\partial P}{\partial r_0}H_{VG}\Big(P, \frac{\partial P}{\partial r_0}\Big)\Big) = \frac{1}{a}(e^{-aT} - e^{-at})H_{VG}(P,Q)\mathcal{K}^{-2}\Big(\sigma^2\tilde{\sigma}^2\sum_{t\leq u\leq T}(\Delta\sqrt{G(u)})^2\Big)$$
$$-\frac{Z\big(\frac{1}{a}(e^{-aT} - e^{-at})\big)H_{VG}(P,Q)}{\mathcal{K}} + 2\big(\frac{1}{a}(e^{-aT} - e^{-at})\big)^2\mathcal{K}^{-4}\Big(\sigma^2\tilde{\sigma}^2\sum_{t\leq u\leq T}(\Delta\sqrt{G(u)})^2\Big)^2$$
$$-\frac{\big(\frac{1}{a}(e^{-aT} - e^{-at})\big)^2}{\mathcal{K}^2} + \big(\frac{1}{a}(e^{-aT} - e^{-at})\big)^2 Z\mathcal{K}^{-3}\Big(-\sigma^2\tilde{\sigma}^2\sum_{t\leq u\leq T}(\Delta\sqrt{G(u)})^2\Big)$$

and $\mathcal{K}$ is given by equation (4.2.17).

Proof. Substituting equations (4.2.22), (4.2.23) and (4.2.24) into the weight function

$$H_{VG}(P, Q_\Gamma) = Q_\Gamma\mathcal{M}(P)^{-1}LP - \mathcal{M}(P)^{-1}\langle DP, DQ_\Gamma\rangle - Q_\Gamma\langle DP, D\mathcal{M}(P)^{-1}\rangle$$
$$= H_{VG}\Big(P, \frac{\partial P}{\partial r_0}H_{VG}\Big(P, \frac{\partial P}{\partial r_0}\Big)\Big)$$

leads to the desired result. □

4.2.6 Computation of *vega* for VG-driven interest rate derivatives

In this subsection, the greek *vega* for VG-driven interest rate derivative is computed.

$$\mathcal{V}^{VG} = \frac{\partial}{\partial\sigma}e^{-r_0 T}\mathbb{E}[\Phi(P)] = e^{-r_0 T}\mathbb{E}\Big[\Phi'(P)\frac{\partial P}{\partial\sigma}\Big] = e^{-r_0 T}\mathbb{E}\Big[\Phi(P)H_{VG}\Big(P, \frac{\partial P}{\partial\sigma}\Big)\Big].$$

We state Lemmas 4.2.13 - 4.2.16 which are needed for Theorem 4.2.3.

Lemma 4.2.13. Let P be the price of the zero-coupon bond driven by the

VG process and $Q_\sigma = \dfrac{\partial P}{\partial \sigma}$. Suppose that DQ_σ is the Malliavin derivative of Q_σ. Then,

$$Q_\sigma = -\Bigg[\frac{\mathbf{w}}{a}\big[T - t + \frac{1}{a}(e^{-aT} - e^{-at})\big] + \sum_{t\le u\le T}\sum_{0\le s\le t}\big(\theta\Delta G(s)e^{-a(u-s)}$$

$$+\widetilde{\sigma}\Delta\sqrt{G(s)}e^{-a(u-s)}Z\big) + \mathbf{w}[T-t] + \sum_{t\le u\le T}(\theta\Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z)$$

$$-\sigma\sum_{t\le u\le T}(\theta\Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z)^2\Bigg]P. \tag{4.2.25}$$

Also,

$$DQ_\sigma = -\Bigg[\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}(\Delta\sqrt{G(s)}e^{-a(u-s)}) + \widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)})$$

$$-2\sigma\widetilde{\sigma}\sum_{t\le u\le T}(\theta\Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Bigg]P$$

$$+\Bigg[\frac{\mathbf{w}}{a}\big[T - t + \frac{1}{a}(e^{-aT} - e^{-at})\big] + \sum_{t\le u\le T}\sum_{0\le s\le t}\big(\theta\Delta G(s)e^{-a(u-s)} + \widetilde{\sigma}\Delta\sqrt{G(s)}e^{-a(u-s)}Z\big)$$

$$+\mathbf{w}[T-t] + \sum_{t\le u\le T}(\theta\Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z)$$

$$-\sigma\sum_{t\le u\le T}(\theta\Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z)^2\Bigg]\mathcal{K}P \tag{4.2.26}$$

where $\mathcal{K}$ is given by equation (4.2.17).

Proof. Since $Q_\sigma = \dfrac{\partial P}{\partial \sigma}$, it follows from equation (4.2.7) that

$$\frac{\partial P}{\partial \sigma} = -\Bigg[\frac{\mathbf{w}}{a}\big[T - t + \frac{1}{a}(e^{-aT} - e^{-at})\big]$$

$$+\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\theta\Delta G(s)e^{-a(u-s)} + \widetilde{\sigma}\Delta\sqrt{G(s)}e^{-a(u-s)}Z\big) + \mathbf{w}[T-t]$$

$$+\sum_{t\le u\le T}(\theta\Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z) - \frac{2\sigma}{2}\Bigg(\sum_{t\le u\le T}(\theta\Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z)^2\Bigg)\Bigg]P = Q_\sigma$$

which is equation (4.2.25).

Furthermore, the Malliavin derivative

$$DQ_\sigma = P \times \Bigg(-\Bigg[\sum_{t\le u\le T}\sum_{0\le s\le t}(\widetilde{\sigma}\Delta\sqrt{G(s)}e^{-a(u-s)}) + \sum_{t\le u\le T}(\widetilde{\sigma}\Delta\sqrt{G(u)})$$

$$-2\sigma\Bigg(\sum_{t\le u\le T}(\theta\Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z)\widetilde{\sigma}\Delta\sqrt{G(u)}\Bigg)\Bigg]\Bigg) + \Bigg(-\Bigg[\frac{\mathbf{w}}{a}\big[T - t + \frac{1}{a}(e^{-aT} - e^{-at})\big]$$

$$+\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\theta\Delta G(s)e^{-a(u-s)}+\widetilde{\sigma}\Delta\sqrt{G(s)}e^{-a(u-s)}Z\big)+\mathbf{w}[T-t]$$

$$+\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)-\sigma\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)^2\Big]\Big)DP$$

$$=-\Big[\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}(\Delta\sqrt{G(s)}e^{-a(u-s)})+\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)})$$

$$-2\sigma\widetilde{\sigma}\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Big]P+-\Big[\frac{\mathbf{w}}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big]$$

$$+\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\theta\Delta G(s)e^{-a(u-s)}+\widetilde{\sigma}\Delta\sqrt{G(s)}e^{-a(u-s)}Z\big)\Big]+\mathbf{w}[T-t]$$

$$+\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)-\sigma\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)^2\Big](-\mathcal{K}P)$$

where $\mathcal{K}$ is given by equation (4.2.17). □

Lemma 4.2.14. Let P be the price of the zero-coupon bond given by equation (4.2.7). Then,

$$Q_\sigma\mathcal{M}(P)^{-1}LP=\Lambda\Big[\mathcal{K}^{-2}\sigma^2\widetilde{\sigma}^2\sum_{t\le u\le T}(\Delta\sqrt{G(u)})^2+1+\frac{Z}{\mathcal{K}}\Big]\qquad(4.2.27)$$

where

$$\Lambda=\frac{\mathbf{w}}{a}\big[T-t+a^{-1}(e^{-aT}-e^{-at})\big]+\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\theta\Delta G(s)e^{-a(u-s)}+\widetilde{\sigma}\Delta\sqrt{G(s)}e^{-a(u-s)}Z\big)$$

$$+\mathbf{w}[T-t]+\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)-\sigma\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)^2.\qquad(4.2.28)$$

and $\mathcal{K}$ is given by equation (4.2.17).

Proof. Let Λ be as given above; from equations (4.2.25), (4.2.12) and (4.2.10), it follows that

$$\begin{aligned}Q_\sigma\mathcal{M}(P)^{-1}LP&\\ &= -\Lambda P\cdot\mathcal{K}^{-2}P^{-2}\cdot\Big(-\Big[\sigma^2\sum_{t\le u\le T}(\widetilde{\sigma}\Delta\sqrt{G(u)})^2+\mathcal{K}^2+Z\mathcal{K}\Big]\Big)P\\ &= \Lambda\mathcal{K}^{-2}\sigma^2\widetilde{\sigma}^2\big(\sum_{t\le u\le T}(\Delta\sqrt{G(u)})^2\big)+\Lambda+\frac{\Lambda Z}{\mathcal{K}}.\qquad\square\end{aligned}$$

Lemma 4.2.15. Let P be the price of the zero-coupon bond given by equation (4.2.7). Then,

$$\mathcal{M}(P)^{-1}\langle DP,DQ_\sigma\rangle=-\Lambda+\Big[\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}(\Delta\sqrt{G(s)}e^{-a(u-s)})+\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)})$$

$$-2\sigma\widetilde{\sigma}\bigg(\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\bigg)\bigg]\mathcal{K}^{-1} \tag{4.2.29}$$

where $\mathcal{K}$ and Λ are given by equations (4.2.17) and (4.2.28), respectively.

Proof. From equations (4.2.12), (4.2.9) and (4.2.26), it follows that

$$\mathcal{M}(P)^{-1}\langle DP, DQ_\sigma\rangle = -(\mathcal{K}P)^{-1}\times\Bigg(-\bigg[\sum_{t\le u\le T}\sum_{0\le s\le t}(\widetilde{\sigma}\Delta\sqrt{G(s)}e^{-a(u-s)})$$

$$+\sum_{t\le u\le T}(\widetilde{\sigma}\Delta\sqrt{G(u)})-2\sigma\bigg(\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\widetilde{\sigma}\Delta\sqrt{G(u)}\bigg)\bigg]P+\Lambda\mathcal{K}P\Bigg)$$

$$=\mathcal{K}^{-1}\bigg[\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}(\Delta\sqrt{G(s)}e^{-a(u-s)})+\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)})$$

$$-2\sigma\widetilde{\sigma}\bigg(\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\bigg)\bigg]-\Lambda. \qquad \square$$

Lemma 4.2.16. Let P be the price of the zero-coupon bond given by equation (4.2.7). Then,

$$Q_\sigma\langle DP, D\mathcal{M}(P)^{-1}\rangle = 2\Lambda\bigg[\mathcal{K}^{-2}\sigma^2\widetilde{\sigma}^2\sum_{t\le u\le T}(\Delta\sqrt{G(u)})^2+1\bigg] \tag{4.2.30}$$

where $\mathcal{K}$ and Λ are given by equations (4.2.17) and (4.2.28) respectively.

Proof. Let $\mathcal{K}$ and Λ be given by equations (4.2.17) and (4.2.28), respectively. Then, from equations (4.2.25), (4.2.9) and (4.2.13), it follows that

$$Q_\sigma\langle DP, D\mathcal{M}(P)^{-1}\rangle = Q_\sigma(DP\cdot D\mathcal{M}(P)^{-1})$$

$$= -\Lambda P\cdot(-\mathcal{K}P)\cdot\frac{2}{\mathcal{K}^3P^2}\bigg[\sigma^2\sum_{t\le u\le T}(\widetilde{\sigma}\Delta\sqrt{G(u)})^2+\mathcal{K}^2\bigg]$$

$$=\frac{2\Lambda}{\mathcal{K}^2}\big[\sigma^2\widetilde{\sigma}^2\sum_{t\le u\le T}(\Delta\sqrt{G(u)})^2\big]+2\Lambda. \qquad \square$$

Theorem 4.2.3

Let P be the price of the zero-coupon bond driven by the VG process. Then, the greek '*vega*' is given by

$$\mathcal{V}^{VG} \quad = \quad e^{-r_0T}\mathbb{E}\bigg[\Phi(P)H_{VG}(P,Q_\sigma)\bigg],$$

the weight function is given by

$$H_{VG}(P,Q_\sigma)=\frac{\Lambda Z}{\mathcal{K}}-\bigg[\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}(\Delta\sqrt{G(s)}e^{-a(u-s)})+\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)})$$

$$-2\sigma\widetilde{\sigma}\Big(\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Big)\Big]\mathcal{K}^{-1}$$

$$-\frac{\Lambda}{\mathcal{K}^2}\Big[\sigma^2\widetilde{\sigma}^2\sum_{t\le u\le T}(\Delta\sqrt{G(u)})^2\Big]$$

where $\mathcal{K}$ and Λ are given by equations (4.2.17) and (4.2.28) respectively.

Proof. From equation (4.2.6), it follows that

$$\mathcal{V}_{VG}=\frac{\partial\mathbb{V}}{\partial\sigma}=e^{-r_0T}\mathbb{E}\Big[\Phi(P)H_{VG}(P,Q_\sigma)\Big].$$

Also, from equation (4.2.8),

$$H_{VG}(P,Q_\sigma)=Q_\sigma\mathcal{M}(P)^{-1}LP-\mathcal{M}(P)^{-1}\langle DP,DQ_\sigma\rangle-Q_\sigma\langle DP,D\mathcal{M}(P)^{-1}\rangle.$$

Substituting equations (4.2.27), (4.2.29) and (4.2.30) into the above equation yields the desired weight function. □

4.2.7 Computation of *drift* for VG-driven interest rate derivatives

In this subsection, we compute the greek '*drift* $\mathcal{D}$' for a VG-driven interest rate derivative.

$$\mathcal{D}=\frac{\partial}{\partial\theta}e^{-r_0T}\mathbb{E}[\Phi(P)]=e^{-r_0T}\mathbb{E}\Big[\Phi'(P)\frac{\partial P}{\partial\theta}\Big]=e^{-r_0T}\mathbb{E}\Big[\Phi(P)H_{VG}\Big(P,\frac{\partial P}{\partial\theta}\Big)\Big].$$

Recall that by equation (4.2.7),

$$P(t,T)=\exp\Big(-\Big(\Big[-\frac{r_0}{a}\big(e^{-aT}-e^{-at}\big)+b\big(T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big)$$

$$+\frac{\sigma\mathbf{w}}{a}\Big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\Big]+\sigma\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\theta\Delta G(s)e^{-a(u-s)}+\widetilde{\sigma}\Delta\sqrt{G(s)}e^{-a(u-s)}Z\big)\Big]$$

$$+\mathbf{w}\sigma[T-t]+\sigma\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)$$

$$-\frac{\sigma^2}{2}\Big(\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)^2\Big)\Big)\Big)$$

where $\mathbf{w}=\frac{1}{\kappa}\ln(1-\theta\kappa-\frac{1}{2}\widetilde{\sigma}^2\kappa)$.

Hence,

$$\frac{\partial\mathbf{w}}{\partial\theta}=-\frac{1}{1-\theta\kappa-\frac{\widetilde{\sigma}^2}{2}\kappa}.$$

Lemma 4.2.17. Let P be the price of the zero-coupon bond driven by VG process and $Q_\theta = \dfrac{\partial P}{\partial \theta}$. Then,

$$Q_\theta = -\Bigg[\frac{\sigma}{a}\big[T-t+a^{-1}(e^{-aT}-e^{-at})\big]\big(\frac{-1}{1-\theta\kappa-\frac{\widetilde{\sigma}^2}{2}\kappa}\big)+\sigma\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta G(s)e^{-a(u-s)}$$

$$+\mathbf{w}\sigma[T-t]\big(\frac{-1}{1-\theta\kappa-\frac{\widetilde{\sigma}^2}{2}\kappa}\big)+\sigma\sum_{t\le u\le T}\Delta G(u)-\sigma^2\Bigg(\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta G(u)\Bigg)\Bigg]P. \quad (4.2.31)$$

Furthermore,

$$DQ_\theta = \Bigg[\sigma^2\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)})\Delta G(u)+\Bigg[\frac{\sigma}{a}\big[T-t+a^{-1}(e^{-aT}-e^{-at})\big]\big(-\frac{1}{1-\theta\kappa-\frac{\widetilde{\sigma}^2}{2}\kappa}\big)$$

$$+\sigma\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta G(s)e^{-a(u-s)}+\mathbf{w}\sigma[T-t]\big(-\frac{1}{1-\theta\kappa-\frac{\widetilde{\sigma}^2}{2}\kappa}\big)+\sigma\sum_{t\le u\le T}\Delta G(u)$$

$$-\sigma^2\Bigg(\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta G(u)\Bigg)\Bigg]\mathcal{K}\Bigg]P \qquad (4.2.32).$$

Proof. From equation (4.2.7), it follows that

$$Q_\theta = \frac{\partial P}{\partial \theta} = -\Bigg[\frac{\sigma}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big]\frac{\partial \mathbf{w}}{\partial \theta}$$

$$+\sigma\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta G(s)e^{-a(u-s)}+\mathbf{w}\sigma[T-t]\frac{\partial \mathbf{w}}{\partial \theta}+\sigma\sum_{t\le u\le T}\Delta G(u)$$

$$-\frac{\sigma^2}{2}\Bigg(2\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta G(u)\Bigg)\Bigg]P$$

which is equation (4.2.29).

Hence, the Malliavin derivative

$$DQ_\theta = -\Bigg[-\sigma^2\sum_{t\le u\le T}(\widetilde{\sigma}\Delta\sqrt{G(u)})\Delta G(u)\Bigg]P$$

$$+-\Bigg[\frac{\sigma}{a}\big[T-t+a^{-1}(e^{-aT}-e^{-at})\big]\big(-\frac{1}{1-\theta\kappa-\frac{\widetilde{\sigma}^2}{2}\kappa}\big)$$

$$+\sigma\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta G(s)e^{-a(u-s)}+\mathbf{w}\sigma[T-t]\big(-\frac{1}{1-\theta\kappa-\frac{\widetilde{\sigma}^2}{2}\kappa}\big)+\sigma\sum_{t\le u\le T}\Delta G(u)$$

$$-\frac{\sigma^2}{2}\Bigg(2\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta G(u)\Bigg)\Bigg]\cdot DP$$

$$=\Bigg[\sigma^2\big(\sum_{t\le u\le T}(\widetilde{\sigma}\Delta\sqrt{G(u)})\Delta G(u)\big)P+-\Bigg[\frac{\sigma}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big]\big(-\frac{1}{1-\theta\kappa-\frac{\widetilde{\sigma}^2}{2}\kappa}\big)$$

$$+\sigma\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta G(s)e^{-a(u-s)}+\mathbf{w}\sigma[T-t]\big(-\frac{1}{1-\theta\kappa-\frac{\widetilde{\sigma}^2}{2}\kappa}\big)+\sigma\sum_{t\le u\le T}\Delta G(u)$$

$$-\frac{\sigma^2}{2}\bigg(2\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta G(u)\bigg)\bigg]\times-\bigg[\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)}$$

$$+\sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)})-\sigma^2\bigg(\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\widetilde{\sigma}\Delta\sqrt{G(u)}\bigg)\bigg]P$$

$$=\bigg[\sigma^2\widetilde{\sigma}\big(\sum_{t\le u\le T}(\Delta\sqrt{G(u)})\Delta G(u)\big)P+\bigg[\frac{\sigma}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big]\big(-\frac{1}{1-\theta\kappa-\frac{\widetilde{\sigma}^2}{2}\kappa}\big)$$

$$+\sigma\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta G(s)e^{-a(u-s)}+\mathbf{w}\sigma[T-t]\big(-\frac{1}{1-\theta\kappa-\frac{\widetilde{\sigma}^2}{2}\kappa}\big)$$

$$+\sigma\sum_{t\le u\le T}\Delta G(u)-\sigma^2\bigg(\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta G(u)\bigg)\bigg]$$

$$\cdot\bigg[\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)}+\sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)})$$

$$-\sigma^2\widetilde{\sigma}\bigg(\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\bigg)\bigg]P$$

$$=\bigg[\sigma^2\widetilde{\sigma}\big(\sum_{t\le u\le T}(\Delta\sqrt{G(u)})\Delta G(u)\big)P+\bigg[\frac{\sigma}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big]\big(-\frac{1}{1-\theta\kappa-\frac{\widetilde{\sigma}^2}{2}\kappa}\big)$$

$$+\sigma\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta G(s)e^{-a(u-s)}+\mathbf{w}\sigma[T-t]\big(-\frac{1}{1-\theta\kappa-\frac{\widetilde{\sigma}^2}{2}\kappa}\big)+\sigma\sum_{t\le u\le T}\Delta G(u)$$

$$-\sigma^2\bigg(\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta G(u)\bigg)\bigg]\mathcal{K}P$$

which is equation (4.2.32). □

Lemma 4.2.18. Let P be the price of the zero-coupon bond driven by the VG process and $Q_\theta=\dfrac{\partial P}{\partial\theta}$. Then,

$$Q_\theta\mathcal{M}(P)^{-1}LP=\mathcal{L}\sigma^2\widetilde{\sigma}^2\big(\sum_{t\le u\le T}(\Delta\sqrt{G(u)})^2\big)\mathcal{K}^{-2}+\mathcal{L}+\frac{Z\mathcal{L}}{\mathcal{K}}\qquad(4.2.33)$$

where

$$\mathcal{L}=\frac{\sigma}{a}\big[T-t+a^{-1}(e^{-aT}-e^{-at})\big]\big(\frac{-1}{1-\theta\kappa-\frac{\widetilde{\sigma}^2}{2}\kappa}\big)+\sigma\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta G(s)e^{-a(u-s)}$$

$$+\mathbf{w}\sigma[T-t]\big(\frac{-1}{1-\theta\kappa-\frac{\widetilde{\sigma}^2}{2}\kappa}\big)+\sigma\sum_{t\le u\le T}\Delta G(u)$$

$$-\sigma^2\bigg(\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta G(u)\bigg). \qquad (4.2.34)$$

Proof. From equations (4.2.31), (4.2.12) and (4.2.10), it follows that

$$Q_\theta\mathcal{M}(P)^{-1}LP=-\bigg[\frac{\sigma}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big]\big(\frac{-1}{1-\theta\kappa-\frac{\widetilde{\sigma}^2}{2}\kappa}\big)$$

$$+\sigma\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta G(s)e^{-a(u-s)}+\mathbf{w}\sigma[T-t]\big(\frac{-1}{1-\theta\kappa-\frac{\widetilde{\sigma}^2}{2}\kappa}\big)$$

$$+\sigma\sum_{t\le u\le T}\Delta G(u)-\sigma^2\bigg(\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta G(u)\bigg)\bigg]P$$

$$\cdot\bigg[\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)}+\sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)})$$

$$-\sigma^2\widetilde{\sigma}\bigg(\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\bigg)\bigg]^{-2}P^{-2}$$

$$\times-\bigg[\sigma^2\widetilde{\sigma}^2\sum_{t\le u\le T}(\Delta\sqrt{G(u)})^2+\bigg[\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)}+\sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)})$$

$$-\sigma^2\widetilde{\sigma}\bigg(\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\bigg)\bigg]^2+Z\bigg[\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)}$$

$$+\sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)})-\sigma^2\widetilde{\sigma}\bigg(\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\bigg)\bigg]\bigg]$$

$$=\bigg[\frac{\sigma}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big]\big(\frac{-1}{1-\theta\kappa-\frac{\widetilde{\sigma}^2}{2}\kappa}\big)+\sigma\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta G(s)e^{-a(u-s)}$$

$$+\mathbf{w}\sigma[T-t]\big(\frac{-1}{1-\theta\kappa-\frac{\widetilde{\sigma}^2}{2}\kappa}\big)+\sigma\sum_{t\le u\le T}\Delta G(u)-\sigma^2\bigg(\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta G(u)\bigg)\bigg]$$

$$\times\sigma^2\widetilde{\sigma}^2\big(\sum_{t\le u\le T}(\Delta\sqrt{G(u)})^2\big)\mathcal{K}^{-2}+\bigg[\frac{\sigma}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big]\big(\frac{-1}{1-\theta\kappa-\frac{\widetilde{\sigma}^2}{2}\kappa}\big)$$

$$+\sigma\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta G(s)e^{-a(u-s)}+\mathbf{w}\sigma[T-t]\big(\frac{-1}{1-\theta\kappa-\frac{\widetilde{\sigma}^2}{2}\kappa}\big)+\sigma\sum_{t\le u\le T}\Delta G(u)$$

$$-\sigma^2\bigg(\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta G(u)\bigg)\bigg]$$

$$+Z\bigg[\frac{\sigma}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big]\big(\frac{-1}{1-\theta\kappa-\frac{\widetilde{\sigma}^2}{2}\kappa}\big)$$

$$+\sigma\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta G(s)e^{-a(u-s)}+\mathbf{w}\sigma[T-t]\big(\frac{-1}{1-\theta\kappa-\frac{\widetilde{\sigma}^2}{2}\kappa}\big)+\sigma\sum_{t\le u\le T}\Delta G(u)$$

$$-\sigma^2\Big(\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta G(u)\Big)\Big]\mathcal{K}^{-1}$$

where $\mathcal{K}$ is given by equation (4.2.17). $\square$

Lemma 4.2.19. Let P be the price of the zero-coupon bond driven by VG process, then

$$\mathcal{M}(P)^{-1}\langle DP,DQ_\theta\rangle=-\sigma^2\widetilde{\sigma}\big(\sum_{t\le u\le T}(\Delta\sqrt{G(u)})\Delta G(u)\big)\mathcal{K}^{-1}-\mathcal{L}\qquad(4.2.35)$$

where $\mathcal{K}$ and $\mathcal{L}$ are given by equations (4.2.17) and (4.2.34), respectively.

Proof. From equations (4.2.12), (4.2.9) and (4.2.32), it follows that

$$M^{-1}(P)\langle DP,DQ_\theta\rangle=-\Big[\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)}+\sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)})$$

$$-\sigma^2\Big(\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\widetilde{\sigma}\Delta\sqrt{G(u)}\Big)\Big]^{-1}P^{-1}$$

$$\times\Big[\sigma^2\sum_{t\le u\le T}(\widetilde{\sigma}\Delta\sqrt{G(u)})\Delta G(u)+\Big[\frac{\sigma}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big]\big(\frac{-1}{1-\theta\kappa-\frac{\widetilde{\sigma}^2}{2}\kappa}\big)$$

$$+\sigma\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta G(s)e^{-a(u-s)}+\mathbf{w}\sigma[T-t]\big(\frac{-1}{1-\theta\kappa-\frac{\widetilde{\sigma}^2}{2}\kappa}\big)$$

$$+\sigma\sum_{t\le u\le T}\Delta G(u)-\sigma^2\Big(\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta G(u)\Big)\Big]$$

$$\times\Big[\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)}+\sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)})$$

$$-\sigma^2\Big(\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\widetilde{\sigma}\Delta\sqrt{G(u)}\Big)\Big]\Big]P$$

$$=-\sigma^2\widetilde{\sigma}\big(\sum_{t\le u\le T}(\Delta\sqrt{G(u)}\Delta G(u)\big)\mathcal{K}^{-1}-\Big[\frac{\sigma}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big]\big(\frac{-1}{1-\theta\kappa-\frac{\widetilde{\sigma}^2}{2}\kappa}\big)$$

$$+\sigma\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta G(s)e^{-a(u-s)}+\mathbf{w}\sigma[T-t]\big(\frac{-1}{1-\theta\kappa-\frac{\widetilde{\sigma}^2}{2}\kappa}\big)+\sigma\sum_{t\le u\le T}\Delta G(u)$$

$$-\sigma^2\Big(\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta G(u)\Big)\Big]$$

which gives equation (4.2.33). $\square$

Lemma 4.2.20. Let P be the price of a zero-coupon bond driven by a VG process. Then

$$Q_\theta\langle DP, D\mathcal{M}(P)^{-1}\rangle = 2\mathcal{L}\Big[\sigma^2\widetilde{\sigma}^2\big(\sum_{t\le u\le T}(\Delta\sqrt{G(u)})^2\big)\cdot\mathcal{K}^{-2}+1\Big] \qquad (4.2.36)$$

where $\mathcal{K}$ and $\mathcal{L}$ are given by equations (4.2.15) and (4.2.32) respectively.

Proof. By equations (4.2.31), (4.2.9) and (4.2.13), we have

$$Q_\theta\langle DP, D\mathcal{M}(P)^{-1}\rangle = -\Big[\frac{\sigma}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big]\big(\frac{-1}{1-\theta\kappa-\frac{\widetilde{\sigma}^2}{2}\kappa}\big)$$

$$+\sigma\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta G(s)e^{-a(u-s)}+\mathbf{w}\sigma[T-t]\big(\frac{-1}{1-\theta\kappa-\frac{\widetilde{\sigma}^2}{2}\kappa}\big)+\sigma\sum_{t\le u\le T}\Delta G(u)$$

$$-\sigma^2\Big(\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta G(u)\Big)\Big]P\times-\Big[\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)}$$

$$+\sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)})-\sigma^2\widetilde{\sigma}\Big(\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Big)\Big]P$$

$$\cdot 2\Big[\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)}+\sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)})$$

$$-\sigma^2\widetilde{\sigma}\Big(\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Big)\Big]^{-3}P^{-2}$$

$$\times\sigma^2\widetilde{\sigma}^2\big(\sum_{t\le u\le T}(\Delta\sqrt{G(u)})^2\big)+\Big[\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)}$$

$$+\sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)})-\sigma^2\widetilde{\sigma}\Big(\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Big)\Big]^2$$

$$= 2\mathcal{L}\sigma^2\widetilde{\sigma}^2\big(\sum_{t\le u\le T}(\Delta\sqrt{G(u)})^2\big)\times\Big[\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)}$$

$$+\sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)})-\sigma^2\widetilde{\sigma}\Big(\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Big)\Big]^{-2}+2\mathcal{L}. \qquad \square$$

Theorem 4.2.4

Let P be the price of the zero-coupon bond driven by VG process and $Q_\theta = \dfrac{\partial P}{\partial\theta}$. Then, the sensitivity *drift* is given by

$$\mathcal{D} = e^{-r_0T}\mathbb{E}\Big[\Phi(P)H_{VG}\Big(P,\frac{\partial P}{\partial\theta}\Big)\Big].$$

where

$$H_{VG}(P,Q_\theta) \;=\; \frac{\mathcal{L}Z}{\mathcal{K}} - \mathcal{L}\sigma^2\widetilde{\sigma}^2\Big(\sum_{t\le u\le T}(\Delta\sqrt{G(u)})^2\Big)\mathcal{K}^{-2} + \sigma^2\widetilde{\sigma}\Big(\sum_{t\le u\le T}(\Delta\sqrt{G(u)})\Delta G(u)\Big)\mathcal{K}^{-1},$$

$\mathcal{K}$ and $\mathcal{L}$ are given by equations (4.2.17) and (4.2.34), respectively.

Proof. From equation (4.2.6),

$$\frac{\partial\mathbb{V}}{\partial\theta} = \mathcal{D} = \frac{\partial}{\partial\theta}e^{-r_0T}\mathbb{E}[\Phi(P)] = e^{-r_0T}\mathbb{E}\Big[\Phi(P)H_{VG}\Big(P,\frac{\partial P}{\partial\theta}\Big)\Big].$$

Also, by substituting equations (4.2.33), (4.2.35) and (4.2.36) into

$$H_{VG}(P,Q_\theta) = Q_\theta\mathcal{M}(P)^{-1}LP - \mathcal{M}(P)^{-1}\langle DP, DQ_\theta\rangle - Q_\theta\langle DP, D\mathcal{M}(P)^{-1}\rangle,$$

the Malliavin weight is obtained. □

4.2.8 Computation of $vega_2$ for VG-driven interest rate derivatives

In this subsection, we compute $vega_2$ for VG-driven interest rate derivative. Recall that the price of the zero-coupon bond driven by VG process is given by equation (4.2.7) as

$$P(t,T) = \exp\Bigg(-\Bigg(\Big[-\frac{r_0}{a}\big(e^{-aT}-e^{-at}\big) + b\big(T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big)$$

$$+\frac{\sigma\mathbf{w}}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big]+\sigma\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\theta\Delta G(s)e^{-a(u-s)}+\widetilde{\sigma}\Delta\sqrt{G(s)}e^{-a(u-s)}Z\big)\Big]$$

$$+\mathbf{w}\sigma[T-t]+\sigma\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)-\frac{\sigma^2}{2}\Big(\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)^2\Big)\Bigg)\Bigg)$$

where

$$\mathbf{w} = \frac{1}{\kappa}\ln(1-\theta\kappa-\frac{1}{2}\widetilde{\sigma}^2\kappa)$$

which implies that

$$\begin{aligned}\tfrac{\partial\mathbf{w}}{\partial\kappa} &= \frac{(-\theta-\frac{1}{2}\widetilde{\sigma}^2)}{\kappa(1-\theta\kappa-\frac{1}{2}\widetilde{\sigma}^2\kappa)} - \frac{\ln(1-\theta\kappa-\frac{1}{2}\widetilde{\sigma}^2\kappa)}{\kappa^2}\\ &= \frac{(-\theta-\frac{1}{2}\widetilde{\sigma}^2)}{\kappa(1-\theta\kappa-\frac{1}{2}\widetilde{\sigma}^2\kappa)} - \frac{\mathbf{w}}{\kappa} = \frac{(-\theta-\frac{1}{2}\widetilde{\sigma}^2)}{\kappa e^{\kappa\mathbf{w}}} - \frac{\mathbf{w}}{\kappa}.\end{aligned}$$

Lemma 4.2.21. Let P be the price of the zero-coupon bond driven by VG process and let $Q_\kappa = \dfrac{\partial P}{\partial \kappa}$. Then,

$$Q_\kappa = -\left(\frac{-(\theta + \frac{1}{2}\widetilde{\sigma}^2)}{\kappa(1-\theta\kappa - \frac{1}{2}\widetilde{\sigma}^2\kappa)} - \frac{\mathbf{w}}{\kappa}\right)\left[\frac{\sigma}{a}\big(T - t + a^{-1}(e^{-aT} - e^{-at})\big) + \sigma[T-t]\right]P \tag{4.2.37}$$

and

$$DQ_\kappa = \left(\frac{-(\theta + \frac{1}{2}\widetilde{\sigma}^2)}{\kappa(1-\theta\kappa - \frac{1}{2}\widetilde{\sigma}^2\kappa)} - \frac{\mathbf{w}}{\kappa}\right)\left[\frac{\sigma}{a}\big(T - t + a^{-1}(e^{-aT} - e^{-at})\big) + \sigma[T-t]\right]\mathcal{K}P \tag{4.2.38}$$

where $\mathcal{K}$ is given by equation (4.2.17).

Proof. From equation (4.2.7), it follows that

$$\begin{aligned} Q_\kappa &= \frac{\partial P}{\partial \kappa} = -\left[\frac{\sigma}{a}\big(T - t + a^{-1}(e^{-aT} - e^{-at})\big)\frac{\partial \mathbf{w}}{\partial \kappa} + \sigma[T-t]\frac{\partial \mathbf{w}}{\partial \kappa}\right]P \\ &= -\left[\frac{\sigma}{a}\big(T - t + a^{-1}(e^{-aT} - e^{-at})\big)\left(\frac{(-\theta - \frac{1}{2}\widetilde{\sigma}^2)}{\kappa(1-\theta\kappa - \frac{1}{2}\widetilde{\sigma}^2\kappa)} - \frac{\mathbf{w}}{\kappa}\right)\right. \\ &\qquad \left. +\sigma[T-t]\left(\frac{(-\theta - \frac{1}{2}\widetilde{\sigma}^2)}{\kappa(1-\theta\kappa - \frac{1}{2}\widetilde{\sigma}^2\kappa)} - \frac{\mathbf{w}}{\kappa}\right)\right]P. \end{aligned}$$

Thus, the Malliavin derivative of Q_κ is

$$\begin{aligned} DQ_\kappa &= -\left(\frac{-(\theta + \frac{1}{2}\widetilde{\sigma}^2)}{\kappa(1-\theta\kappa - \frac{1}{2}\widetilde{\sigma}^2\kappa)} - \frac{\mathbf{w}}{\kappa}\right)\left[\frac{\sigma}{a}\big(T - t + a^{-1}(e^{-aT} - e^{-at})\big) + \sigma[T-t]\right]DP \\ &= -\left(\frac{-(\theta + \frac{1}{2}\widetilde{\sigma}^2)}{\kappa(1-\theta\kappa - \frac{1}{2}\widetilde{\sigma}^2\kappa)} - \frac{\mathbf{w}}{\kappa}\right)\left[\frac{\sigma}{a}\big(T - t + a^{-1}(e^{-aT} - e^{-at})\big) + \sigma[T-t]\right] \\ &\quad \times - \left[\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)} + \sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)})\right. \\ &\quad \left. -\sigma^2\left(\sum_{t\le u\le T}(\theta\Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z)\widetilde{\sigma}\Delta\sqrt{G(u)}\right)\right]P \\ &= \left(\frac{-(\theta + \frac{1}{2}\widetilde{\sigma}^2)}{\kappa(1-\theta\kappa - \frac{1}{2}\widetilde{\sigma}^2\kappa)} - \frac{\mathbf{w}}{\kappa}\right)\left[\frac{\sigma}{a}\big(T - t + a^{-1}(e^{-aT} - e^{-at})\big) + \sigma[T-t]\right]\mathcal{K}P. \qquad \square \end{aligned}$$

Lemma 4.2.22. Let P be the price of the zero-coupon bond driven by VG process, then the following results hold:

1. $Q_\kappa \mathcal{M}(P)^{-1}LP$

$$= \left(\frac{-(\theta + \frac{1}{2}\widetilde{\sigma}^2)}{\kappa(1-\theta\kappa - \frac{1}{2}\widetilde{\sigma}^2\kappa)} - \frac{\mathbf{w}}{\kappa}\right)\left[\frac{\sigma}{a}\big(T - t + a^{-1}(e^{-aT} - e^{-at})\big) + \sigma[T-t]\right]$$

$$\times\Big[\mathcal{K}^{-2}\sigma^2\widetilde{\sigma}^2\big(\sum_{t\le u\le T}(\Delta\sqrt{G(u)})^2\big)+1+\frac{Z}{\mathcal{K}}\Big]. \tag{4.2.39}$$

2. $\mathcal{M}(P)^{-1}\langle DP, DQ_\kappa\rangle$

$$=-\Big(\frac{-(\theta+\frac{1}{2}\widetilde{\sigma}^2)}{\kappa(1-\theta\kappa-\frac{1}{2}\widetilde{\sigma}^2\kappa)}-\frac{\mathbf{w}}{\kappa}\Big)\Big[\frac{\sigma}{a}\big(T-t+a^{-1}(e^{-aT}-e^{-at})\big)+\sigma[T-t]\Big]. \tag{4.2.40}$$

3. $Q_\kappa\langle DP, D\mathcal{M}(P)^{-1}\rangle$

$$=2\Big(\frac{-(\theta+\frac{1}{2}\widetilde{\sigma}^2)}{\kappa(1-\theta\kappa-\frac{1}{2}\widetilde{\sigma}^2\kappa)}-\frac{\mathbf{w}}{\kappa}\Big)\Big[\frac{\sigma}{a}\big(T-t+a^{-1}(e^{-aT}-e^{-at})\big)+\sigma[T-t]\Big]$$

$$\cdot\Big[\sigma^2\widetilde{\sigma}^2\big(\sum_{t\le u\le T}(\Delta\sqrt{G(u)})^2\big)\mathcal{K}^{-2}+1\Big]. \tag{4.2.41}$$

Proof.

1. From equations (4.2.37), (4.2.12) and (4.2.10); it follows that

$$Q_\kappa\mathcal{M}(P)^{-1}LP=-\Big(\frac{-(\theta+\frac{1}{2}\widetilde{\sigma}^2)}{\kappa(1-\theta\kappa-\frac{1}{2}\widetilde{\sigma}^2\kappa)}-\frac{\mathbf{w}}{\kappa}\Big)\Big[\frac{\sigma}{a}\big(T-t+a^{-1}(e^{-aT}-e^{-at})\big)+\sigma[T-t]\Big]P$$

$$\times\Big[\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)}+\sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)})$$

$$-\sigma^2\widetilde{\sigma}\Big(\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Big)\Big]^{-2}P^{-2}$$

$$\times-\Big[\sigma^2\widetilde{\sigma}^2\sum_{t\le u\le T}(\Delta\sqrt{G(u)})^2+\Big[\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)}$$

$$+\sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)})-\sigma^2\widetilde{\sigma}\Big(\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Big)\Big]^2$$

$$+Z\Big[\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)}+\sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)})$$

$$-\sigma^2\widetilde{\sigma}\Big(\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Big)\Big]\Big]P$$

$$=\Big(\frac{-(\theta+\frac{1}{2}\widetilde{\sigma}^2)}{\kappa(1-\theta\kappa-\frac{1}{2}\widetilde{\sigma}^2\kappa)}-\frac{\mathbf{w}}{\kappa}\Big)\Big[\frac{\sigma}{a}\big(T-t+a^{-1}(e^{-aT}-e^{-at})\big)+\sigma[T-t]\Big]\mathcal{K}^{-2}$$

$$\cdot\sigma^2\widetilde{\sigma}^2\sum_{t\le u\le T}(\Delta\sqrt{G(u)})^2+\Big(\frac{-(\theta+\frac{1}{2}\widetilde{\sigma}^2)}{\kappa(1-\theta\kappa-\frac{1}{2}\widetilde{\sigma}^2\kappa)}-\frac{\mathbf{w}}{\kappa}\Big)\Big[\frac{\sigma}{a}\big(T-t+a^{-1}(e^{-aT}-e^{-at})\big)$$

$$+\sigma[T-t]\Big]+Z\Big(\frac{-(\theta+\frac{1}{2}\widetilde{\sigma}^2)}{\kappa(1-\theta\kappa-\frac{1}{2}\widetilde{\sigma}^2\kappa)}-\frac{\mathbf{w}}{\kappa}\Big)$$

$$\cdot\Bigg[\frac{\sigma}{a}\big(T-t+a^{-1}(e^{-aT}-e^{-at})\big)+\sigma[T-t]\Bigg]\mathcal{K}^{-1}$$

where $\mathcal{K}$ is given by equation (4.2.17) $\square$

2. By equations (4.2.12), (4.2.9) and (4.2.38), we have

$$(\mathcal{M}(P))^{-1}\langle DP, DQ_\kappa\rangle = -\Bigg[\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)}+\sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)})$$

$$-\sigma^2\widetilde{\sigma}\Bigg(\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Bigg)\Bigg]^{-1}P^{-1}$$

$$\times\Bigg(\frac{-(\theta+\frac{1}{2}\widetilde{\sigma}^2)}{\kappa(1-\theta\kappa-\frac{1}{2}\widetilde{\sigma}^2\kappa)}-\frac{\mathbf{w}}{\kappa}\Bigg)\Bigg[\frac{\sigma}{a}\big(T-t+a^{-1}(e^{-aT}-e^{-at})\big)+\sigma[T-t]\Bigg]$$

$$\cdot\Bigg[\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)}+\sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)})$$

$$-\sigma^2\widetilde{\sigma}\Bigg(\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Bigg)\Bigg]P$$

$$= -\Bigg(\frac{-(\theta+\frac{1}{2}\widetilde{\sigma}^2)}{\kappa(1-\theta\kappa-\frac{1}{2}\widetilde{\sigma}^2\kappa)}-\frac{\mathbf{w}}{\kappa}\Bigg)\Bigg[\frac{\sigma}{a}\big(T-t+a^{-1}(e^{-aT}-e^{-at})\big)+\sigma[T-t]\Bigg],$$

which is equation (4.2.38).

3. From equations (4.2.37), (4.2.9) and (4.2.13); it follows that

$Q_\kappa\langle DP, D\mathcal{M}(P)^{-1}\rangle$

$$= -\Bigg(\frac{-(\theta+\frac{1}{2}\widetilde{\sigma}^2)}{\kappa(1-\theta\kappa-\frac{1}{2}\widetilde{\sigma}^2\kappa)}-\frac{\mathbf{w}}{\kappa}\Bigg)\Bigg[\frac{\sigma}{a}\big(T-t+a^{-1}(e^{-aT}-e^{-at})\big)+\sigma[T-t]\Bigg]P$$

$$\times-\Bigg[\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)}+\sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)})$$

$$-\sigma^2\widetilde{\sigma}\Bigg(\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Bigg)\Bigg]P$$

$$\times 2\Bigg[\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)}+\sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)})$$

$$-\sigma^2\widetilde{\sigma}\Bigg(\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Bigg)\Bigg]^{-3}P^{-2}$$

$$\times\Bigg[\sigma^2\widetilde{\sigma}^2\sum_{t\le u\le T}(\Delta\sqrt{G(u)})^2+\Bigg[\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)}$$

$$+\sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)})-\sigma^2\widetilde{\sigma}\Bigg(\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Bigg)\Bigg]^2\Bigg]$$

$$= 2\left(\frac{-(\theta + \frac{1}{2}\widetilde{\sigma}^2)}{\kappa(1-\theta\kappa - \frac{1}{2}\widetilde{\sigma}^2\kappa)} - \frac{\mathbf{w}}{\kappa}\right)\left[\frac{\sigma}{a}\big(T - t + a^{-1}(e^{-aT} - e^{-at})\big) + \sigma[T-t]\right]$$

$$\cdot\sigma^2\widetilde{\sigma}^2\big(\sum_{t\le u\le T}(\Delta\sqrt{G(u)})^2\big)\mathcal{K}^{-2}$$

$$+2\left(\frac{-(\theta + \frac{1}{2}\widetilde{\sigma}^2)}{\kappa(1-\theta\kappa - \frac{1}{2}\widetilde{\sigma}^2\kappa)} - \frac{\mathbf{w}}{\kappa}\right)\left[\frac{\sigma}{a}\big(T - t + a^{-1}(e^{-aT} - e^{-at})\big) + \sigma[T-t]\right]. \qquad \square$$

Theorem 4.2.5

Let P be the price of the zero-coupon bond driven by VG process, then the greek

$$\mathcal{V}_2^{VG} = e^{-r_0T}\left(\mathbb{E}\left[\Phi(P)H_{VG}(P, Q_\kappa)\right] + \mathbb{E}_{(\kappa)}[\Phi(P)]\right)$$

where

$$H_{VG}(P, Q_\kappa) = \left(\frac{-(\theta + \frac{1}{2}\widetilde{\sigma}^2)}{\kappa(1-\theta\kappa - \frac{1}{2}\widetilde{\sigma}^2\kappa)} - \frac{\mathbf{w}}{\kappa}\right)\left[\frac{\sigma}{a}\big(T - t + \frac{1}{a}(e^{-aT} - e^{-at})\big) + \sigma[T-t]\right]$$

$$\cdot\left(\frac{Z}{\mathcal{K}} - \frac{\sigma^2\widetilde{\sigma}^2\big(\sum_{t\le u\le T}(\Delta\sqrt{G(u)})^2\big)}{\mathcal{K}^2}\right)$$

and $\mathcal{K}$ is given by equation (4.2.17).

Proof. From equation (4.2.6), it follows that

$$\frac{\partial\mathbb{V}}{\partial\kappa} = \mathcal{V}_2 = \frac{\partial}{\partial\kappa}e^{-r_0T}\mathbb{E}[\Phi(P)] = e^{-r_0T}\mathbb{E}\left[\Phi(P)H_{VG}\left(P, \frac{\partial P}{\partial\theta}\right)\right].$$

Also, by substituting equations (4.2.39), (4.2.40) and (4.2.41), the Malliavin weight becomes

$$H_{VG}(P, Q_\kappa) = Z\left(\frac{-(\theta + \frac{1}{2}\widetilde{\sigma}^2)}{\kappa(1-\theta\kappa - \frac{1}{2}\widetilde{\sigma}^2\kappa)} - \frac{\mathbf{w}}{\kappa}\right)\left[\frac{\sigma}{a}\big(T - t + \frac{1}{a}(e^{-aT} - e^{-at})\big) + \sigma[T-t]\right]\mathcal{K}^{-1}$$

$$-\left(\frac{-(\theta + \frac{1}{2}\widetilde{\sigma}^2)}{\kappa(1-\theta\kappa - \frac{1}{2}\widetilde{\sigma}^2\kappa)} - \frac{\mathbf{w}}{\kappa}\right)\left[\frac{\sigma}{a}\big(T - t + \frac{1}{a}(e^{-aT} - e^{-at})\big) + \sigma[T-t]\right]$$

$$\cdot\sigma^2\widetilde{\sigma}^2\big(\sum_{t\le u\le T}(\Delta\sqrt{G(u)})^2\big)\mathcal{K}^{-2}.$$

The computation of $E_{(\kappa)}[\Phi(P)]$ is given in the Appendix. $\square$

4.2.9 Computation of *Theta* for VG-driven interest rate derivatives

We compute the greek 'Θ^{VG}' for VG-driven interest rate derivative.

$$\begin{aligned}\Theta^{VG} &= \frac{\partial}{\partial T} e^{-r_0 T}\mathbb{E}[\Phi(P)] = -r_0 e^{-r_0 T}\mathbb{E}[\Phi(P)] + \mathbb{E}\left[\Phi'(P)\frac{\partial P}{\partial T}\right] \\ &= -r_0 e^{-r_0 T}\mathbb{E}[\Phi(P)] + \mathbb{E}\left[\Phi(P) H_{VG}\left(P, \frac{\partial P}{\partial T}\right)\right] \\ &= -r_0 e^{-r_0 T}\mathbb{E}[\Phi(P)] + \mathbb{E}\big[\Phi(P) H_{VG}(P, Q_T)\big].\end{aligned}$$

Lemma 4.2.23. Let P be the price of the zero-coupon bond driven by VG process and $Q_T = \dfrac{\partial P}{\partial T}$. Then,

$$Q_T = -\big(r_0 e^{-aT} + b(1-e^{-aT}) + \frac{\sigma \mathbf{w}}{a}(1-e^{-aT}) + \mathbf{w}\sigma\big)P \tag{4.2.42}$$

and

$$DQ_T = \big(r_0 e^{-aT} + b(1-e^{-aT}) + \frac{\sigma \mathbf{w}}{a}(1-e^{-aT}) + \mathbf{w}\sigma\big)\mathcal{K}P \tag{4.2.43}$$

where $\mathcal{K}$ is given by equation (4.2.17).

Proof. Applying partial derivative to equation (4.2.7) with respect to maturity time T will give equation (4.2.42).

Furthermore, the Malliavin derivative of Q_T is given by

$$DQ_T = -\big(r_0 e^{-aT} + b(1-e^{-aT}) + \frac{\sigma \mathbf{w}}{a}(1-e^{-aT}) + \mathbf{w}\sigma\big)DP.$$

Substituting DP from equation (4.2.9) into the above equation implies that

$$\begin{aligned} DQ_T = -\big(r_0 e^{-aT} + b(1-e^{-aT}) + \frac{\sigma \mathbf{w}}{a}(1-e^{-aT}) + \mathbf{w}\sigma\big) \\ \times -\Bigg[\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)} + \sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)}) \\ -\sigma^2\Big(\sum_{t\le u\le T}(\theta\Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z)\widetilde{\sigma}\Delta\sqrt{G(u)}\Big)\Bigg]P \end{aligned}$$

$$= \big(r_0 e^{-aT} + b(1-e^{-aT}) + \frac{\sigma \mathbf{w}}{a}(1-e^{-aT}) + \mathbf{w}\sigma\big)\cdot\Bigg[\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)}$$

$$+\sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)}) - \sigma^2\widetilde{\sigma}\Big(\sum_{t\le u\le T}(\theta\Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Big)\Bigg]P$$

where

$$\mathcal{K} = \sigma\widetilde{\sigma} \sum_{t\leq u\leq T} \sum_{0\leq s\leq t} \Delta\sqrt{G(s)}e^{-a(u-s)} + \sigma\widetilde{\sigma} \sum_{t\leq u\leq T} (\Delta\sqrt{G(u)})$$
$$-\sigma^2\widetilde{\sigma}\Big(\sum_{t\leq u\leq T} (\theta\Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Big). \qquad \square$$

Lemma 4.2.24. Let P be the price of the zero-coupon bond driven by VG process and $Q_T = \dfrac{\partial P}{\partial T}$. Then,

$$Q_T\mathcal{M}(P)^{-1}LP = \big(r_0e^{-aT} + b(1-e^{-aT}) + \frac{\sigma\mathbf{w}}{a}(1-e^{-aT}) + \mathbf{w}\sigma\big)$$
$$\cdot\Big[\frac{\sigma^2\widetilde{\sigma}^2\big(\sum_{t\leq u\leq T}(\Delta\sqrt{G(u)})^2\big)}{\mathcal{K}^2} + 1 + \frac{Z}{\mathcal{K}}\Big]. \qquad (4.2.44)$$

Proof. From equations (4.2.42), (4.2.12) and (4.2.10), we have

$Q_T\mathcal{M}(P)^{-1}LP$

$$= -\big(r_0e^{-aT} + b(1-e^{-aT}) + \frac{\sigma\mathbf{w}}{a}(1-e^{-aT}) + \mathbf{w}\sigma\big)P$$
$$\cdot\Big[\sigma\widetilde{\sigma} \sum_{t\leq u\leq T} \sum_{0\leq s\leq t} \Delta\sqrt{G(s)}e^{-a(u-s)} + \sigma\widetilde{\sigma} \sum_{t\leq u\leq T} (\Delta\sqrt{G(u)})$$
$$-\sigma^2\widetilde{\sigma}\Big(\sum_{t\leq u\leq T} (\theta\Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Big)\Big]^{-2} P^{-2}$$
$$\times - \Big[\sigma^2 \sum_{t\leq u\leq T} (\widetilde{\sigma}\Delta\sqrt{G(u)})^2 + \Big[\sigma\widetilde{\sigma} \sum_{t\leq u\leq T} \sum_{0\leq s\leq t} \Delta\sqrt{G(s)}e^{-a(u-s)}$$
$$+\sigma\widetilde{\sigma} \sum_{t\leq u\leq T} (\Delta\sqrt{G(u)}) - \sigma^2\widetilde{\sigma}\big(\sum_{t\leq u\leq T} (\theta\Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\big)\Big]^2$$
$$+Z\Big[\sigma\widetilde{\sigma} \sum_{t\leq u\leq T} \sum_{0\leq s\leq t} \Delta\sqrt{G(s)}e^{-a(u-s)} + \sigma\widetilde{\sigma} \sum_{t\leq u\leq T} (\Delta\sqrt{G(u)})$$
$$-\sigma^2\widetilde{\sigma}\Big(\sum_{t\leq u\leq T} (\theta\Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Big)\Big]\Big]P$$
$$= \big(r_0e^{-aT} + b(1-e^{-aT}) + \frac{\sigma\mathbf{w}}{a}(1-e^{-aT}) + \mathbf{w}\sigma\big) \times \Big[\sigma\widetilde{\sigma} \sum_{t\leq u\leq T} \sum_{0\leq s\leq t} \Delta\sqrt{G(s)}e^{-a(u-s)}$$
$$+\sigma\widetilde{\sigma} \sum_{t\leq u\leq T} (\Delta\sqrt{G(u)}) - \sigma^2\widetilde{\sigma}\Big(\sum_{t\leq u\leq T} (\theta\Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Big)\Big]^{-2}$$
$$\cdot\sigma^2\big(\sum_{t\leq u\leq T} (\widetilde{\sigma}\Delta\sqrt{G(u)})^2\big) + \big(r_0e^{-aT} + b(1-e^{-aT}) + \frac{\sigma\mathbf{w}}{a}(1-e^{-aT}) + \mathbf{w}\sigma\big)$$

$$+Z\big(r_0e^{-aT}+b(1-e^{-aT})+\frac{\sigma\mathbf{w}}{a}(1-e^{-aT})+\mathbf{w}\sigma\big)\cdot\Big[\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)}$$

$$+\sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)})-\sigma^2\widetilde{\sigma}\big(\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\big)\Big]^{-1}. \qquad \square$$

Lemma 4.2.25. Let P be the price of the zero-coupon bond driven by VG process and $Q_T=\dfrac{\partial P}{\partial T}$. Then,

$$\mathcal{M}(P)^{-1}\langle DP,DQ_T\rangle=-\big(r_0e^{-aT}+b(1-e^{-aT})+\frac{\sigma\mathbf{w}}{a}(1-e^{-aT})+\mathbf{w}\sigma\big). \quad (4.2.45)$$

Proof. From equations (4.2.12), (4.2.9) and (4.2.43), it follows that

$\mathcal{M}(P)^{-1}\langle DP,DQ_T\rangle=$

$$=-\Big[\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)}+\sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)})$$

$$-\sigma^2\widetilde{\sigma}\Big(\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Big)\Big]P$$

$$\cdot\Big[\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)}+\sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)})$$

$$-\sigma^2\widetilde{\sigma}\Big(\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Big)\Big]^{-2}P^{-2}$$

$$\cdot\big(r_0e^{-aT}+b(1-e^{-aT})+\frac{\sigma\mathbf{w}}{a}(1-e^{-aT})+\mathbf{w}\sigma\big)\cdot\Big[\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)}$$

$$+\sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)})-\sigma^2\widetilde{\sigma}\Big(\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Big)\Big]P$$

$$=-\big(r_0e^{-aT}+b(1-e^{-aT})+\frac{\sigma\mathbf{w}}{a}(1-e^{-aT})+\mathbf{w}\sigma\big). \qquad \square$$

Lemma 4.2.26. Let P be the price of the zero-coupon bond driven by VG process and $Q_T=\dfrac{\partial P}{\partial T}$. Then,

$$Q_T\langle DP,D\mathcal{M}(P)^{-1}\rangle=2\big(r_0e^{-aT}+b(1-e^{-aT})+\frac{\sigma\mathbf{w}}{a}(1-e^{-aT})+\mathbf{w}\sigma\big)$$

$$\cdot\Big[\frac{\sigma^2\widetilde{\sigma}^2\big(\sum_{t\le u\le T}(\Delta\sqrt{G(u)})^2\big)}{\mathcal{K}^2}+1\Big]. \quad (4.2.46)$$

Proof. From equations (4.2.42), (4.2.9) and (4.2.13), it follows that

$Q_T\langle DP,D\mathcal{M}(P)^{-1}\rangle$

$$=-\big(r_0e^{-aT}+b(1-e^{-aT})+\frac{\sigma\mathbf{w}}{a}(1-e^{-aT})+\mathbf{w}\sigma\big)P$$

$$\times - \Bigg[\sigma\widetilde{\sigma} \sum_{t\leq u\leq T} \sum_{0\leq s\leq t} \Delta\sqrt{G(s)}e^{-a(u-s)} + \sigma\widetilde{\sigma} \sum_{t\leq u\leq T} (\Delta\sqrt{G(u)})$$

$$-\sigma^2\widetilde{\sigma}\Bigg(\sum_{t\leq u\leq T} (\theta\Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Bigg)\Bigg] P$$

$$\times 2\Bigg[\sigma\widetilde{\sigma} \sum_{t\leq u\leq T} \sum_{0\leq s\leq t} \Delta\sqrt{G(s)}e^{-a(u-s)} + \sigma\widetilde{\sigma} \sum_{t\leq u\leq T} (\Delta\sqrt{G(u)})$$

$$-\sigma^2\widetilde{\sigma}\Bigg(\sum_{t\leq u\leq T} (\theta\Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Bigg)\Bigg]^{-3} P^{-2}$$

$$\cdot\Bigg[\sigma^2 \sum_{t\leq u\leq T} (\widetilde{\sigma}\Delta\sqrt{G(u)})^2 + \Bigg[\sigma\widetilde{\sigma} \sum_{t\leq u\leq T} \sum_{0\leq s\leq t} \Delta\sqrt{G(s)}e^{-a(u-s)}$$

$$+\sigma\widetilde{\sigma} \sum_{t\leq u\leq T} (\Delta\sqrt{G(u)}) - \sigma^2\widetilde{\sigma}\Bigg(\sum_{t\leq u\leq T} (\theta\Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Bigg)\Bigg]^2\Bigg]$$

$$= 2\big(r_0e^{-aT} + b(1-e^{-aT}) + \frac{\sigma\mathbf{w}}{a}(1-e^{-aT}) + \mathbf{w}\sigma\big)\sigma^2\big(\sum_{t\leq u\leq T} (\widetilde{\sigma}\Delta\sqrt{G(u)})^2\big)\mathcal{K}^{-2}$$

$$+2\big(r_0e^{-aT} + b(1-e^{-aT}) + \frac{\sigma\mathbf{w}}{a}(1-e^{-aT}) + \mathbf{w}\sigma\big). \qquad \square$$

Theorem 4.2.6

Let P be the price of the zero-coupon bond driven by VG process and $Q_T = \dfrac{\partial P}{\partial T}$. Then,

$$\Theta^{VG} = -r_0e^{-r_0T}\mathbb{E}[\Phi(P)] + e^{-r_0T}\mathbb{E}\big[\Phi(P)H_{VG}(P,Q_T)\big]$$

where

$$\begin{aligned} H_{VG}(P,Q_T) &= \big(r_0e^{-aT} + b(1-e^{-aT}) + \frac{\sigma\mathbf{w}}{a}(1-e^{-aT}) + \mathbf{w}\sigma\big) \\ &\quad \times\Bigg[\frac{Z}{\mathcal{K}} - \frac{\sigma^2\widetilde{\sigma}^2\big(\sum_{t\leq u\leq T}(\Delta\sqrt{G(u)})^2\big)}{\mathcal{K}^2}\Bigg] \end{aligned}$$

and $\mathcal{K}$ is given by equation (4.2.17).

Proof. From equation (4.2.6), it follows that

$$\begin{aligned} \Theta &= \frac{\partial\mathbb{V}}{\partial T} = -r_0e^{-r_0T}\mathbb{E}[\Phi(P)] + e^{-r_0T}\mathbb{E}\Bigg[\Phi(P)H_{VG}\Bigg(P,\frac{\partial P}{\partial T}\Bigg)\Bigg] \\ &= -r_0e^{-r_0T}\mathbb{E}[\Phi(P)] + e^{-r_0T}\mathbb{E}\big[\Phi(P)H_{VG}(P,Q_T)\big]. \end{aligned}$$

From equation (4.2.8),

$$H_{VG}(P,Q_T) = Q_T\mathcal{M}(P)^{-1}LP - \mathcal{M}(P)^{-1}\langle DP, DQ_T\rangle - Q_T\langle DP, D\mathcal{M}(P)^{-1}\rangle.$$

Substituting equations (4.2.44), (4.2.45) and (4.2.46), we get

$$H_{VG}(P, Q_T) \quad = \quad \frac{Z}{\mathcal{K}}\big(r_0 e^{-aT} + b(1-e^{-aT}) + \frac{\sigma \mathbf{w}}{a}(1-e^{-aT}) + \mathbf{w}\sigma\big)$$

$$-\frac{\sigma^2\widetilde{\sigma}^2\big(\sum_{t\le u\le T}(\Delta\sqrt{G(u)})^2\big)}{\mathcal{K}^2}\cdot\big(r_0 e^{-aT} + b(1-e^{-aT}) + \frac{\sigma \mathbf{w}}{a}(1-e^{-aT}) + \mathbf{w}\sigma\big). \qquad \square$$

4.2.10 Computation of *vega*$_3$ for VG-driven interest rate derivatives

In this subsection, we compute *vega*$_3$ for VG-driven interest rate derivative.

$$\mathcal{V}_3^{VG} = \frac{\partial}{\partial\widetilde{\sigma}} e^{-r_0 T}\mathbb{E}[\Phi(P)] = e^{-r_0 T}\mathbb{E}\Big[\Phi'(P)\frac{\partial P}{\partial \widetilde{\sigma}}\Big] = e^{-r_0 T}\mathbb{E}\Big[\Phi(P) H_{VG}\Big(P, \frac{\partial P}{\partial\widetilde{\sigma}}\Big)\Big].$$

Recall that by equation (4.2.6),

$$P(t,T) = \exp\Bigg(- \Bigg(\Bigg[- \frac{r_0}{a}\big(e^{-aT} - e^{-at}\big) + b\big(T - t + \frac{1}{a}(e^{-aT} - e^{-at})\big)$$

$$+\frac{\sigma \mathbf{w}}{a}\big[T - t + \frac{1}{a}(e^{-aT} - e^{-at})\big] + \sigma \sum_{t\le u\le T}\sum_{0\le s\le t}\big(\theta\Delta G(s) e^{-a(u-s)} + \widetilde{\sigma}\Delta\sqrt{G(s)} e^{-a(u-s)} Z\big)\Bigg]$$

$$+\mathbf{w}\sigma[T-t] + \sigma\sum_{t\le u\le T}(\theta\Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z)$$

$$-\frac{\sigma^2}{2}\big(\sum_{t\le u\le T}(\theta\Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z)^2\big)\Bigg)\Bigg)$$

where $\mathbf{w} = \kappa^{-1}\ln(1 - \theta\kappa - \frac{1}{2}\widetilde{\sigma}^2\kappa)$

and

$$\frac{\partial \mathbf{w}}{\partial\widetilde{\sigma}} = \frac{-\widetilde{\sigma}}{1 - \theta\kappa - \frac{1}{2}\widetilde{\sigma}^2\kappa}.$$

Lemma 4.2.27. Let P be the price of the zero-coupon bond driven by VG process and $Q_{\widetilde{\sigma}} = \dfrac{\partial P}{\partial\widetilde{\sigma}}$. Then,

$$Q_{\widetilde{\sigma}} = -\Bigg[\Bigg(\frac{\sigma}{a}\big[T - t + a^{-1}(e^{-aT} - e^{-at})\big] + \sigma[T-t]\Bigg)\cdot\big(\frac{-\widetilde{\sigma}}{1-\theta\kappa - \frac{1}{2}\widetilde{\sigma}^2\kappa}\big)$$

$$+\sigma\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{G(s)}e^{-a(u-s)}Z\big)$$

$$+\sigma\sum_{t\le u\le T}(\Delta\sqrt{G(u)}Z) - \sigma^2\big(\sum_{t\le u\le T}(\theta\Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}Z\big)\Bigg]P \tag{4.2.47}$$

and

$$DQ_{\widetilde{\sigma}} = \Bigg(- \Bigg[\sigma \sum_{t\le u\le T}\sum_{0\le s\le t} \Delta\sqrt{G(s)}e^{-a(u-s)} + \sigma \sum_{t\le u\le T} \Delta\sqrt{G(u)}$$

$$-\sigma^2\Big[\sum_{t\le u\le T}\Big((\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}+\widetilde{\sigma}(\Delta\sqrt{G(u)})^2 Z\Big)\Big]\Bigg]\Bigg)P+\widetilde{L}\mathcal{K}P \tag{4.2.48}$$

where $\mathcal{K}$ is given by equation (4.2.17), and

$$\widetilde{L} = \Big(\frac{\sigma}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big] + \sigma[T-t]\Big)\cdot\big(\frac{-\widetilde{\sigma}}{1-\theta\kappa-\frac{1}{2}\widetilde{\sigma}^2\kappa}\big)$$

$$+\sigma \sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{G(s)}e^{-a(u-s)}Z\big) + \sigma\sum_{t\le u\le T}(\Delta\sqrt{G(u)}Z)$$

$$-\sigma^2\big(\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}Z\big). \tag{4.2.49}$$

Proof. Applying partial derivative with respect to $\widetilde{\sigma}$ to equation (4.2.7), we get

$$Q_{\widetilde{\sigma}} = \frac{\partial P}{\partial\widetilde{\sigma}} = -\Bigg[\frac{\sigma}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big]\frac{\partial\mathbf{w}}{\partial\widetilde{\sigma}} + \sigma[T-t]\frac{\partial\mathbf{w}}{\partial\widetilde{\sigma}}$$

$$+\sigma \sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{G(s)}e^{-a(u-s)}Z\big) + \sigma\sum_{t\le u\le T}(\Delta\sqrt{G(u)}Z)$$

$$-\sigma^2\big(\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}Z\big)\Bigg]P$$

which is $Q_{\widetilde{\sigma}}$. Thus, the Malliavin derivative

$$DQ_{\widetilde{\sigma}} = P\cdot\Bigg(- \Bigg[\sigma\big(\sum_{t\le u\le T}\sum_{0\le s\le t} \Delta\sqrt{G(s)}e^{-a(u-s)} + \sigma \sum_{t\le u\le T} \Delta\sqrt{G(u)}$$

$$-\sigma^2\Big[\sum_{t\le u\le T}\Big((\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}+\widetilde{\sigma}(\Delta\sqrt{G(u)})^2 Z\Big)\Big)\Big]\Bigg]\Bigg)$$

$$+-\Bigg[\Big(\frac{\sigma}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big] + \sigma[T-t]\Big)\cdot\big(\frac{-\widetilde{\sigma}}{1-\theta\kappa-\frac{1}{2}\widetilde{\sigma}^2\kappa}\big)$$

$$+\sigma \sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{G(s)}e^{-a(u-s)}Z\big) + \sigma\sum_{t\le u\le T}(\Delta\sqrt{G(u)}Z)$$

$$-\sigma^2\big(\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}Z\big)\Bigg]$$

$$\times - \Bigg[\sigma\widetilde{\sigma} \sum_{t\le u\le T}\sum_{0\le s\le t} \Delta\sqrt{G(s)}e^{-a(u-s)} + \sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)})$$

$$-\sigma^2\Big(\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\widetilde{\sigma}\Delta\sqrt{G(u)}\Big)\Bigg]P$$

$$= \Big(- \Big[\sigma \sum_{t \le u \le T} \sum_{0 \le s \le t} \Delta\sqrt{G(s)} e^{-a(u-s)} + \sigma \sum_{t \le u \le T} \Delta\sqrt{G(u)}$$

$$-\sigma^2 \Big[\sum_{t \le u \le T} \Big((\theta \Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)} + \widetilde{\sigma}(\Delta\sqrt{G(u)})^2 Z \Big) \Big] \Big] \Big) P$$

$$+ \Big[\Big(\frac{\sigma}{a}\big[T - t + \frac{1}{a}(e^{-aT} - e^{-at})\big] + \sigma[T-t] \Big) \cdot \big(\frac{-\widetilde{\sigma}}{1 - \theta\kappa - \frac{1}{2}\widetilde{\sigma}^2\kappa} \big)$$

$$+ \sigma \sum_{t \le u \le T} \sum_{0 \le s \le t} \big(\Delta\sqrt{G(s)} e^{-a(u-s)} Z \big) + \sigma \sum_{t \le u \le T} (\Delta\sqrt{G(u)} Z)$$

$$-\sigma^2 \big(\sum_{t \le u \le T} (\theta \Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}Z \big) \Big] \mathcal{K} P$$

$$= \Big(- \Big[\sigma \sum_{t \le u \le T} \sum_{0 \le s \le t} \Delta\sqrt{G(s)} e^{-a(u-s)} + \sigma \sum_{t \le u \le T} \Delta\sqrt{G(u)}$$

$$-\sigma^2 \Big[\sum_{t \le u \le T} \Big((\theta \Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)} + \widetilde{\sigma}(\Delta\sqrt{G(u)})^2 Z \Big) \Big] \Big] \Big) P$$

$$+ \Big[\Big(\frac{\sigma}{a}\big[T - t + \frac{1}{a}(e^{-aT} - e^{-at})\big] + \sigma[T-t] \Big) \cdot \big(\frac{-\widetilde{\sigma}}{1 - \theta\kappa - \frac{1}{2}\widetilde{\sigma}^2\kappa} \big)$$

$$+ \sigma \sum_{t \le u \le T} \sum_{0 \le s \le t} \big(\Delta\sqrt{G(s)} e^{-a(u-s)} Z \big) + \sigma \sum_{t \le u \le T} (\Delta\sqrt{G(u)} Z)$$

$$-\sigma^2 \big(\sum_{t \le u \le T} (\theta \Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}Z \big) \Big] \mathcal{K} P$$

where $\mathcal{K}$ is given by equation (4.2.17). □

Lemma 4.2.28. Let P be the price of the zero-coupon bond driven by VG process. Then,

$$Q_{\widetilde{\sigma}} \mathcal{M}(P)^{-1} L P = \widetilde{L} \Big[\frac{\big(\sigma^2 \widetilde{\sigma}^2 \sum_{t \le u \le T} (\Delta\sqrt{G(u)})^2\big)}{\mathcal{K}^2} + 1 + \frac{Z}{\mathcal{K}} \Big] \tag{4.2.50}$$

where $\mathcal{K}$ and $\widetilde{L}$ are given by equations (4.2.17) and (4.2.49), respectively.

Proof. From equations (4.2.47), (4.2.12) and (4.2.10), it follows that

$$Q_{\widetilde{\sigma}} \mathcal{M}(P)^{-1} L P = - \Big[\frac{\sigma}{a}\big[T - t + \frac{1}{a}(e^{-aT} - e^{-at})\big] \frac{\partial \mathbf{w}}{\partial \widetilde{\sigma}} + \sigma[T-t] \frac{\partial \mathbf{w}}{\partial \widetilde{\sigma}}$$

$$+ \sigma \sum_{t \le u \le T} \sum_{0 \le s \le t} \big(\Delta\sqrt{G(s)} e^{-a(u-s)} Z \big) + \sigma \sum_{t \le u \le T} (\Delta\sqrt{G(u)} Z)$$

$$-\sigma^2 \big(\sum_{t \le u \le T} (\theta \Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}Z \big) \Big] P \cdot \Big[\sigma\widetilde{\sigma} \sum_{t \le u \le T} \sum_{0 \le s \le t} \Delta\sqrt{G(s)} e^{-a(u-s)}$$

$$+\sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)})-\sigma^2\Big(\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\widetilde{\sigma}\Delta\sqrt{G(u)}\Big)\Big]^{-2}P^{-2}$$

$$\cdot\Big[-\sigma^2\sum_{t\le u\le T}(\widetilde{\sigma}\Delta\sqrt{G(u)})^2+\Big[\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)}$$

$$+\sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)})-\sigma^2\Big(\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\widetilde{\sigma}\Delta\sqrt{G(u)}\Big)\Big]^2$$

$$+Z\Big[\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)}+\sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)})$$

$$-\sigma^2\Big(\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\widetilde{\sigma}\Delta\sqrt{G(u)}\Big)\Big]P$$

$$=\widetilde{L}\big(\sigma^2\widetilde{\sigma}^2\sum_{t\le u\le T}(\Delta\sqrt{G(u)})^2\big)\mathcal{K}^{-2}+\widetilde{L}+\frac{\widetilde{L}Z}{\mathcal{K}}.$$

Lemma 4.2.29. Let P be the price of the zero-coupon bond driven by VG process. Then,

$$\mathcal{M}(P)^{-1}\langle DP,DQ_{\widetilde{\sigma}}\rangle=\frac{1}{\mathcal{K}}\Bigg(\Big[\sigma\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)}+\sigma\sum_{t\le u\le T}\Delta\sqrt{G(u)}$$

$$-\sigma^2\Big[\sum_{t\le u\le T}\Big((\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}+\widetilde{\sigma}(\Delta\sqrt{G(u)})^2Z\Big)\Big]\Big]\Bigg)-\widetilde{L} \tag{4.2.51}$$

where $\mathcal{K}$ and $\widetilde{L}$ are given by equations (4.2.17) and (4.2.49), respectively.

Proof. From equations (4.2.12), (4.2.9) and (4.2.48), it follows that

$$\mathcal{M}(P)^{-1}\langle DP,DQ_{\widetilde{\sigma}}\rangle=-\Big[\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)}+\sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)})$$

$$-\sigma^2\Big(\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\widetilde{\sigma}\Delta\sqrt{G(u)}\Big)\Big]^{-1}P^{-1}$$

$$\Bigg(-\Big[\sigma\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)}+\sigma\sum_{t\le u\le T}\Delta\sqrt{G(u)}$$

$$-\sigma^2\Big[\sum_{t\le u\le T}\Big((\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}+\widetilde{\sigma}(\Delta\sqrt{G(u)})^2Z\Big)\Big]\Big]\Bigg)P+\widetilde{L}\mathcal{K}P$$

$$=\Big[\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)}+\sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)})$$

$$-\sigma^2\Big(\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\widetilde{\sigma}\Delta\sqrt{G(u)}\Big)\Big]^{-1}$$

$$\cdot\Big(\Big[\sigma\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)}+\sigma\sum_{t\le u\le T}\Delta\sqrt{G(u)}$$

$$-\sigma^2\Big[\sum_{t\le u\le T}\Big((\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}+\widetilde{\sigma}(\Delta\sqrt{G(u)})^2 Z\Big)\Big]\Big]\Big)-\widetilde{L}. \qquad \square$$

Lemma 4.2.30. Let P be the price of the zero-coupon bond driven by VG process. Then,

$$Q_{\widetilde{\sigma}}\langle DP, D\mathcal{M}(P)^{-1}\rangle = 2\widetilde{L}\mathcal{K}^{-2}\big(\sigma^2\widetilde{\sigma}^2\sum_{t\le u\le T}(\Delta\sqrt{G(u)})^2\big)+2\widetilde{L} \tag{4.2.52}$$

where $\mathcal{K}$ and $\widetilde{L}$ are given by equations (4.2.17) and (4.2.49), respectively.

Proof.

$$Q_{\widetilde{\sigma}}\langle DP, D\mathcal{M}(P)^{-1}\rangle = -\Big[\Big(\frac{\sigma}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big]-\sigma[T-t]\Big)\frac{-\widetilde{\sigma}}{1-\theta\kappa-\frac{1}{2}\widetilde{\sigma}^2\kappa}$$

$$+\sigma\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{G(s)}e^{-a(u-s)}Z\big)+\sigma\sum_{t\le u\le T}(\Delta\sqrt{G(u)}Z)$$

$$-\sigma^2\big(\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}Z\big)\Big]P$$

$$\times-\Big[\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)}+\sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)})$$

$$-\sigma^2\Big(\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\widetilde{\sigma}\Delta\sqrt{G(u)}\Big)\Big]P$$

$$\cdot 2\Big[\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)}+\sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)})$$

$$-\sigma^2\Big(\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\widetilde{\sigma}\Delta\sqrt{G(u)}\Big)\Big]^{-3}P^{-2}$$

$$\cdot\Big[\sigma^2\sum_{t\le u\le T}(\widetilde{\sigma}\Delta\sqrt{G(u)})^2+\Big[\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)}$$

$$+\sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)})-\sigma^2\Big(\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\widetilde{\sigma}\Delta\sqrt{G(u)}\Big)\Big]^2\Big]$$

$$=2\Big[\Big(\frac{\sigma}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big]-\sigma[T-t]\Big)\frac{-\widetilde{\sigma}}{1-\theta\kappa-\frac{1}{2}\widetilde{\sigma}^2\kappa}$$

$$+\sigma\sum_{t\leq u\leq T}\sum_{0\leq s\leq t}\big(\Delta\sqrt{G(s)}e^{-a(u-s)}Z\big)+\sigma\sum_{t\leq u\leq T}(\Delta\sqrt{G(u)}Z)$$

$$-\sigma^2\Big(\sum_{t\leq u\leq T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}Z\Big)\Bigg]$$

$$\cdot\Bigg[\sigma\widetilde{\sigma}\sum_{t\leq u\leq T}\sum_{0\leq s\leq t}\Delta\sqrt{G(s)}e^{-a(u-s)}+\sigma\widetilde{\sigma}\sum_{t\leq u\leq T}(\Delta\sqrt{G(u)})$$

$$-\sigma^2\Bigg(\sum_{t\leq u\leq T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\widetilde{\sigma}\Delta\sqrt{G(u)}\Bigg)\Bigg]^{-2}$$

$$\cdot\big(\sigma^2\sum_{t\leq u\leq T}(\widetilde{\sigma}\Delta\sqrt{G(u)})^2\big)+2\Bigg[\Big(\frac{\sigma}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big]-\sigma[T-t]\Big)\frac{-\widetilde{\sigma}}{1-\theta\kappa-\frac{1}{2}\widetilde{\sigma}^2\kappa}$$

$$+\sigma\sum_{t\leq u\leq T}\sum_{0\leq s\leq t}\big(\Delta\sqrt{G(s)}e^{-a(u-s)}Z\big)+\sigma\sum_{t\leq u\leq T}(\Delta\sqrt{G(u)}Z)$$

$$-\sigma^2\Big(\sum_{t\leq u\leq T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}Z\Big)\Bigg].$$

Substituting $\mathcal{K}$ and $\widetilde{L}$ will give the result. □

Theorem 4.2.7

Let P be the price of the zero-coupon bond driven by VG process. Then,

$$\mathcal{V}_3^{VG}=e^{-r_0T}\mathbb{E}\big[\Phi(P)H_{VG}\big(P,Q_{\widetilde{\sigma}}\big)\big]$$

where

$$H_{VG}\big(P,Q_{\widetilde{\sigma}}\big)=\frac{Z\widetilde{L}}{\mathcal{K}}-\frac{\widetilde{L}\big(\sigma^2\widetilde{\sigma}^2\sum_{t\leq u\leq T}(\Delta\sqrt{G(u)})^2\big)}{\mathcal{K}^2}-\frac{1}{\mathcal{K}}\Bigg(\Bigg[\sigma\Big(\sum_{t\leq u\leq T}\sum_{0\leq s\leq t}\Delta\sqrt{G(s)}e^{-a(u-s)}\Big)$$

$$+\sigma\Big(\sum_{t\leq u\leq T}\Delta\sqrt{G(u)}\Big)-\sigma^2\Bigg[\sum_{t\leq u\leq T}\Bigg((\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}+\widetilde{\sigma}(\Delta\sqrt{G(u)})^2Z\Bigg)\Bigg]\Bigg]\Bigg),$$

$\mathcal{K}$ and $\widetilde{L}$ are given by equations (4.2.15) and (4.2.47), respectively.

Proof. From equation (4.2.6), it follows that

$$\mathcal{V}_3=\frac{\partial\mathbb{V}}{\partial\widetilde{\sigma}}=e^{-r_0T}\mathbb{E}\big[\Phi(P)H_{VG}\big(P,Q_{\widetilde{\sigma}}\big)\big].$$

Substituting equations (4.2.50), (4.2.51) and (4.2.52) into (4.2.8) will give the desired result.

Remark 4.2.3.

The extended Vasicek model is:

$$\begin{aligned} dr_t &= a(b-r_t)dt + \sigma dX_t = a(b-r_t)dt + \sigma d(\mathbf{w}t + \theta G_t + \widetilde{\sigma}\sqrt{G}Z) \\ &= a(b-r_t)dt + \sigma(\mathbf{w}dt + \theta\Delta G_t + \widetilde{\sigma}\Delta\sqrt{G}Z). \end{aligned}$$

Substituting $\theta = 0$, $\widetilde{\sigma} = 1$, $G = t$ and $\mathbf{w} = 0$, we obtain

$$\begin{aligned} dr_t &= a(b-r_t)dt + \sigma(\mathbf{w}dt + \theta\Delta G_t + \widetilde{\sigma}\Delta\sqrt{G}Z) \\ &= a(b-r_t)dt + \sigma(0 + 0 + \Delta\sqrt{t}Z) = a(b-r_t)dt + \sigma\Delta\sqrt{t}Z. \end{aligned}$$

As $W_t = \sqrt{t}Z$, this gives

$$dr_t = a(b-r_t)dt + \sigma dW_t,$$

which is the original Vasicek model.

4.3 Sensitivity analysis of zero-coupon bond price under NIG-driven Lévy market

In this section, we extend the Vasicek short rate model to an interest rate derivative market driven by NIG process and derive an expression for the price of zero-coupon bond. The price driven by NIG process will enable the excess kurtosis of the model to be captured. We are to derive expressions for the greeks of the price of the zero-coupon bond by means of the Malliavin calculus.

4.3.1 Short rate model under NIG process

In this subsection, we develop a modified Vasicek (1977) interest rate model driven by NIG process. The rate r satisfies the stochastic differential equation given by

$$dr_t = a(b-r_t)dt + \sigma dX_t$$

where X_t is a Lévy process, b is long-term mean rate, a is speed of mean reversion and σ is the volatility of the interest rate.

Let $f(s,x) = xe^{as}$, $\dfrac{\partial f}{\partial s} = axe^{as}$, $\dfrac{\partial f}{\partial x} = e^{as}$, $\dfrac{\partial^2 f}{\partial x^2} = 0$, and $r(t) = r_t$, then applying Itô's formula,

$$f(t,r_t) = f(0,r_0) + \int_0^t \frac{\partial f}{\partial s}(s,r_s)ds + \int_0^t \frac{\partial f}{\partial x}(s,r_{s_-})dr_s + \sum_{0\le s\le t}^{\Delta r_s \neq 0} | \, r_s e^{as} - e^{as} r_{s_-} - \Delta r_s e^{as} \, |$$

$$= f(0,r_0) + \int_0^t a r_s e^{as} ds + \int_0^t e^{as} dr_s$$

$$= f(0,r_0) + \int_0^t a e^{as} r_s ds + \int_0^t e^{as}(a(b-r_s))ds + \int_0^t \sigma e^{as} dX_s$$

$$= f(0,r_0) + \int_0^t a e^{as} r_s ds + \int_0^t ab e^{as} ds - \int_0^t a e^{as} r_s ds + \int_0^t \sigma e^{as} dX_s.$$

Hence,

$$\begin{aligned} r_t e^{at} &= r_0 + ab\int_0^t e^{as} ds + \int_0^t \sigma e^{as} dX_s = r_0 + be^{as}|_0^t + \int_0^t \sigma e^{as} dX_s \\ &= r_0 + b(e^{at} - 1) + \int_0^t \sigma e^{as} dX_s \end{aligned}$$

which implies that

$$r_t = r_0 e^{-at} + b(1 - e^{-at}) + \sigma \int_0^t e^{-a(t-s)} dX_s. \tag{4.3.1}$$

We adopt the NIG model given by $X_t = \mathbf{w}t + \beta\delta^2 IG_t + \delta W(IG_t)$ (Nicoletta (2011)) where $\mathbf{w}$ is the cumulant generating function given by

$$\mathbf{w} = \delta((\alpha^2 - (\beta+1)^2)^{0.5} - (\alpha^2 - \beta^2)^{0.5}).$$

The parameter α manages the behaviour of the tail of the distribution, β controls skewness and δ is the scale parameter.

Thus,

$$X_t = \mathbf{w}t + \delta\sqrt{IG(t)}Z + \beta\delta^2 IG(t), \implies dX_t = \mathbf{w}dt + \delta\Delta\sqrt{IG(t)}Z + \beta\delta^2\Delta IG(t).$$

Hence, equation (4.3.1) becomes

$$r_t = r_0 e^{-at} + b(1-e^{-at}) + \sigma\big(\mathbf{w}\int_0^t e^{-a(t-s)}ds + \delta\sum_{0\le s\le t}(\Delta\sqrt{IG(s)}Z + \beta\delta\Delta IG(s))e^{-a(t-s)}\big)$$

$$= r_0 e^{-at} + b(1-e^{-at}) + \frac{\sigma\mathbf{w}}{a}(1-e^{-at}) + \sigma\delta\big(\sum_{0\le s\le t}(\Delta\sqrt{IG(s)}Z + \beta\delta\Delta IG(s))e^{-a(t-s)}\big). \tag{4.3.2}$$

We adopt the above expression to derive an expression for the price of the zero-coupon bond driven by NIG process.

4.3.2 Expression for the price of a zero-coupon bond with a Vasicek short rate model under NIG process

In this subsection, we proceed to obtain an expression for the price of a zero-coupon bond driven by NIG process by adopting the improved Vasicek short rate model obtained in the previous subsection.

By equation (4.2.3), the dynamics of the zero-coupon bond price under risk neutral measure is given by

$$dP = r_t P dt + \sigma P dX_t.$$

Applying Itô's lemma, we have

$$\begin{aligned} d\ln P &= \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial x}dP + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}d[P,P] \\ &= \frac{1}{P}\cdot P(rdt + \sigma dX_t) - \frac{1}{2P^2}(rdt + \sigma dX_t)^2 P^2 \\ &= r_t dt + \sigma dX_t - \frac{1}{2}(r_t dt + \sigma dX_t)^2 \\ &= r_t dt + \sigma dX_t - \frac{1}{2}\sigma^2 (dX_t)^2 \ \text{ where } \ (dt)^2 = 0, dtdX = 0. \end{aligned}$$

But

$$(dX)^2 = (\mathbf{w}dt + \delta\Delta\sqrt{IG(t)}Z + \beta\delta^2\Delta IG(t))^2 = (\delta\Delta\sqrt{IG(t)}Z + \beta\delta^2\Delta IG(t))^2.$$

Hence,

$$d\ln P = r_t dt + \sigma\mathbf{w}dt + \sigma(\delta\Delta\sqrt{IG(t)}Z + \beta\delta^2\Delta IG(t)) - \frac{1}{2}\sigma^2(\delta\Delta\sqrt{IG(t)}Z + \beta\delta^2\Delta IG(t))^2. \tag{4.3.3}$$

Integrating equation (4.3.3), we get

$$\begin{aligned} \ln P(T,T) - \ln P(t,T) = \int_t^T r_u du + \sigma\mathbf{w}\int_t^T du + \sigma\Big(\sum_{0\le u\le T}(\delta\Delta\sqrt{IG(u)}Z + \beta\delta^2\Delta IG(u)) \\ - \sum_{0\le u\le t}(\delta\Delta\sqrt{IG(u)}Z + \beta\delta^2\Delta IG(u))\Big) - \frac{1}{2}\sigma^2\Big(\sum_{0\le u\le T}(\delta\Delta\sqrt{IG(u)}Z \\ + \beta\delta^2\Delta IG(u))^2 - \sum_{0\le u\le t}(\delta\Delta\sqrt{IG(u)}Z + \beta\delta^2\Delta IG(u))^2\Big) \end{aligned}$$

and

$$\ln P(t,T) = -\Big(\int_t^T r_u du + \sigma \mathbf{w}\int_t^T du + \sigma\Big(\sum_{0\le u\le T}(\delta\Delta\sqrt{IG(u)}Z + \beta\delta^2\Delta IG(u))$$

$$-\sum_{0\le u\le t}(\delta\Delta\sqrt{IG(u)}Z + \beta\delta^2\Delta IG(u))\Big) - \frac{1}{2}\sigma^2\Big(\sum_{0\le u\le T}(\delta\Delta\sqrt{IG(u)}Z$$

$$+\beta\delta^2\Delta IG(u))^2 - \sum_{0\le u\le t}(\delta\Delta\sqrt{IG(u)}Z + \beta\delta^2\Delta IG(u))^2\Big)\Big).$$

By equation (4.3.2),

$$r_t = r_0 e^{-at} + b(1-e^{-at}) + \frac{\sigma\mathbf{w}}{a}(1-e^{-at}) + \sigma\delta\big(\sum_{0\le s\le t}(\Delta\sqrt{IG(s)}Z + \beta\delta\Delta IG(s))e^{-a(t-s)}\big).$$

Hence,

$$\begin{aligned}\int_t^T r_u du &= r_0\int_t^T e^{-au}du + b\int_t^T(1-e^{-au})du + \frac{\sigma\mathbf{w}}{a}\int_u^T(1-e^{-au})\\ &\quad +\sigma\delta\big(\sum_{0\le u\le T}\sum_{0\le s\le t}(\Delta\sqrt{IG(s)}Z + \beta\delta\Delta IG(s))e^{-a(u-s)}\big)\\ &\quad -\sigma\delta\big(\sum_{0\le u\le t}\sum_{0\le s\le t}(\Delta\sqrt{IG(s)}Z + \beta\delta\Delta IG(s))e^{-a(u-s)}\big)\end{aligned}$$

$$\begin{aligned}= \; & \frac{-r_0}{a}(e^{-aT} - e^{-at}) + b\big(T - t + \frac{1}{a}(e^{-aT} - e^{-at})\big) + \frac{\sigma\mathbf{w}}{a}\big(T - t + \frac{1}{a}(e^{-aT} - e^{-at})\big)\\ & +\sigma\delta\big(\sum_{0\le u\le T}\sum_{0\le s\le t}(\Delta\sqrt{IG(s)}Z + \beta\delta\Delta IG(s))e^{-a(u-s)}\big)\\ & -\sigma\delta\big(\sum_{0\le u\le t}\sum_{0\le s\le t}(\Delta\sqrt{IG(s)}Z + \beta\delta\Delta IG(s))e^{-a(u-s)}\big).\end{aligned}$$

Thus, the price of the zero-coupon bond driven by NIG process is given by

$$P(t,T) = \exp\Big(-\Big[\frac{-r_0}{a}(e^{-aT}-e^{-at}) + b\big(T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big) + \frac{\sigma\mathbf{w}}{a}\big(T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big)$$

$$+\sigma\delta\big(\sum_{0\le u\le T}\sum_{0\le s\le t}(\Delta\sqrt{IG(s)}Z+\beta\delta\Delta IG(s))e^{-a(u-s)}\big) - \sigma\delta\big(\sum_{0\le u\le t}\sum_{0\le s\le t}(\Delta\sqrt{IG(s)}Z$$

$$+\beta\delta\Delta IG(s))e^{-a(u-s)}\big) + \sigma\mathbf{w}[T-t] + \sigma\delta\Big(\sum_{0\le u\le T}(\Delta\sqrt{IG(u)}Z + \beta\delta\Delta IG(u))$$

$$-\sum_{0\le u\le t}(\Delta\sqrt{IG(u)}Z + \beta\delta\Delta IG(u))\Big) - \frac{1}{2}\sigma^2\delta^2\Big(\sum_{0\le u\le T}(\Delta\sqrt{IG(u)}Z$$

$$+\beta\delta\Delta IG(u))^2 - \sum_{0\le u\le t}(\Delta\sqrt{IG(u)}Z + \beta\delta\Delta IG(u))^2\Big)\Big]\Big]\Big). \qquad (4.3.4)$$

Besides being a function of t and T, the expression on the right hand side of the above equation also depends on r_0, σ, β, δ, $\mathbf{w}$ and Z. Thus, in the sequel, we shall regard P as a function of t, T, r_0, σ, β, δ, $\mathbf{w}$ and Z.

Remark 4.3.1. Recall that the call option price with P as the underlying is given in equation (4.2.6) as

$$\mathbb{V} = e^{-r_0 T}\mathbb{E}[\Phi(P)]$$

where $\Phi(P)$ is the payoff.

In the subsequent subsections, we employ the price of the call option given by equation (4.2.6).

4.3.3 The greeks of zero-coupon bonds driven by NIG Lévy process

In this subsection, we compute the greeks of the price of an interest rate derivative driven by NIG process.

By equation (4.3.4) and Remark 4.2.2, the price of the zero-coupon bond driven by NIG Lévy process can be written as

$$P(t,T) = \exp\Big(-\Big([\frac{-r_0}{a}(e^{-aT}-e^{-at})+b\big(T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big)+\frac{\sigma\mathbf{w}}{a}[T-t+\frac{1}{a}(e^{-aT}-e^{-at})]$$
$$+\mathbf{w}\sigma[T-t] + \sigma\delta\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}Z + \beta\delta\Delta IG(s)e^{-a(u-s)}\big)$$
$$+\sigma\delta\sum_{t\le u\le T}\big(\Delta\sqrt{IG(u)}Z+\beta\delta\Delta IG(u)\big)-\frac{\sigma^2\delta^2}{2}\Big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)^2\Big)\Big)\Big). \qquad (4.3.5)$$

where

$$\mathbf{w} = \delta(\sqrt{\alpha^2-(\beta+1)^2} - \sqrt{\alpha^2-\beta^2}).$$

List of greeks under the NIG process

Let $\mathbb{V} = e^{-r_0 T}\mathbb{E}[\Phi(P)]$ be the call option price as given by equation (4.2.6), with P as the underlying driven by NIG process. Let $\Phi(P) = \max(P-K, 0)$ be the payoff with strike price K.

The greeks are: (i) $\text{Delta}^{NIG} := \triangle^{NIG} = \dfrac{\partial \mathbb{V}}{\partial r_0}$,

(ii) $\text{Gamma}^{NIG} := \Gamma^{NIG} = \dfrac{\partial^2 \mathbb{V}}{\partial {r_0}^2}$, (iii) $\text{Theta}^{NIG} := \Theta^{NIG} = \dfrac{\partial \mathbb{V}}{\partial T}$,

(iv) $\text{Vega}^{NIG} := \mathcal{V}^{NIG} = \dfrac{\partial \mathbb{V}}{\partial \sigma}$, (v) $\text{Vega}_{\tilde{2}}^{NIG} := \mathcal{V}_{\tilde{2}}^{NIG} = \dfrac{\partial \mathbb{V}}{\partial \delta}$,

(vi) $\text{Vega}_{\tilde{3}}^{NIG} := \mathcal{V}_{\tilde{3}}^{NIG} = \dfrac{\partial \mathbb{V}}{\partial \alpha}$, (vii) $\text{Vega}_{\tilde{4}}^{NIG} := \mathcal{V}_{\tilde{4}}^{NIG} = \dfrac{\partial \mathbb{V}}{\partial \beta}$.

$\triangle^{NIG}$ measures the sensitivity of the NIG-driven zero-coupon bond option price to changes in the initial interest rate. Γ^{NIG} measures the sensitivity of the delta to changes in the underlying, that is, the initial interest rate. Θ^{NIG} measures how the option value changes as there is decrease in time remaining for the option to expire. $\mathcal{V}^{NIG}$ measures the sensitivity of the bond option price to changes in the volatility of the short rate model driven by NIG process. $\text{Vega}_{\tilde{2}}^{NIG}$ describes the option price sensitivity to changes in the scale of the distribution. $\text{Vega}_{\tilde{3}}^{NIG}$ describes the option price sensitivity to changes in the tail heaviness of the distribution. $\text{Vega}_{\tilde{4}}^{NIG}$ measures the option price sensitivity to changes in the skewness of the distribution. We shall derive the expressions for the above greeks in the case of NIG-driven interest rate derivative.

We state the necessary lemmas for the computation of the greeks.

Lemma 4.3.1. Let P be the price of the zero-coupon bond driven by NIG process. Then, the Malliavin derivative of P is given by

$$DP = -\Bigg[\sigma\delta \sum_{t\leq u\leq T}\sum_{0\leq s\leq t}\left(\Delta\sqrt{IG(s)}e^{-a(u-s)}\right) + \sigma\delta\sum_{t\leq u\leq T}(\Delta\sqrt{IG(u)}) \\ -\sigma^2\delta^2\Bigg(\sum_{t\leq u\leq T}(\beta\delta\Delta IG(u) + \Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Bigg)\Bigg]P. \tag{4.3.6}$$

Proof. From equation (4.3.5), it follows that the Malliavin derivative

$$DP = D\exp\Bigg(-\Bigg(\Big[\frac{-r_0}{a}(e^{-aT}-e^{-at})+b\big(T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big)+\frac{\sigma\mathbf{w}}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big] \\ +\mathbf{w}\sigma[T-t] + \sigma\delta\sum_{t\leq u\leq T}\sum_{0\leq s\leq t}\left(\Delta\sqrt{IG(s)}e^{-a(u-s)}Z + \beta\delta\Delta IG(s)e^{-a(u-s)}\right) \\ +\sigma\delta\sum_{t\leq u\leq T}\left(\Delta\sqrt{IG(u)}Z + \beta\delta\Delta IG(u)\right) - \frac{\sigma^2\delta^2}{2}\Bigg(\sum_{t\leq u\leq T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)^2\Bigg)\Bigg)\Bigg)$$

$$= -\Big[\sigma\delta \sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big) + \sigma\delta \sum_{t\le u\le T}(\Delta\sqrt{IG(u)})$$

$$-\sigma^2\delta^2\Big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u) + \Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Big)\Big]P. \qquad \square$$

Lemma 4.3.2. Let P be the price of the zero-coupon bond driven by NIG process. Then, the Ornstein Uhlenbeck operator L on P is given by

$$LP = -\Big[\sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big) + \widetilde{K}^2 + \widetilde{K}Z\Big]P \tag{4.3.7}$$

where

$$\widetilde{K} = \sigma\delta \sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big) + \sigma\delta \sum_{t\le u\le T}(\Delta\sqrt{IG(u)})$$

$$-\sigma^2\delta^2\Big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u) + \Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Big). \tag{4.3.8}$$

Proof. The Malliavin derivative of equation (4.3.6) is given by

$$DDP = D\Big(-\Big[\sigma\delta \sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big) + \sigma\delta \sum_{t\le u\le T}(\Delta\sqrt{IG(u)})$$

$$-\sigma^2\delta^2\Big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u) + \Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Big)\Big]P\Big)$$

$$= -\Big[-\sigma^2\delta^2 \sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\Big]P + \Big(-\Big[\sigma\delta \sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big)$$

$$+\sigma\delta \sum_{t\le u\le T}(\Delta\sqrt{IG(u)}) - \sigma^2\delta^2\Big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u) + \Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Big)\Big]\Big)^2 P$$

$$= \sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)P + \Big[\sigma\delta \sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big)$$

$$+\sigma\delta \sum_{t\le u\le T}(\Delta\sqrt{IG(u)}) - \sigma^2\delta^2\Big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u) + \delta\Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Big)\Big]^2 P.$$

By equation (4.2.11) and Remark 4.2.2,

$$LP = -[DDP + \varphi DP] = -[DDP - ZDP].$$

Substituting DDP and equation (4.3.6) into LP yields

$$LP = -\Big[\sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)P + \Big[\sigma\delta \sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big)$$

$$+\sigma\delta\sum_{t\leq u\leq T}(\Delta\sqrt{IG(u)})-\sigma^2\delta^2\Bigg(\sum_{t\leq u\leq T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)\delta\Delta\sqrt{IG(u)}\Bigg)\Bigg]^2 P$$

$$+(-Z)\Bigg(-\Bigg[\sigma\delta\sum_{t\leq u\leq T}\sum_{0\leq s\leq t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big)+\sigma\delta\sum_{t\leq u\leq T}(\Delta\sqrt{IG(u)})$$

$$-\sigma^2\delta^2\Bigg(\sum_{t\leq u\leq T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Bigg)\Bigg]\Bigg)P\Bigg]$$

$$=-\Bigg[\sigma^2\delta^2\sum_{t\leq u\leq T}(\Delta\sqrt{IG(u)})^2+\Bigg[\sigma\delta\sum_{t\leq u\leq T}\sum_{0\leq s\leq t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big)$$

$$+\sigma\delta\sum_{t\leq u\leq T}(\Delta\sqrt{IG(u)})-\sigma^2\delta^2\Bigg(\sum_{t\leq u\leq T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Bigg)\Bigg]^2$$

$$+Z\Bigg[\sigma\delta\sum_{t\leq u\leq T}\sum_{0\leq s\leq t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big)+\sigma\delta\sum_{t\leq u\leq T}(\Delta\sqrt{IG(u)})$$

$$-\sigma^2\delta^2\Bigg(\sum_{t\leq u\leq T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Bigg)\Bigg]\Bigg]P. \qquad \square$$

Lemma 4.3.3

Let P be the price of the zero-coupon bond driven by NIG process and $\mathcal{M}(P)$ be its Malliavin covariance matrix. Then,

$$\mathcal{M}(P)^{-1}=\Bigg[\sigma\delta\sum_{t\leq u\leq T}\sum_{0\leq s\leq t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big)+\sigma\delta\sum_{t\leq u\leq T}(\Delta\sqrt{IG(u)})$$

$$-\sigma^2\delta^2\Bigg(\sum_{t\leq u\leq T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Bigg)\Bigg]^{-2}P^{-2} \tag{4.3.9}$$

with the assumption that

$$\sigma\neq 0,\ \delta\neq 0 \text{ and } P\neq 0.$$

Proof. From equation (4.3.6), it follows that

$\mathcal{M}(P)=\langle DP,DP\rangle=(DP\cdot DP)$

$$=\Bigg[\sigma\delta\sum_{t\leq u\leq T}\sum_{0\leq s\leq t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big)+\sigma\delta\sum_{t\leq u\leq T}(\Delta\sqrt{IG(u)})$$

$$-\sigma^2\delta^2\Bigg(\sum_{t\leq u\leq T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Bigg)\Bigg]^2P^2.$$

Hence,

$$\mathcal{M}(P)^{-1} = \Bigg(\Bigg[\sigma\delta \sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big) + \sigma\delta \sum_{t\le u\le T}(\Delta\sqrt{IG(u)})$$

$$-\sigma^2\delta^2\Big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u) + \Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Big)\Bigg]P\Bigg)^{-2}. \qquad \square$$

Lemma 4.3.4. Let P be the price of the zero-coupon bond driven by NIG process. Then, the Malliavin derivative of the inverse Malliavin covariance matrix of P is given by

$$D\mathcal{M}(P)^{-1} = \frac{2}{\widetilde{K}^3P^2}\Big[\widetilde{K}^2 + \sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)\Big] \tag{4.3.10}$$

where $\widetilde{K}$ is given by equation (4.3.8).

Proof. Applying Malliavin derivative to equation (4.3.8) gives

$$D\mathcal{M}(P)^{-1} = -2\Bigg(\Bigg[\sigma\delta \sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big) + \sigma\delta \sum_{t\le u\le T}(\Delta\sqrt{IG(u)})$$

$$-\sigma^2\delta^2\Big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u) + \Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Big)\Bigg]P\Bigg)^{-3}$$

$$\cdot\Bigg(\Bigg[\sigma\delta \sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big) + \sigma\delta \sum_{t\le u\le T}(\Delta\sqrt{IG(u)})$$

$$-\sigma^2\delta^2\Big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u) + \Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Big)\Bigg]DP$$

$$+PD\Bigg(\Bigg[\sigma\delta \sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big) + \sigma\delta \sum_{t\le u\le T}(\Delta\sqrt{IG(u)})$$

$$-\sigma^2\delta^2\Big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u) + \Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Big)\Bigg]\Bigg)\Bigg)$$

$$= -2\Bigg(\Bigg[\sigma\delta \sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big) + \sigma\delta \sum_{t\le u\le T}(\Delta\sqrt{IG(u)})$$

$$-\sigma^2\delta^2\Big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u) + \Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Big)\Bigg]P\Bigg)^{-3}$$

$$\cdot\Bigg[-\Bigg[\sigma\delta \sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big) + \sigma\delta \sum_{t\le u\le T}(\Delta\sqrt{IG(u)})$$

$$-\sigma^2\delta^2\Big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u) + \Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Big)\Bigg]^2 P$$

$$-\sigma^2\delta^2\Big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\Big)P\Big]$$

$$= 2\Bigg[\sigma\delta\sum_{t\le u\le T}\sum_{0\le s\le t}\Big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\Big)+\sigma\delta\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})$$

$$-\sigma^2\delta^2\Bigg(\sum_{t\le u\le T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Bigg)\Bigg]^{-3}P^{-2}$$

$$\cdot\Bigg[\Bigg[\sigma\delta\sum_{t\le u\le T}\sum_{0\le s\le t}\Big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\Big)+\sigma\delta\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})$$

$$-\sigma^2\delta^2\Bigg(\sum_{t\le u\le T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Bigg)\Bigg]^2+\sigma^2\delta^2\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\Bigg]$$

which yields the desired result. □

4.3.4 Computation of *delta* for NIG-driven interest rate derivatives

In this subsection, we compute the greek *delta* for interest rate derivative driven by NIG process. Let P be the price of the zero-coupon bond given by equation (4.3.5) and $\Phi(P)$ be the payoff. Then,

$$\triangle^{NIG} = \frac{\partial}{\partial r_0}[e^{-r_0T}\mathbb{E}(\Phi(P))] = -Te^{-r_0T}\mathbb{E}(\Phi(P)) + e^{-r_0T}\mathbb{E}\Big[\Phi'(P)\frac{\partial P}{\partial r_0}\Big]$$

$$= -Te^{-r_0T}\mathbb{E}(\Phi(P)) + e^{-r_0T}\mathbb{E}[\Phi(P)H_{NIG}(P,Q)]$$

where $Q = \dfrac{\partial P}{\partial r_0}$.

Next, Lemmas 4.3.5 - 4.3.8 will be stated in order to obtain the Malliavin weight $H_{NIG}(P,Q)$.

Lemma 4.3.5. Let P be the price of the zero-coupon bond driven by NIG process and $Q = \dfrac{\partial P}{\partial r_0}$. Then the following holds:

$$Q = \frac{1}{a}(e^{-aT} - e^{-at})P \tag{4.3.11}$$

and

$$DQ = -\frac{1}{a}(e^{-aT}-e^{-at})\Bigg(\sigma\delta\sum_{t\le u\le T}\sum_{0\le s\le t}\Big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\Big)+\sigma\delta\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})$$

$$-\sigma^2\delta^2\Big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)})\Big)P \qquad (4.3.12).$$

Proof. Applying partial derivative to equation (4.3.5) with respect to r_0, we obtain

$$Q=\frac{\partial}{\partial r_0}\exp\Bigg(-\Big([\frac{-r_0}{a}(e^{-aT}-e^{-at})+b\big(T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big)$$

$$+\frac{\sigma\mathbf{w}}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big]+\mathbf{w}\sigma[T-t]$$

$$+\sigma\delta\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}Z+\beta\delta\Delta IG(s)e^{-a(u-s)}\big)$$

$$+\sigma\delta\sum_{t\le u\le T}\big(\Delta\sqrt{IG(u)}Z+\beta\delta\Delta IG(u)\big)-\frac{\sigma^2\delta^2}{2}\big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)^2\big)\Big)\Bigg)$$

$$=\frac{1}{a}(e^{-aT}-e^{-at})P.$$

Hence, the Malliavin derivative

$$DQ=\frac{1}{a}(e^{-aT}-e^{-at})DP.$$

Substituting DP from equation (4.3.6) into the above equation yields

$$DQ=-\frac{1}{a}(e^{-aT}-e^{-at})\Bigg[\sigma\delta\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big)+\sigma\delta\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})$$

$$-\sigma^2\delta^2\Bigg(\sum_{t\le u\le T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Bigg)\Bigg]P. \qquad \square$$

Lemma 4.3.6. Let P be the price of the zero-coupon bond driven by NIG process and L be the Ornstein-Uhlenbeck operator on P. Then,

$$Q\mathcal{M}(P)^{-1}LP=-\frac{1}{a}(e^{-aT}-e^{-at})\Bigg[\frac{\sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)}{\widetilde{K}^2}+1+\frac{Z}{\widetilde{K}}\Bigg] \qquad (4.3.13)$$

where $\widetilde{K}$ is given by equation (4.3.8).

Proof. By equations (4.3.11), (4.3.9) and (4.3.7), we have

$$Q\mathcal{M}(P)^{-1}LP=\frac{1}{a}(e^{-aT}-e^{-at})P\cdot\Bigg[\sigma\delta\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big)$$

$$+\sigma\delta\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})-\sigma^2\delta^2\Bigg(\sum_{t\le u\le T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Bigg)\Bigg]^{-2}P^{-2}$$

$$\times\Big(-\Big[\sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)+\widetilde{K}^2+\widetilde{K}Z\Big]P\Big).$$

$$=-\frac{1}{a}(e^{-aT}-e^{-at})\sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)\widetilde{K}^{-2}$$

$$-\frac{1}{a}(e^{-aT}-e^{-at})-\frac{Z\big(\frac{1}{a}(e^{-aT}-e^{-at})\big)}{\widetilde{K}}. \qquad \square$$

Lemma 4.3.7. Given that P is the price of the zero-coupon bond driven by NIG process. Then,

$$\mathcal{M}(P)^{-1}\langle DP, DQ\rangle=\frac{1}{a}(e^{-aT}-e^{-at}). \tag{4.3.14}$$

Proof. From equations (4.3.9), (4.3.6) and (4.3.12), it follows that

$\mathcal{M}(P)^{-1}\langle DP, DQ\rangle$

$$=-\Big[\sigma\delta\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big)+\sigma\delta\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})$$

$$-\sigma^2\delta^2\Big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Big)\Big]P$$

$$\cdot\Big(\Big[\sigma\delta\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big)+\sigma\delta\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})$$

$$-\sigma^2\delta^2\Big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Big)\Big]P\Big)^{-2}$$

$$\times-\frac{1}{a}(e^{-aT}-e^{-at})\Big[\sigma\delta\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big)+\sigma\delta\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})$$

$$-\sigma^2\delta^2\Big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Big)\Big]P$$

$$=\frac{1}{a}(e^{-aT}-e^{-at}). \qquad \square$$

Lemma 4.3.8. Let P be the price of the zero-coupon bond driven by NIG process. Then,

$Q\langle DP, D\mathcal{M}(P)^{-1}\rangle=$

$$-2\big(\frac{1}{a}(e^{-aT}-e^{-at})\big)\Big[\frac{\sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)}{\widetilde{K}^2}+1\Big] \tag{4.3.15}$$

where $\widetilde{K}$ satisfies equation (4.3.8).

Proof. The result follows from Lemmas 4.3.1, 4.3.5 and 4.3.4;

$$Q\langle DP, D\mathcal{M}(P)^{-1}\rangle = \frac{1}{a}(e^{-aT} - e^{-at})P \times -\Bigg[\sigma\delta \sum_{t\leq u\leq T}\sum_{0\leq s\leq t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big)$$
$$+\sigma\delta \sum_{t\leq u\leq T}(\Delta\sqrt{IG(u)}) - \sigma^2\delta^2\Bigg(\sum_{t\leq u\leq T}(\beta\delta\Delta IG(u) + \Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Bigg)\Bigg]P$$
$$\cdot\frac{2}{\widetilde{K}^3P^2}\Bigg[\widetilde{K}^2 + \sigma^2\delta^2\big(\sum_{t\leq u\leq T}(\Delta\sqrt{IG(u)})^2\big)\Bigg]$$
$$= -2\big(\frac{1}{a}(e^{-aT} - e^{-at})\big)\Bigg[\sigma\delta \sum_{t\leq u\leq T}\sum_{0\leq s\leq t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big) + \sigma\delta \sum_{t\leq u\leq T}(\Delta\sqrt{IG(u)})$$
$$-\sigma^2\delta^2\Bigg(\sum_{t\leq u\leq T}(\beta\delta\Delta IG(u) + \Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Bigg)\Bigg]^{-2}$$
$$\cdot\Bigg[\widetilde{K}^2 + \sigma^2\delta^2\big(\sum_{t\leq u\leq T}(\Delta\sqrt{IG(u)})^2\big)\Bigg]$$
$$= -2\big(\frac{1}{a}(e^{-aT} - e^{-at})\big) - 2\big(\frac{1}{a}(e^{-aT} - e^{-at})\big)\widetilde{K}^{-2}\sigma^2\delta^2\big(\sum_{t\leq u\leq T}(\Delta\sqrt{IG(u)})^2\big). \quad \square$$

Theorem 4.3.1

Let P be the price of the zero-coupon bond driven by NIG process and $Q = \dfrac{\partial P}{\partial r_0}$, then

$$\triangle^{NIG} = -Te^{-r_0T}\mathbb{E}(\Phi(P))$$
$$+e^{-r_0T}\mathbb{E}\Bigg[\Phi(P)\frac{1}{a}(e^{-aT} - e^{-at})\Bigg(\frac{\sigma^2\delta^2\sum_{t\leq u\leq T}(\Delta\sqrt{IG(u)})^2}{\widetilde{K}^2} - \frac{Z}{\widetilde{K}}\Bigg)\Bigg]$$

where the Malliavin weight for the delta is given by

$$H_{NIG}(P,Q) = \frac{1}{a}(e^{-aT} - e^{-at})\Bigg(\frac{\sigma^2\delta^2\sum_{t\leq u\leq T}(\Delta\sqrt{IG(u)})^2}{\widetilde{K}^2} - \frac{Z}{\widetilde{K}}\Bigg)$$

and $\widetilde{K}$ is given by equation (4.3.8).

Proof. From equation (4.2.6),

$$\triangle^{NIG} = \frac{\partial}{\partial r_0}[e^{-r_0T}\mathbb{E}(\Phi(P))] = -Te^{-r_0T}\mathbb{E}(\Phi(P)) + e^{-r_0T}\mathbb{E}[\Phi(P)H_{NIG}(P,Q)].$$

Substituting equations (4.3.13), (4.3.14) and (4.3.15) into $H_{NIG}(P,Q)$ in equation (4.2.8) given by

$$H_{NIG}(P,Q) = Q\mathcal{M}(P)^{-1}LP - \mathcal{M}(P)^{-1}\langle DP, DQ\rangle - Q\langle DP, D\mathcal{M}(P)^{-1}\rangle,$$

we get

$$H_{NIG}(P,Q) = -\frac{1}{a}(e^{-aT} - e^{-at})\sigma^2\delta^2\Big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\Big)\widetilde{K}^{-2}$$

$$-\frac{1}{a}(e^{-aT} - e^{-at}) - \frac{Z\big(\frac{1}{a}(e^{-aT} - e^{-at})\big)}{\widetilde{K}} - \frac{1}{a}(e^{-aT} - e^{-at})$$

$$+2\Big(\frac{1}{a}(e^{-aT} - e^{-at})\Big) + 2\Big(\frac{\sigma^2}{a}(e^{-aT} - e^{-at})\Big)\widetilde{K}^{-2}\delta^2\Big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\Big)$$

$$= \frac{1}{a}(e^{-aT} - e^{-at})\sigma^2\delta^2\Big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\Big)\widetilde{K}^{-2} - \frac{Z\big(\frac{1}{a}(e^{-aT} - e^{-at})\big)}{\widetilde{K}}. \square$$

4.3.5 Computation of *gamma* for NIG-driven interest rate derivatives

In this subsection, we compute the greek '*gamma* Γ' for interest rate derivative driven by NIG process.

$$\begin{aligned}
\Gamma^{NIG} &= \frac{\partial^2}{\partial r_0^2}(e^{-r_0T}\mathbb{E}[\Phi(P)]) = \frac{\partial}{\partial r_0}\Big(-Te^{-r_0T}\mathbb{E}(\Phi(P)) + e^{-r_0T}\mathbb{E}[\Phi'(P)Q]\Big) \\
&= \frac{\partial}{\partial r_0}\Big(-Te^{-r_0T}\mathbb{E}(\Phi(P)) + e^{-r_0T}\mathbb{E}[\Phi(P)H_{NIG}(P,Q)]\Big) \\
&= T^2e^{-r_0T}\mathbb{E}[\Phi(P)] - Te^{-r_0T}\mathbb{E}[\Phi(P)H_{NIG}(P,Q)] \\
&\quad -Te^{-r_0T}\mathbb{E}[\Phi(P)H_{NIG}(P,Q)] + e^{-r_0T}\mathbb{E}[\Phi(P)H_{NIG}(P,QH_{NIG}(P,Q))] \\
&= T^2e^{-r_0T}\mathbb{E}[\Phi(P)] - 2Te^{-r_0T}\mathbb{E}[\Phi(P)H_{NIG}(P,Q)] \\
&\quad +e^{-r_0T}\mathbb{E}[\Phi(P)H_{NIG}(P,QH_{NIG}(P,Q))]
\end{aligned}$$

where $H_{NIG}(P,Q) = H_{NIG}\left(P, \frac{\partial P}{\partial r_0}\right)$ is given in Theorem 4.3.1.

Lemma 4.3.9

Let P be the price of the zero-coupon bond driven by NIG process. Then,

$$Q_\Gamma = \frac{\partial P}{\partial r_0}H_{NIG}(P,Q) = \frac{1}{a}(e^{-aT} - e^{-at})PH_{NIG}(P,Q) \tag{4.3.16}$$

and

$$DQ_\Gamma = -\frac{1}{a}(e^{-aT}-e^{-at})P\Big[H_{NIG}(P,Q)\widetilde{K} + \frac{\big(\frac{1}{a}(e^{-aT}-e^{-at})\big)}{\widetilde{K}}$$

$$-2\big(\frac{1}{a}(e^{-aT}-e^{-at})\big)\frac{\big[\sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)\big]^2}{\widetilde{K}^3} \tag{4.3.17}$$

$$+\frac{Z\big(\frac{1}{a}(e^{-aT}-e^{-at})\big)}{\widetilde{K}^2}\sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)\Big]$$

where $H_{NIG}(P,Q)$ is given in Theorem 4.3.1 and $\widetilde{K}$ is given by equation (4.3.8).

Proof. By partial derivative of P given by equation (4.3.5) with respect to r_0, it follows that

$$Q_\Gamma = \frac{\partial P}{\partial r_0}H_{NIG}(P,Q) = \frac{\partial P}{\partial r_0}H_{NIG}\big(P,\frac{\partial P}{\partial r_0}\big) = \frac{1}{a}(e^{-aT}-e^{-at})PH_{NIG}(P,Q)$$

Furthermore, the Malliavin derivative

$$DQ_\Gamma = H_{NIG}(P,Q)D\big(\frac{1}{a}(e^{-aT}-e^{-at})P\big) + \frac{1}{a}(e^{-aT}-e^{-at})PD\big(H_{NIG}(P,Q)\big)$$

$$= -\frac{1}{a}(e^{-aT}-e^{-at})H_{NIG}(P,Q)\Big[\sigma\delta\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}$$

$$+\sigma\delta\sum_{t\le u\le T}(\Delta\sqrt{IG(u)}) - \sigma^2\delta^2\Big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Big)\Big]P$$

$$+\frac{1}{a}(e^{-aT}-e^{-at})PD\big(H_{NIG}(P,Q)\big).$$

But

$$D(H_{NIG}(P,Q)) = -2\big(\frac{1}{a}(e^{-aT}-e^{-at})\big)\sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)$$

$$\cdot\Big[\sigma\delta\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big) + \sigma\delta\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})$$

$$-\sigma^2\delta^2\Big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Big)\Big]^{-3}$$

$$\times D\Big(\Big[\sigma\delta\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big) + \sigma\delta\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})$$

$$-\sigma^2\delta^2\Big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Big)\Big]\Big)$$

$$-\Big(\frac{1}{a}(e^{-aT}-e^{-at}\Big)\Big[\sigma\delta\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big)+\sigma\delta\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})$$

$$-\sigma^2\delta^2\Big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Big)\Big]$$

$$-\frac{1}{a}(e^{-aT}-e^{-at})Z\cdot D\Big(\Big[\sigma\delta\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big)+\sigma\delta\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})$$

$$-\sigma^2\delta^2\Big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Big)\Big]\Big)\Big)$$

$$\cdot\Big[\sigma\delta\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big)+\sigma\delta\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})$$

$$-\sigma^2\delta^2\Big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Big)\Big]^{-2}$$

$$=2\big(\frac{1}{a}(e^{-aT}-e^{-at})\big)\frac{\big[\sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)\big]^2}{\widetilde{K}^3}-\frac{\big(\frac{1}{a}(e^{-aT}-e^{-at})\big)}{\widetilde{K}}$$

$$-\frac{Z\big(\frac{1}{a}(e^{-aT}-e^{-at})\big)}{\widetilde{K}^2}\sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big).$$

Hence,

$$DQ_\Gamma=-\frac{1}{a}(e^{-aT}-e^{-at})H_{NIG}(P,Q)\widetilde{K}P$$

$$+2\big(\frac{1}{a}(e^{-aT}-e^{-at})\big)^2\frac{\big[\sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)\big]^2}{\widetilde{K}^3}$$

$$-\frac{\big(\frac{1}{a}(e^{-aT}-e^{-at})\big)^2}{\widetilde{K}}-\frac{Z\big(\frac{1}{a}(e^{-aT}-e^{-at})\big)^2}{\widetilde{K}^2}\sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)\Big)P$$

where $\widetilde{K}$ is given by equation (4.3.8) □

Lemma 4.3.10

Suppose that P is the price of the zero-coupon bond driven by NIG process. Then, the following results hold:

(i) $Q_\Gamma\mathcal{M}(P)^{-1}LP=$

$$-\big(\frac{1}{a}(e^{-aT}-e^{-at})\big)H_{NIG}(P,Q)\Big[\frac{\sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)}{\widetilde{K}^2}+1+\frac{Z}{\widetilde{K}}\Big]. \tag{4.3.18}$$

(ii) $\mathcal{M}(P)^{-1}\langle DP, DQ_\Gamma\rangle =$

$$\Big(\frac{1}{a}(e^{-aT}-e^{-at})\Big)^2\Bigg[\frac{1}{\widetilde{K}^2}-\frac{2\big[\sigma^2\delta^2\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big]^2}{\widetilde{K}^4}$$

$$+\frac{Z}{\widetilde{K}^3}\sigma^2\delta^2\Big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\Big)\Bigg]+\frac{1}{a}(e^{-aT}-e^{-at})H_{NIG}(P,Q). \quad (4.3.19)$$

(iii) $Q_\Gamma\langle DP, D\mathcal{M}(P)^{-1}\rangle =$

$$-2\Big(\frac{1}{a}(e^{-aT}-e^{-at})\Big)H_{NIG}(P,Q)\Bigg[\frac{\sigma^2\delta^2}{\widetilde{K}^2}\Big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\Big)-1\Bigg]. \quad (4.3.20)$$

Proof.

(i) From equations (4.3.16), (4.3.9) and (4.3.7), it follows that

$Q_\Gamma\mathcal{M}(P)^{-1}LP =$

$$\frac{1}{a}(e^{-aT}-e^{-at})H_{NIG}(P,Q)P\times-\big[\sigma^2\delta^2\Big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\Big)+\widetilde{K}^2+\widetilde{K}Z\big]P$$

$$\cdot\Bigg[\sigma\delta\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big)+\sigma\delta\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})$$

$$-\sigma^2\delta^2\Bigg(\sum_{t\le u\le T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Bigg)\Bigg]^{-2}P^{-2}$$

$$=-\frac{\sigma^2}{a}(e^{-aT}-e^{-at})\frac{\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)}{\widetilde{K}^2}H_{NIG}(P,Q)$$

$$-\frac{1}{a}(e^{-aT}-e^{-at})H_{NIG}(P,Q)-\frac{Z\big(\frac{1}{a}(e^{-aT}-e^{-at})\big)}{\widetilde{K}}H_{NIG}(P,Q),$$

which is equation (4.3.18).

(ii) From equations (4.3.9), (4.3.6) and (4.3.17), we get

$\mathcal{M}(P)^{-1}\langle DP, DQ_\Gamma\rangle =$

$$-\Bigg[\sigma\delta\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big)+\sigma\delta\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})$$

$$-\sigma^2\delta^2\Bigg(\sum_{t\le u\le T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Bigg)\Bigg]^{-1}P^{-1}$$

$$\times\Bigg(-\frac{1}{a}(e^{-aT}-e^{-at})H_{NIG}(P,Q)\widetilde{K}+2\Big(\frac{1}{a}(e^{-aT}-e^{-at})\Big)^2-\frac{\big(\frac{1}{a}(e^{-aT}-e^{-at})\big)^2}{\widetilde{K}}$$

$$\cdot\frac{\big[\sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)\big]^2}{\widetilde{K}^3}-\frac{Z\big(\frac{1}{a}(e^{-aT}-e^{-at})\big)^2}{\widetilde{K}^2}\sigma^2\delta^2\Big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\Big)\Bigg)P$$

$$=\frac{1}{a}(e^{-aT}-e^{-at})H_{NIG}(P,Q)+\frac{\big(\frac{1}{a}(e^{-aT}-e^{-at})\big)^2}{\widetilde{K}^2}$$

$$-2\Big(\frac{1}{a}(e^{-aT}-e^{-at})\Big)^2\frac{\big[\sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)\big]^2}{\widetilde{K}^4}$$

$$+\frac{Z\big(\frac{1}{a}(e^{-aT}-e^{-at})\big)^2}{\widetilde{K}^3}\sigma^2\delta^2\Big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\Big).$$

(iii) From equations (4.3.16), (4.3.6) and (4.3.10), it follows that

$$Q_\Gamma\langle DP, D\mathcal{M}(P)^{-1}\rangle=-\frac{1}{a}(e^{-aT}-e^{-at})PH_{NIG}(P,Q)$$

$$\cdot\Bigg[\sigma\delta\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big)+\sigma\delta\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})$$

$$-\sigma^2\delta^2\Bigg(\sum_{t\le u\le T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Bigg)\Bigg]P$$

$$\cdot\frac{2}{\widetilde{K}^3P^2}\big[\widetilde{K}^2+\sigma^2\delta^2\Big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\Big)\big]$$

$$=\frac{-2\big(\frac{1}{a}(e^{-aT}-e^{-at})\big)}{\widetilde{K}^2}H_{NIG}(P,Q)\big[\widetilde{K}^2+\sigma^2\delta^2\Big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\Big)\big]$$

$$=-\frac{2\big(\frac{1}{a}(e^{-aT}-e^{-at})\big)}{\widetilde{K}^2}\sigma^2\delta^2\Big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\Big)H_{NIG}(P,Q)$$

$$-2\Big(\frac{1}{a}(e^{-aT}-e^{-at})\Big)H_{NIG}(P,Q).\ \Box$$

Theorem 4.3.2

Let P be the price of the zero-coupon bond driven by NIG process, then

$$\Gamma_{NIG}=T^2e^{-r_0T}\mathbb{E}[\Phi(P)]-2Te^{-r_0T}\mathbb{E}[\Phi(P)H_{NIG}(P,Q)]$$

$$+e^{-r_0T}\mathbb{E}[\Phi(P)H_{NIG}(P,QH_{NIG}(P,Q))]$$

where $H_{NIG}(P,Q)=H_{NIG}\left(P,\dfrac{\partial P}{\partial r_0}\right)$ is given by Theorem 4.3.1, and the following holds for the Malliavin weight:

$$H_{NIG}(P,Q_\Gamma)=\frac{1}{a}(e^{-aT}-e^{-at})H_{NIG}(P,Q)\Bigg(\frac{\sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)}{\widetilde{K}^2}-\frac{Z}{\widetilde{K}}\Bigg)$$

$$+\Big(\frac{1}{a}(e^{-aT}-e^{-at})\Big)^2\Bigg(\frac{\big[2\sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)\big]^2}{\widetilde{K}^4}$$

$$-\frac{1}{\widetilde{K}^2}-\frac{Z}{\widetilde{K}^3}\sigma^2\delta^2\Big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\Big)\Bigg).$$

Proof. From equation (4.2.6), it follows that

$$\Gamma_{NIG}=\frac{\partial^2}{\partial r_0^2}\mathbb{V}=\frac{\partial^2}{\partial r_0^2}(e^{-r_0T}\mathbb{E}[\Phi(P)])$$

$$=T^2e^{-r_0T}\mathbb{E}[\Phi(P)]-2Te^{-r_0T}\mathbb{E}[\Phi(P)H_{NIG}(P,Q)]+e^{-r_0T}\mathbb{E}[\Phi(P)H_{NIG}(P,QH_{NIG}(P,Q))].$$

Substituting equations (4.3.18), (4.3.19) and (4.3.20) into $H_{NIG}(P,Q_\Gamma)$, we get

$$H_{NIG}(P,Q_\Gamma)=Q_\Gamma\mathcal{M}(P)^{-1}LP-\mathcal{M}(P)^{-1}\langle DP,DQ_\Gamma\rangle-Q_\Gamma\langle DP,D\mathcal{M}(P)^{-1}\rangle$$

$$=-\frac{\sigma^2}{a}(e^{-aT}-e^{-at})\frac{\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)}{\widetilde{K}^2}H_{NIG}(P,Q)$$

$$-\frac{1}{a}(e^{-aT}-e^{-at})H_{NIG}(P,Q)-\frac{Z\big(\frac{1}{a}(e^{-aT}-e^{-at})\big)}{\widetilde{K}}H_{NIG}(P,Q)$$

$$-\frac{1}{a}(e^{-aT}-e^{-at})H_{NIG}(P,Q)-\frac{\big(\frac{1}{a}(e^{-aT}-e^{-at})\big)^2}{\widetilde{K}^2}$$

$$+2\Big(\frac{1}{a}(e^{-aT}-e^{-at})\Big)^2\frac{\big[\sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)\big]^2}{\widetilde{K}^4}$$

$$-\frac{Z\big(\frac{1}{a}(e^{-aT}-e^{-at})\big)^2}{\widetilde{K}^3}\sigma^2\delta^2\Big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\Big)$$

$$+\frac{2\big(\frac{1}{a}(e^{-aT}-e^{-at})\big)}{\widetilde{K}^2}\sigma^2\delta^2\Big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\Big)H_{NIG}(P,Q)$$

$$+2\Big(\frac{1}{a}(e^{-aT}-e^{-at})\Big)H_{NIG}(P,Q)$$

$$=\frac{\sigma^2}{a}(e^{-aT}-e^{-at})\frac{\delta^2\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2}{\widetilde{K}^2}H_{NIG}(P,Q)$$

$$-\frac{Z\big(\frac{1}{a}(e^{-aT}-e^{-at})\big)}{\widetilde{K}}H_{NIG}(P,Q)-\frac{\big(\frac{1}{a}(e^{-aT}-e^{-at})\big)^2}{\widetilde{K}^2}$$

$$+2\Big(\frac{1}{a}(e^{-aT}-e^{-at})\Big)^2\frac{\big[\sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)\big]^2}{\widetilde{K}^4}$$

$$-\frac{Z\big(\frac{1}{a}(e^{-aT}-e^{-at})\big)^2}{\widetilde{K}^3}\sigma^2\delta^2\Big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\Big).\qquad\square$$

4.3.6 Computation of *vega* for NIG-driven interest rate derivatives

In this subsection, we compute the greek '*vega* $\mathcal{V}$' for the interest rate derivative driven by NIG process.

$$\mathcal{V}_{NIG} = \frac{\partial}{\partial\sigma} e^{r_0 T}\mathbb{E}[\Phi(P)] = e^{-r_0 T}\mathbb{E}\Big[\Phi'(P)\frac{\partial P}{\partial \sigma}\Big] = e^{-r_0 T}\mathbb{E}\Big[\Phi(P) H_{NIG}\Big(P, \frac{\partial P}{\partial\sigma}\Big)\Big].$$

Lemma 4.3.11. Suppose that P is the price of the zero-coupon bond driven by NIG process and $Q_\sigma = \dfrac{\partial P}{\partial\sigma}$. Then,

$$\begin{aligned} Q_\sigma &= -\Big[\frac{\mathbf{w}}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big] + \mathbf{w}[T-t] \\ &\quad +\delta \sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}Z + \beta\delta\Delta IG(s)e^{-a(u-s)}\big) \\ &\quad +\delta\sum_{t\le u\le T}\big(\Delta\sqrt{IG(u)}Z + \beta\delta\Delta IG(u)\big) \\ &\quad -\sigma\delta^2\sum_{t\le u\le T}(\beta\delta\Delta IG(u) + \Delta\sqrt{IG(u)}Z)^2\Big]P \end{aligned} \tag{4.3.21}$$

and

$$\begin{aligned} DQ_\sigma &= -\Big[\delta\sum_{t\le u\le T}\sum_{0\le s\le t}(\Delta\sqrt{IG(s)}e^{-a(u-s)}) + \delta\sum_{t\le u\le T}(\Delta\sqrt{IG(u)}) \\ &\quad -2\sigma\delta^2\Big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u) + \Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Big)\Big]P + \widetilde{\Lambda}\widetilde{K}P \end{aligned} \tag{4.3.22}$$

where $\widetilde{K}$ is given by equation (4.3.8) and

$$\begin{aligned} \widetilde{\Lambda} &= \frac{\mathbf{w}}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big] + \mathbf{w}[T-t] + \delta\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}Z \\ &\quad +\beta\delta\Delta IG(s)e^{-a(u-s)}\big) + \delta\sum_{t\le u\le T}\big(\Delta\sqrt{IG(u)}Z + \beta\delta\Delta IG(u)\big) \\ &\quad -\sigma\delta^2\Big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u) + \Delta\sqrt{IG(u)}Z)^2\Big). \end{aligned} \tag{4.3.23}$$

Proof. Applying partial derivative to equation (4.3.5), we have

$$Q_\sigma = \frac{\partial P}{\partial\sigma} = \frac{\partial P}{\partial \sigma}$$

$$= -\Big[\frac{\mathbf{w}}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big] + \mathbf{w}[T-t] + \delta\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}Z$$

$$+\beta\delta\Delta IG(s)e^{-a(u-s)}\Big) + \delta \sum_{t\le u\le T} \big(\Delta\sqrt{IG(u)}Z + \beta\delta\Delta IG(u)\big)$$

$$-\sigma\delta^2\Big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u) + \Delta\sqrt{IG(u)}Z)^2\Big)\Big]P.$$

Hence, the Malliavin derivative

$$DQ_\sigma = -\Big[\delta \sum_{t\le u\le T}\sum_{0\le s\le t}(\Delta\sqrt{IG(s)}e^{-a(u-s)}) + \delta\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})$$

$$-\sigma\delta^2\Big(2\sum_{t\le u\le T}(\beta\delta\Delta IG(u) + \Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Big)\Big]P$$

$$+\Big(-\Big[\frac{\mathbf{w}}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big]+\mathbf{w}[T-t]+\delta\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}Z$$

$$+\beta\delta\Delta IG(s)e^{-a(u-s)}\big) + \delta\sum_{t\le u\le T}\big(\Delta\sqrt{IG(u)}Z + \beta\delta\Delta IG(u)\big)$$

$$-\sigma\delta^2\Big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)^2\Big)\Big]\cdot\Big(-\Big[\sigma\delta\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big)$$

$$+\sigma\delta\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})-\sigma^2\delta^2\Big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Big)\Big]\Big)\Big)P$$

$$= -\Big[\delta\sum_{t\le u\le T}\sum_{0\le s\le t}(\Delta\sqrt{IG(s)}e^{-a(u-s)}) + \delta\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})$$

$$-\sigma\delta^2\Big(2\sum_{t\le u\le T}(\beta\delta\Delta IG(u) + \Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Big)\Big]P$$

$$+\Big(\Big[\frac{\mathbf{w}}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big]+\mathbf{w}[T-t]+\delta\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}Z$$

$$+\beta\delta\Delta IG(s)e^{-a(u-s)}\big) + \delta\sum_{t\le u\le T}\big(\Delta\sqrt{IG(u)}Z + \beta\delta\Delta IG(u)\big)$$

$$-\sigma\delta^2\Big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u) + \Delta\sqrt{IG(u)}Z)^2\Big)\Big]$$

$$\cdot\Big[\sigma\delta\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big) + \sigma\delta\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})\big)$$

$$-\sigma^2\delta^2\Big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u) + \Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Big)\Big]\Big)P. \qquad \square$$

Lemma 4.3.12. Let P be the price of the zero-coupon bond driven by NIG process. The following holds for the sensitivity with respect to σ:

$$Q_\sigma \mathcal{M}(P)^{-1}LP = \widetilde{\Lambda}\Big[\frac{\sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)}{\widetilde{K}^2} + 1 + \frac{Z}{\widetilde{K}}\Big] \tag{4.3.24}$$

where $\widetilde{\Lambda}$ and $\widetilde{K}$ are given by equations (4.3.23) and (4.3.8), respectively.

Proof.

From equations (4.3.21), (4.3.9) and (4.3.7), it is obvious that

$Q_\sigma \mathcal{M}(P)^{-1}LP$

$$= -\Bigg[\frac{\mathbf{w}}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big]+\mathbf{w}[T-t]+\delta\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}Z$$

$$+\beta\delta\Delta IG(s)e^{-a(u-s)}\big)+\delta\sum_{t\le u\le T}\big(\Delta\sqrt{IG(u)}Z+\beta\delta\Delta IG(u)\big)$$

$$-\sigma\delta^2\Bigg(\sum_{t\le u\le T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)^2\Bigg)\Bigg]P\times\Bigg[\sigma\delta\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big)$$

$$+\sigma\delta\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})-\sigma^2\delta^2\Bigg(\sum_{t\le u\le T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Bigg)\Bigg]^{-2}P^{-2}$$

$$\times-\Bigg[\sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)+\widetilde{K}^2+\widetilde{K}Z\Bigg]P$$

$$=\widetilde{\Lambda}\frac{\sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)}{\widetilde{K}^2}+\widetilde{\Lambda}+\frac{\widetilde{\Lambda}Z}{\widetilde{K}}. \qquad \square$$

Lemma 4.3.13. Let P be the price of the zero-coupon bond driven by NIG process. Then

$\mathcal{M}(P)^{-1}\langle DP, DQ_\sigma\rangle =$

$$\widetilde{K}^{-1}\Bigg[\delta\sum_{t\le u\le T}\sum_{0\le s\le t}(\Delta\sqrt{IG(s)}e^{-a(u-s)})+\delta\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})$$

$$-2\sigma\delta^2\Bigg(\sum_{t\le u\le T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Bigg)\Bigg]-\widetilde{\Lambda} \qquad (4.3.25)$$

where $\widetilde{\Lambda}$ and $\widetilde{K}$ are given by equations (4.3.23) and (4.3.8), respectively.

Proof. From equations (4.3.9), (4.3.6) and (4.3.15), it follows that

$\mathcal{M}(P)^{-1}\langle DP, DQ_\sigma\rangle =$

$$-\Bigg[\sigma\delta\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big)+\sigma\delta\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})$$

$$-\sigma^2\delta^2\Bigg(\sum_{t\le u\le T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Bigg)\Bigg]^{-1}P^{-1}$$

$$\times\Bigg(-\Bigg[\delta\sum_{t\le u\le T}\sum_{0\le s\le t}(\Delta\sqrt{IG(s)}e^{-a(u-s)})+\delta\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})$$

$$-\sigma\delta^2\Big(2\sum_{t\le u\le T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Big)\Big]P$$

$$+\Bigg(\Big[\frac{\mathbf{w}}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big]+\mathbf{w}[T-t]+\delta\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}Z$$

$$+\beta\delta\Delta IG(s)e^{-a(u-s)}\big)+\delta\sum_{t\le u\le T}\big(\Delta\sqrt{IG(u)}Z+\beta\delta\Delta IG(u)\big)$$

$$-\sigma\delta^2\Big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)^2\Big)\Big]\cdot\Big[\sigma\delta\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big)$$

$$+\sigma\delta\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})-\sigma^2\delta^2\Big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Big)\Big]\Bigg)P\Bigg)$$

$$=\widetilde{K}^{-1}\Big[\Big[\delta\sum_{t\le u\le T}\sum_{0\le s\le t}(\Delta\sqrt{IG(s)}e^{-a(u-s)})+\delta\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})$$

$$-\sigma\delta^2\Big(2\sum_{t\le u\le T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Big)\Big]$$

$$-\Bigg(\Big[\frac{\mathbf{w}}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big]+\mathbf{w}[T-t]+\delta\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}Z$$

$$+\beta\delta\Delta IG(s)e^{-a(u-s)}\big)+\delta\sum_{t\le u\le T}\big(\Delta\sqrt{IG(u)}Z+\beta\delta\Delta IG(u)\big)$$

$$-\sigma\delta^2\Big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)^2\Big)\Big]$$

$$\cdot\Big[\sigma\delta\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big)+\sigma\delta\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})$$

$$-\sigma^2\delta^2\Big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Big)\Big]\Bigg)\Big]. \qquad \square$$

Lemma 4.3.14

Let P be the price of the zero-coupon bond driven by NIG process. Then, the following holds:

$$Q_\sigma\langle DP, D\mathcal{M}(P)^{-1}\rangle = 2\widetilde{\Lambda}\Big[1+\frac{\sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)}{\widetilde{K}^2}\Big] \qquad (4.3.26)$$

where $\widetilde{\Lambda}$ and $\widetilde{K}$ are given by equations (4.3.23) and (4.3.8), respectively.

Proof. From equations (4.3.21), (4.3.6) and (4.3.10), it follows that

$Q_\sigma\langle DP, D\mathcal{M}(P)^{-1}\rangle =$

$$-\Big[\frac{\mathbf{w}}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big]+\mathbf{w}[T-t]+\delta\sum_{t\le u\le T}\sum_{0\le s\le t}(\Delta\sqrt{IG(s)}e^{-a(u-s)}Z$$

$$+\beta\delta\Delta IG(s)e^{-a(u-s)}\Big) + \delta \sum_{t\le u\le T} \big(\Delta\sqrt{IG(u)}Z + \beta\delta\Delta IG(u)\big)$$

$$-\sigma\delta^2\bigg(\sum_{t\le u\le T}(\beta\delta\Delta IG(u) + \Delta\sqrt{IG(u)}Z)^2\bigg)\bigg]P\times - \bigg[\sigma\delta\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big)$$

$$+\sigma\delta\sum_{t\le u\le T}(\Delta\sqrt{IG(u)}) - \sigma^2\delta^2\bigg(\sum_{t\le u\le T}(\beta\delta\Delta IG(u) + \Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\bigg)\bigg]P$$

$$\cdot\frac{2}{\widetilde{K}^3P^2}\bigg[\widetilde{K}^2 + \sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)\bigg]$$

$$= 2\widetilde{\Lambda} + 2\widetilde{\Lambda}\frac{\sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)}{\widetilde{K}^2}. \qquad \square$$

Theorem 4.3.3. Let P be the price of the zero-coupon bond driven by NIG process. Then, the Malliavin weight for the sensitivity with respect to volatility is given by

$$H_{NIG}(P, Q_\sigma) = \frac{\widetilde{\Lambda}Z}{\widetilde{K}} - \widetilde{\Lambda}\frac{\sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)}{\widetilde{K}^2}$$

$$-\frac{1}{\widetilde{K}}\bigg[\delta\sum_{t\le u\le T}\sum_{0\le s\le t}(\Delta\sqrt{IG(s)}e^{-a(u-s)}) + \delta\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})$$

$$-2\sigma\delta^2\bigg(\sum_{t\le u\le T}(\beta\delta\Delta IG(u) + \Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\bigg)\bigg]$$

where $\widetilde{\Lambda}$ and $\widetilde{K}$ are given by equations (4.3.23) and (4.3.8), respectively.

Proof. Substituting equations (4.3.24), (4.3.25) and (4.3.26) into

$$H_{NIG}(P, Q_\sigma) = Q_\sigma\mathcal{M}(P)^{-1}LP - \mathcal{M}(P)^{-1}\langle DP, DQ_\sigma\rangle - Q_\sigma\langle DP, D\mathcal{M}(P)^{-1}\rangle,$$

the Malliavin weight is obtained. $\square$

4.3.7 Computation of *Theta* for NIG-driven interest rate derivatives

In this subsection, we compute the greek *Theta* for the interest rate derivative driven by NIG process. Let P be the price of the zero-coupon bond driven by NIG process as given by equation (4.3.5). Then, the sensitivity with respect to T is given by

$$\Theta^{NIG} = \frac{\partial}{\partial T}e^{-r_0T}\mathbb{E}[\Phi(P)] = -r_0e^{-r_0T}\mathbb{E}[\Phi(P)] + e^{-r_0T}\mathbb{E}[\Phi(P)H_{NIG}(P, Q_T)]$$

where $H_{NIG}(P, Q_T) = H_{NIG}\left(P, \frac{\partial P}{\partial T}\right)$ is the Malliavin weight to be determined.

Lemma 4.3.15. Let P be the price of a zero-coupon bond driven by NIG process and $Q_T = \frac{\partial P}{\partial T}$. Then,

$$Q_T = -\left(r_0 e^{-aT} + b(1 - e^{-aT}) + \frac{\sigma \mathbf{w}}{a}(1 - e^{-aT}) + \mathbf{w}\sigma\right)P \tag{4.3.27}$$

and

$$DQ_T = \big(r_0 e^{-aT} + b(1 - e^{-aT}) + \frac{\sigma \mathbf{w}}{a}(1 - e^{-aT}) + \mathbf{w}\sigma\big)\widetilde{K}P \tag{4.3.28}$$

where $\widetilde{K}$ is given by equation (4.3.8).

Proof. Applying partial derivative to equation (4.3.5) yields

$$Q_T = \frac{\partial P}{\partial T} = \frac{\partial}{\partial T}\exp\Bigg(- \Bigg(\Big[\frac{-r_0}{a}(e^{-aT} - e^{-at}) + b\big(T - t + \frac{1}{a}(e^{-aT} - e^{-at})\big)$$

$$+\frac{\sigma \mathbf{w}}{a}\big[T - t + \frac{1}{a}(e^{-aT} - e^{-at})\big] + \mathbf{w}\sigma[T - t]$$

$$+\sigma\delta \sum_{t \le u \le T}\sum_{0 \le s \le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}Z + \beta\delta\Delta IG(s)e^{-a(u-s)}\big)$$

$$+\sigma\delta \sum_{t \le u \le T}(\Delta\sqrt{IG(u)}Z + \beta\delta\Delta IG(u)) - \frac{\sigma^2\delta^2}{2}\Bigg(\sum_{t \le u \le T}(\beta\delta\Delta IG(u) + \Delta\sqrt{IG(u)}Z)^2\Bigg)\Bigg)\Bigg)$$

$$= -\left(r_0 e^{-aT} + b(1 - e^{-aT}) + \frac{\sigma \mathbf{w}}{a}(1 - e^{-aT}) + \mathbf{w}\sigma\right)P.$$

Hence, the Malliavin derivative

$$DQ_T = -\left(r_0 e^{-aT} + b(1 - e^{-aT}) + \frac{\sigma \mathbf{w}}{a}(1 - e^{-aT}) + \mathbf{w}\sigma\right)DP$$

$$= -\left(r_0 e^{-aT} + b(1 - e^{-aT}) + \frac{\sigma \mathbf{w}}{a}(1 - e^{-aT}) + \mathbf{w}\sigma\right)$$

$$\times - \Bigg[\sigma\delta \sum_{t \le u \le T}\sum_{0 \le s \le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big) + \sigma\delta\sum_{t \le u \le T}(\delta\Delta\sqrt{IG(u)})$$

$$-\sigma^2\delta^2\Bigg(\sum_{t \le u \le T}(\beta\delta\Delta IG(u) + \Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Bigg)\Bigg]P$$

$$= \left(r_0 e^{-aT} + b(1 - e^{-aT}) + \frac{\sigma \mathbf{w}}{a}(1 - e^{-aT}) + \mathbf{w}\sigma\right)$$

$$\cdot\Bigg[\sigma\delta \sum_{t \le u \le T}\sum_{0 \le s \le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big) + \sigma\delta\sum_{t \le u \le T}(\Delta\sqrt{IG(u)})$$

$$-\sigma^2\delta^2\Big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Big)\Big]P. \qquad \square$$

Lemma 4.3.16. Let P be the price of the zero-coupon bond driven by NIG process and $Q_T = \dfrac{\partial P}{\partial T}$. Then,

$$Q_T\mathcal{M}(P)^{-1}LP = \Big(r_0e^{-aT}+b(1-e^{-aT})+\frac{\sigma\mathbf{w}}{a}(1-e^{-aT})+\mathbf{w}\sigma\Big)\Big[1+\frac{Z}{\widetilde{K}}$$
$$+\frac{\sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)}{\widetilde{K}^2}\Big] \tag{4.3.29}$$

where $\widetilde{K}$ is given by equation (4.3.8).

Proof. From equations (4.3.27), (4.3.9) and (4.3.7), it follows that

$$Q_T\mathcal{M}(P)^{-1}LP = -\Big(r_0e^{-aT}+b(1-e^{-aT})+\frac{\sigma\mathbf{w}}{a}(1-e^{-aT})+\mathbf{w}\sigma\Big)P$$
$$\cdot\Big[\sigma\delta\sum_{t\le u\le T}\sum_{0\le s\le t}(\Delta\sqrt{IG(s)}e^{-a(u-s)})+\sigma\delta\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})$$
$$-\sigma^2\delta^2\Big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Big)\Big]^{-2}P^{-2}$$
$$\times-\Big[\sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)+\widetilde{K}^2+\widetilde{K}Z\Big]P$$
$$=\Big(r_0e^{-aT}+b(1-e^{-aT})+\frac{\sigma\mathbf{w}}{a}(1-e^{-aT})+\mathbf{w}\sigma\Big)\sigma^2$$
$$\cdot\frac{\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)}{\widetilde{K}^2}+\Big(r_0e^{-aT}+b(1-e^{-aT})$$
$$+\frac{\sigma\mathbf{w}}{a}(1-e^{-aT})+\mathbf{w}\sigma\Big)+\frac{Z}{\widetilde{K}}\Big(r_0e^{-aT}+b(1-e^{-aT})+\frac{\sigma\mathbf{w}}{a}(1-e^{-aT})+\mathbf{w}\sigma\Big). \qquad \square$$

Lemma 4.3.17. Let P be the price of the zero-coupon bond driven by NIG process and $Q_T = \dfrac{\partial P}{\partial T}$. Then,

$$\mathcal{M}(P)^{-1}\langle DP, DQ_T\rangle = -\big(r_0e^{-aT}+b(1-e^{-aT})+\frac{\sigma\mathbf{w}}{a}(1-e^{-aT})+\mathbf{w}\sigma\big). \tag{4.3.30}$$

Proof. By equations (4.3.9), (4.3.6) and (4.3.28), we get

$$\mathcal{M}(P)^{-1}\langle DP, DQ_T\rangle = \Big[\sigma\delta\sum_{t\le u\le T}\sum_{0\le s\le t}(\Delta\sqrt{IG(s)}e^{-a(u-s)})+\sigma\delta\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})$$
$$-\sigma^2\delta^2\Big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Big)\Big]^{-1}P^{-1}$$

$$\cdot\Big(r_0e^{-aT}+b(1-e^{-aT})+\frac{\sigma\mathbf{w}}{a}(1-e^{-aT})+\mathbf{w}\sigma\Big)$$

$$\cdot\Big[\sigma\delta\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big)+\sigma\delta\sum_{0\le u\le T}(\Delta\sqrt{IG(u)})$$

$$-\sigma^2\delta^2\Big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Big)\Big]P$$

$$=-\Big(r_0e^{-aT}+b(1-e^{-aT})+\frac{\sigma\mathbf{w}}{a}(1-e^{-aT})+\mathbf{w}\sigma\Big). \qquad \square$$

Lemma 4.3.18. Let P be the price of the zero-coupon bond driven by NIG process and $Q_T=\dfrac{\partial P}{\partial T}$. Then,

$$Q_T\langle DP,D\mathcal{M}(P)^{-1}\rangle=2\big(r_0e^{-aT}+b(1-e^{-aT})+\frac{\sigma\mathbf{w}}{a}(1-e^{-aT})$$

$$+\mathbf{w}\sigma\big)\Big[1+\frac{\sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)}{\widetilde{K}^2}\Big] \qquad (4.3.31)$$

where $\widetilde{K}$ is given by equation (4.3.8).

Proof. By equations (4.3.27), (4.3.6) and (4.3.10), we get

$$Q_T\langle DP,D\mathcal{M}(P)^{-1}\rangle=-\Big(r_0e^{-aT}+b(1-e^{-aT})+\frac{\sigma\mathbf{w}}{a}(1-e^{-aT})+\mathbf{w}\sigma\Big)P$$

$$\times-\Big[\sigma\delta\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big)+\sigma\delta\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})$$

$$-\sigma^2\delta^2\Big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Big)\Big]P$$

$$\cdot\frac{2}{\widetilde{K}^3P^2}\Big[\widetilde{K}^2+\sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)\Big]$$

$$=\frac{2}{\widetilde{K}^2}\big(r_0e^{-aT}+b(1-e^{-aT})+\frac{\sigma\mathbf{w}}{a}(1-e^{-aT})+\mathbf{w}\sigma\big)$$

$$\cdot\Big[\widetilde{K}^2+\sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)\Big]$$

$$=2\Big(r_0e^{-aT}+b(1-e^{-aT})+\frac{\sigma\mathbf{w}}{a}(1-e^{-aT})+\mathbf{w}\sigma\Big)+\frac{2\sigma^2}{\widetilde{K}^2}\Big(r_0e^{-aT}+b(1-e^{-aT})$$

$$+\frac{\sigma\mathbf{w}}{a}(1-e^{-aT})+\mathbf{w}\sigma\Big)\cdot\big(\delta^2\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big). \qquad \square$$

Theorem 4.3.4

Let P be the price of the zero-coupon bond driven by NIG process. Then, the Malliavin weight for the greek 'Θ^{NIG}' is given by

$$H_{NIG}(P,Q_T) = \left(r_0e^{-aT} + b(1-e^{-aT}) + \frac{\sigma\mathbf{w}}{a}(1-e^{-aT}) + \mathbf{w}\sigma\right)\left[\frac{Z}{\widetilde{K}} - \frac{\sigma^2\delta^2\left(\sum_{t\leq u\leq T}(\Delta\sqrt{IG(u)})^2\right)}{\widetilde{K}^2}\right]$$

where $\widetilde{K}$ is given by equation (4.3.8).

Proof. Substituting equations (4.3.29), (4.3.30) and (4.3.31) into

$$H_{NIG}(P,Q_T) = Q_T\mathcal{M}(P)^{-1}LP - \mathcal{M}(P)^{-1}\langle DP, DQ_T\rangle - Q_T\langle DP, D\mathcal{M}(P)^{-1}\rangle,$$

it follows that

$$\begin{aligned}
H_{NIG}(P,Q_T) = &\left(r_0e^{-aT} + b(1-e^{-aT}) + \frac{\sigma\mathbf{w}}{a}(1-e^{-aT}) + \mathbf{w}\sigma\right)\left[1 + \frac{Z}{\widetilde{K}} + \frac{\sigma^2\delta^2\left(\sum_{t\leq u\leq T}(\Delta\sqrt{IG(u)})^2\right)}{\widetilde{K}^2}\right]\\
&+\left(r_0e^{-aT} + b(1-e^{-aT}) + \frac{\sigma\mathbf{w}}{a}(1-e^{-aT}) + \mathbf{w}\sigma\right)\\
&-2\left(r_0e^{-aT} + b(1-e^{-aT}) + \frac{\sigma\mathbf{w}}{a}(1-e^{-aT}) + \mathbf{w}\sigma\right)\\
&+\frac{2\sigma^2}{\widetilde{K}^2}\left(r_0e^{-aT} + b(1-e^{-aT}) + \frac{\sigma\mathbf{w}}{a}(1-e^{-aT}) - \mathbf{w}\sigma\right)\cdot\left(\delta^2\sum_{t\leq u\leq T}(\Delta\sqrt{IG(u)})^2\right)\\
= &\frac{Z}{\widetilde{K}}\left(r_0e^{-aT} + b(1-e^{-aT}) + \frac{\sigma\mathbf{w}}{a}(1-e^{-aT}) + \mathbf{w}\sigma\right) - \frac{\sigma^2}{\widetilde{K}^2}\left(r_0e^{-aT} + b(1-e^{-aT})\right.\\
&\left.+\frac{\sigma\mathbf{w}}{a}(1-e^{-aT}) + \mathbf{w}\sigma\right)\frac{\delta^2\left(\sum_{t\leq u\leq T}(\Delta\sqrt{IG(u)})^2\right)}{\widetilde{K}^2}. \qquad \square
\end{aligned}$$

4.3.8 Computation of *vega*$_2$ for NIG-driven interest rate derivatives

In this subsection, we compute the greek *vega*$_2$ for the interest rate derivative driven by NIG process. Recall that by equation (4.3.5),

$$P(t,T) = \exp\left(-\left(\left[\frac{-r_0}{a}(e^{-aT}-e^{-at}) + b\left(T-t+\frac{1}{a}(e^{-aT}-e^{-at})\right) + \frac{\sigma\mathbf{w}}{a}\left[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\right]\right.\right.\right.$$

$$+\mathbf{w}\sigma[T-t]+\sigma\delta\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}Z+\beta\delta\Delta IG(s)e^{-a(u-s)}\big)$$

$$+\sigma\delta\sum_{t\le u\le T}\big(\Delta\sqrt{IG(u)}Z+\beta\delta\Delta IG(u)\big)$$

$$-\frac{\sigma^2\delta^2}{2}\bigg(\sum_{t\le u\le T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)^2\bigg)\bigg)\bigg).$$

where

$$\mathbf{w}=\delta(\sqrt{\alpha^2-(\beta+1)^2}-\sqrt{\alpha^2-\beta^2}) \qquad\Longrightarrow\qquad \frac{\partial\mathbf{w}}{\partial\delta}=\sqrt{\alpha^2-(\beta+1)^2}-\sqrt{\alpha^2-\beta^2}.$$

The greek is given by

$$\frac{\partial}{\partial\delta}e^{-r_0T}\mathbb{E}[\Phi(P)]=e^{-r_0T}\mathbb{E}\left[\Phi'(P)\frac{\partial P}{\partial\delta}\right]+e^{-r_0T}\mathbb{E}_{(\delta)}[\Phi(P)]$$

$$=e^{-r_0T}\mathbb{E}\left[\Phi(P)H_{NIG}\left(P,\frac{\partial P}{\partial\delta}\right)\right]+e^{-r_0T}\mathbb{E}_{(\delta)}[\Phi(P)]$$

$$=e^{-r_0T}\mathbb{E}\big[\Phi(P)H_{NIG}\big(P,Q_\delta\big)\big]+e^{-r_0T}\mathbb{E}_{(\delta)}[\Phi(P)],$$

where $\mathbb{E}_{(\delta)}[\Phi(P)]$ is given in the Appendix.

Lemma 4.3.19. Let P be the zero-coupon bond price driven by NIG process and $Q_\delta=\dfrac{\partial P}{\partial\delta}$. Then,

$$Q_\delta=-\bigg[\frac{\sigma}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big]\big(\sqrt{\alpha^2-(\beta+1)^2}-\sqrt{\alpha^2-\beta^2}\big)$$
$$+\sigma[T-t]\big(\sqrt{\alpha^2-(\beta+1)^2}-\sqrt{\alpha^2-\beta^2}\big)+\sigma\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}Z$$
$$+2\beta\delta\Delta IG(s)e^{-a(u-s)}\big)+\sigma\sum_{t\le u\le T}\big(\Delta\sqrt{IG(u)}Z+2\beta\delta\Delta IG(u)\big)$$
$$-\sigma^2\delta\bigg(\sum_{t\le u\le T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)(\Delta\sqrt{IG(u)}Z+2\beta\delta\Delta IG(u))\bigg)\bigg]P \tag{4.3.32}$$

where $Q_\delta=\dfrac{\partial P}{\partial\delta}$.

Proof. From equation (4.3.5), one obtains

$$Q_\delta=Q_\delta=\frac{\partial P}{\partial\delta}=-\bigg[\frac{\sigma}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big]\frac{\partial\mathbf{w}}{\partial\delta}+\sigma[T-t]\frac{\partial\mathbf{w}}{\partial\delta}$$

$$+\sigma\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}Z+2\beta\delta\Delta IG(s)e^{-a(u-s)}\big)$$

$$+\sigma\sum_{t\le u\le T}\big(\Delta\sqrt{IG(u)}Z+2\beta\delta\Delta IG(u)\big)$$

$$-\sigma^2\delta\bigg(\sum_{t\le u\le T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)(\Delta\sqrt{IG(u)}Z+2\beta\delta\Delta IG(u))\bigg)\bigg]P$$

where $\dfrac{\partial \mathbf{w}}{\partial\delta}=\sqrt{\alpha^2-(\beta+1)^2}-\sqrt{\alpha^2-\beta^2}$. □

Lemma 4.3.20. Let P be the zero-coupon bond price driven by NIG process and $Q_\delta=\dfrac{\partial P}{\partial\delta}$. Then,

$$DQ_\delta=-\bigg[\sigma\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{IG(s)}e^{-a(u-s)}+\sigma\sum_{t\le u\le T}\Delta\sqrt{IG(u)}$$

$$-\sigma^2\delta\bigg(\sum_{t\le u\le T}(2\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}$$

$$+(\Delta\sqrt{IG(u)}Z+\beta\delta\Delta IG(u))\Delta\sqrt{IG(u)}\bigg)\bigg]P+\widehat{\Lambda}\widetilde{K}P \tag{4.3.33}$$

where

$$\widehat{\Lambda}=\bigg[\frac{\sigma}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big]\big(\sqrt{\alpha^2-(\beta+1)^2}-\sqrt{\alpha^2-\beta^2}\big)$$

$$+\sigma[T-t]\big(\sqrt{\alpha^2-(\beta+1)^2}-\sqrt{\alpha^2-\beta^2}\big)+\sigma\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}Z$$

$$+2\beta\delta\Delta IG(s)e^{-a(u-s)}\big)+\sigma\sum_{t\le u\le T}\big(\Delta\sqrt{IG(u)}Z+2\beta\delta\Delta IG(u)\big)$$

$$-\sigma^2\delta\bigg(\sum_{t\le u\le T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)(\Delta\sqrt{IG(u)}Z+2\beta\delta\Delta IG(u))\bigg)\bigg] \tag{4.3.34}$$

and $\widetilde{K}$ is given by equation (4.3.8).

Proof. It follows from equation (4.3.32) that the Malliavin derivative

$$DQ_\delta=P\cdot\bigg(-\bigg[\sigma\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{IG(s)}e^{-a(u-s)}+\sigma\sum_{t\le u\le T}\Delta\sqrt{IG(u)}$$

$$-\sigma^2\delta\bigg(\sum_{t\le u\le T}(2\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}$$

$$+(\Delta\sqrt{IG(u)}Z+\beta\delta\Delta IG(u))\Delta\sqrt{IG(u)}\bigg)\bigg]\bigg)$$

$$+\bigg(-\bigg[\frac{\sigma}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big]\big(\sqrt{\alpha^2-(\beta+1)^2}-\sqrt{\alpha^2-\beta^2}\big)$$

$$
\begin{aligned}
&+\sigma[T-t]\big(\sqrt{\alpha^2-(\beta+1)^2}-\sqrt{\alpha^2-\beta^2}\big)+\sigma\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}Z\\
&+2\beta\delta\Delta IG(s)e^{-a(u-s)}\big)+\sigma\sum_{t\le u\le T}\big(\Delta\sqrt{IG(u)}Z+2\beta\delta\Delta IG(u)\big)\\
&-\sigma^2\delta\bigg(\sum_{t\le u\le T}\big((\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)(2\beta\delta\Delta IG(u))+\Delta\sqrt{IG(u)}Z\big)\bigg)\bigg]\bigg)\\
&\cdot\bigg(-\bigg[\sigma\delta\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big)+\sigma\delta\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})\\
&-\sigma^2\delta^2\bigg(\sum_{t\le u\le T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\bigg)\bigg]\bigg)P
\end{aligned}
$$

$$
\begin{aligned}
&=-\bigg[\sigma\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{IG(s)}e^{-a(u-s)}+\sigma\sum_{t\le u\le T}\Delta\sqrt{IG(u)}\\
&-\sigma^2\delta\bigg(\sum_{t\le u\le T}(2\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\\
&+(\Delta\sqrt{IG(u)}Z+\beta\delta\Delta IG(u))\Delta\sqrt{IG(u)})\bigg)\bigg]P\\
&+\bigg[\frac{\sigma}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big]\big(\sqrt{\alpha^2-(\beta+1)^2}-\sqrt{\alpha^2-\beta^2}\big)\\
&+\sigma[T-t]\big(\sqrt{\alpha^2-(\beta+1)^2}-\sqrt{\alpha^2-\beta^2}\big)+\sigma\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}Z\\
&+2\beta\delta\Delta IG(s)e^{-a(u-s)}\big)+\sigma\sum_{t\le u\le T}\big(\Delta\sqrt{IG(u)}Z+2\beta\delta\Delta IG(u)\big)\\
&-\sigma^2\delta\bigg(\sum_{t\le u\le T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)(\Delta\sqrt{IG(u)}Z+2\beta\delta\Delta IG(u))\bigg)\bigg]\\
&\cdot\bigg[\sigma\delta\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big)+\sigma\delta\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})\\
&-\sigma^2\delta^2\bigg(\sum_{t\le u\le T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\bigg)\bigg]P\\
&=-\bigg[\sigma\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{IG(s)}e^{-a(u-s)}+\sigma\sum_{t\le u\le T}\Delta\sqrt{IG(u)}\\
&-\sigma^2\delta\bigg(\sum_{t\le u\le T}(2\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\\
&+(\Delta\sqrt{IG(u)}Z+\beta\delta\Delta IG(u))\Delta\sqrt{IG(u)})\bigg)\bigg]P\\
&+\bigg[\frac{\sigma}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big]\big(\sqrt{\alpha^2-(\beta+1)^2}-\sqrt{\alpha^2-\beta^2}\big)
\end{aligned}
$$

$$+\sigma[T-t]\big(\sqrt{\alpha^2-(\beta+1)^2}-\sqrt{\alpha^2-\beta^2}\big)+\sigma\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}Z$$

$$+2\beta\delta\Delta IG(s)e^{-a(u-s)}\big)+\sigma\sum_{t\le u\le T}\big(\Delta\sqrt{IG(u)}Z+2\beta\delta\Delta IG(u)\big)$$

$$-\sigma^2\delta\bigg(\sum_{t\le u\le T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)(\Delta\sqrt{IG(u)}Z+2\beta\delta\Delta IG(u))\bigg)\bigg]\widetilde{K}P$$

where $\widetilde{K}$ is given by equation (4.3.8) $\square$

Lemma 4.3.21. Let P be the zero-coupon bond price driven by NIG process. Then,

$$Q_\delta\mathcal{M}(P)^{-1}LP=\widehat{\Lambda}\bigg[\frac{\sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)}{\widetilde{K}^2}+1+\frac{Z}{\widetilde{K}}\bigg] \tag{4.3.35}$$

where $\widetilde{K}$ and $\widehat{\Lambda}$ are given by equation (4.3.8) and (4.3.34), respectively.

Proof. The result follows from equations (4.3.32), (4.3.9) and (4.3.7). Hence,

$$Q_\delta\mathcal{M}(P)^{-1}LP=-\bigg[\frac{\sigma}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big]\big(\sqrt{\alpha^2-(\beta+1)^2}-\sqrt{\alpha^2-\beta^2}\big)$$

$$+\sigma[T-t]\big(\sqrt{\alpha^2-(\beta+1)^2}-\sqrt{\alpha^2-\beta^2}\big)+\sigma\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}Z$$

$$+2\beta\delta\Delta IG(s)e^{-a(u-s)}\big)+\sigma\sum_{t\le u\le T}\big(\Delta\sqrt{IG(u)}Z+2\beta\delta\Delta IG(u)\big)$$

$$-\sigma^2\delta\bigg(\sum_{t\le u\le T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)(\Delta\sqrt{IG(u)}Z+2\beta\delta\Delta IG(u))\bigg)\bigg]P$$

$$\cdot\bigg[\sigma\delta\bigg(\sum_{t\le u\le T}\sum_{0\le s\le t}(\Delta\sqrt{IG(s)}e^{-a(u-s)})-\sum_{0\le u\le t}\sum_{0\le s\le t}(\Delta\sqrt{IG(s)}e^{-a(u-s)})\bigg)$$

$$+\sigma\delta\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})-\sigma^2\delta^2\bigg(\sum_{t\le u\le T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\bigg)\bigg]^{-2}P^{-2}$$

$$\times-\bigg[\sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)+\widetilde{K}^2+\widetilde{K}Z\bigg]P$$

$$=\bigg[\frac{\sigma}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big]\big(\sqrt{\alpha^2-(\beta+1)^2}-\sqrt{\alpha^2-\beta^2}\big)$$

$$+\sigma[T-t]\big(\sqrt{\alpha^2-(\beta+1)^2}-\sqrt{\alpha^2-\beta^2}\big)+\sigma\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}Z$$

$$+2\beta\delta\Delta IG(s)e^{-a(u-s)}\big)+\sigma\sum_{t\le u\le T}\big(\Delta\sqrt{IG(u)}Z+2\beta\delta\Delta IG(u)\big)$$

$$-\sigma^2\delta\bigg(\sum_{t\le u\le T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)(\Delta\sqrt{IG(u)}Z+2\beta\delta\Delta IG(u))\bigg)\bigg]$$

$$\cdot \widetilde{K}^{-2}\Big[\sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)+\widetilde{K}^2+\widetilde{K}Z\Big]$$
$$=\widehat{\Lambda}\widetilde{K}^{-2}\Big[\sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)+\widetilde{K}^2+\widetilde{K}Z\Big].$$

Thus,

$$Q_\delta\mathcal{M}(P)^{-1}LP=\frac{\widehat{\Lambda}}{\widetilde{K}^2}\sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)+\widehat{\Lambda}+\frac{Z\widehat{\Lambda}}{\widetilde{K}}. \qquad \square$$

Lemma 4.3.22. Let P represent the price of the zero-coupon bond driven by NIG process. Then,

$$\mathcal{M}(P)^{-1}\langle DP,DQ_\delta\rangle=\frac{1}{\widetilde{K}}\Big[\sigma\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{IG(s)}e^{-a(u-s)}+\sigma\sum_{t\le u\le T}\Delta\sqrt{IG(u)}$$
$$-\sigma^2\delta\Big(\sum_{t\le u\le T}(2\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}$$
$$+(\Delta\sqrt{IG(u)}Z+\beta\delta\Delta IG(u))\Delta\sqrt{IG(u)}\Big)\Big]-\widehat{\Lambda} \qquad (4.3.36)$$

where $\widetilde{K}$ and $\widehat{\Lambda}$ are given by equation (4.3.8) and (4.3.34), respectively.

Proof. From equations (4.3.9), (4.3.6) and (4.3.33), one gets

$$\mathcal{M}(P)^{-1}\langle DP,DQ_\delta\rangle=-\Big[\sigma\delta\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big)+\sigma\delta\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})$$
$$-\sigma^2\delta^2\Big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Big)\Big]^{-1}P^{-1}$$
$$\cdot\Big(-\Big[\sigma\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{IG(s)}e^{-a(u-s)}+\sigma\sum_{t\le u\le T}\Delta\sqrt{IG(u)}$$
$$-\sigma^2\delta\Big(\sum_{t\le u\le T}(2\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}$$
$$+(\Delta\sqrt{IG(u)}Z+\beta\delta\Delta IG(u))\Delta\sqrt{IG(u)}\Big)\Big]P+\widehat{\Lambda}\widetilde{K}P\Big)$$
$$=-\widetilde{K}^{-1}\cdot\Big(-\Big[\sigma\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{IG(s)}e^{-a(u-s)}+\sigma\sum_{t\le u\le T}\Delta\sqrt{IG(u)}$$
$$-\sigma^2\delta\Big(\sum_{t\le u\le T}(2\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}$$
$$+(\Delta\sqrt{IG(u)}Z+\beta\delta\Delta IG(u))\Delta\sqrt{IG(u)}\Big)\Big]+\widehat{\Lambda}\widetilde{K}\Big)$$

$$= \frac{1}{\widetilde{K}}\Bigg(\Big[\sigma\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{IG(s)}e^{-a(u-s)} + \sigma\sum_{t\le u\le T}\Delta\sqrt{IG(u)}$$

$$-\sigma^2\delta\Big(\sum_{t\le u\le T}(2\beta\delta\Delta IG(u) + \Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}$$

$$+(\Delta\sqrt{IG(u)}Z + \beta\delta\Delta IG(u))\Delta\sqrt{IG(u)}\Big)\Big] - \widehat{\Lambda}\widetilde{K}\Bigg). \quad \square$$

Lemma 4.3.23. Suppose that P is the price of the zero-coupon bond driven by NIG process, then

$$Q_\delta\langle DP, D\mathcal{M}(P)^{-1}\rangle = 2\widehat{\Lambda}\Big[\frac{\sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)}{\widetilde{K}^2} + 1\Big] \qquad (4.3.37)$$

where $\widetilde{K}$ is given by equation (4.3.8) and $\widehat{\Lambda}$ is given by equation (4.3.33).

Proof. From equations (4.3.32), (4.3.6) and (4.3.10), we obtain

$$Q_\delta\langle DP, D\mathcal{M}(P)^{-1}\rangle = -\Bigg[\frac{\sigma}{a}\big[T - t + \frac{1}{a}(e^{-aT} - e^{-at})\big]\big(\sqrt{\alpha^2-(\beta+1)^2} - \sqrt{\alpha^2-\beta^2}\big)$$

$$+\sigma[T-t]\big(\sqrt{\alpha^2-(\beta+1)^2} - \sqrt{\alpha^2-\beta^2}\big) + \sigma\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}Z$$

$$+2\beta\delta\Delta IG(s)e^{-a(u-s)}\big) + \sigma\sum_{t\le u\le T}\big(\Delta\sqrt{IG(u)}Z + 2\beta\delta\Delta IG(u)\big)$$

$$-\sigma^2\delta\Big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u) + \Delta\sqrt{IG(u)}Z)(\Delta\sqrt{IG(u)}Z + 2\beta\delta\Delta IG(u))\Big)\Bigg]P$$

$$\cdot\Bigg(-\Big[\sigma\delta\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big) + \sigma\delta\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})$$

$$-\sigma^2\delta^2\Big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u) + \Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Big)\Big]P\Bigg)$$

$$\cdot\frac{2}{\widetilde{K}^3P^2}\Big[\widetilde{K}^2 + \sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)\Big]$$

$$= \frac{2\widehat{\Lambda}}{\widetilde{K}^2}\sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big) + 2\widehat{\Lambda},$$

which is equation (4.3.37). $\square$

Theorem 4.3.5

The Malliavin weight of the sensitivity with respect to δ of the zero-coupon bond price driven by NIG process is given by

$$H_{NIG}(P, Q_\delta) = \frac{Z\widehat{\Lambda}}{\widetilde{K}} - \frac{1}{\widetilde{K}}\Big[\sigma\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{IG(s)}e^{-a(u-s)} + \sigma\sum_{t\le u\le T}\Delta\sqrt{IG(u)}$$

$$-\sigma^2\delta\Big(\sum_{t\le u\le T}(2\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}$$

$$+(\Delta\sqrt{IG(u)}Z+\beta\delta\Delta IG(u))\Delta\sqrt{IG(u)}\Big)\Big]-\frac{\widehat{\Lambda}}{\widetilde{K}^2}\sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)$$

where $\widetilde{K}$ is given by equation (4.3.8) and $\widehat{\Lambda}$ is given by equation (4.3.34).

Proof. Substituting equations (4.3.35), (4.3.36) and (4.3.37), it follows that

$$\begin{aligned}
H_{NIG}(P,Q_\delta) &= Q_\delta\mathcal{M}(P)^{-1}LP-\mathcal{M}(P)^{-1}\langle DP,DQ_\delta\rangle-Q_\delta\langle DP,D\mathcal{M}(P)^{-1}\rangle\\
&=\frac{\widehat{\Lambda}}{\widetilde{K}^2}\sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)+\widehat{\Lambda}+\frac{Z\widehat{\Lambda}}{\widetilde{K}}\\
&-\frac{1}{\widetilde{K}}\Big[\sigma\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{IG(s)}e^{-a(u-s)}+\sigma\sum_{t\le u\le T}\Delta\sqrt{IG(u)}\\
&-\sigma^2\delta\Big(\sum_{t\le u\le T}(2\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}+(\Delta\sqrt{IG(u)}Z\\
&+\beta\delta\Delta IG(u))\Delta\sqrt{IG(u)}\Big)\Big]+\widehat{\Lambda}-\frac{2\widehat{\Lambda}}{\widetilde{K}^2}\sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)-2\widehat{\Lambda}
\end{aligned}$$

which ends the proof. □

4.3.9 Computation of *vega*$_3$ for NIG-driven interest rate derivatives

In this subsection, we compute *vega*$_3$ for the interest rate derivative driven by NIG process. Recall that by equation (4.3.5),

$$P(t,T)=\exp\Big(-\Big(\big[\frac{-r_0}{a}(e^{-aT}-e^{-at})+b\big(T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big)+\frac{\sigma\mathbf{w}}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big]$$

$$+\mathbf{w}\sigma[T-t]+\sigma\delta\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}Z+\beta\delta\Delta IG(s)e^{-a(u-s)}\big)$$

$$+\sigma\delta\sum_{t\le u\le T}\big(\Delta\sqrt{IG(u)}Z+\beta\delta\Delta IG(u)\big)-\frac{\sigma^2\delta^2}{2}\big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)^2\big)\Big)\Big).$$

where

$$\mathbf{w}=\delta((\alpha^2-(\beta+1)^2)^{0.5}-(\alpha^2-\beta^2)^{0.5})$$

which implies

$$\frac{\partial \mathbf{w}}{\partial \alpha} = \frac{\delta\alpha}{\sqrt{\alpha^2-(\beta+1)^2}} - \frac{\delta\alpha}{\sqrt{\alpha^2-\beta^2}}.$$

Moreover,

$$\begin{aligned}\text{vega}_3^{NIG} = \mathcal{V}_3^{NIG} = \frac{\partial}{\partial\alpha} e^{-r_0T}\mathbb{E}[\Phi(P)] &= e^{-r_0T}\mathbb{E}\Big[\Phi'(P)\frac{\partial P}{\partial \alpha}\Big] + e^{-r_0T}\mathbb{E}_{(\alpha)}[\Phi(P)] \\ &= e^{-r_0T}\mathbb{E}\Big[\Phi(P)H_{NIG}\Big(P, \frac{\partial P}{\partial\alpha}\Big)\Big] + e^{-r_0T}\mathbb{E}_{(\alpha)}[\Phi(P)],\end{aligned}$$

where $\mathbb{E}_{(\alpha)}[\Phi(P)]$ is given in the Appendix.

Lemma 4.3.24. Let P be the zero-coupon bond price driven by NIG process. Then the following hold:

$$Q_\alpha = -\Big(\frac{\sigma}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big]+\sigma[T-t]\Big)\Big(\frac{\delta\alpha}{\sqrt{\alpha^2-(\beta+1)^2}} - \frac{\delta\alpha}{\sqrt{\alpha^2-\beta^2}}\Big)P \tag{4.3.38}$$

and

$$DQ_\alpha = -\Big(\frac{\sigma}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big]+\sigma[T-t]\Big)\Big[\frac{\delta\alpha}{\sqrt{\alpha^2-(\beta+1)^2}} - \frac{\delta\alpha}{\sqrt{\alpha^2-\beta^2}}\Big]\widetilde{K}P \tag{4.3.39}$$

where $\widetilde{K}$ is given by equation (4.3.8).

Proof. From equation (4.3.5), it follows that

$$\begin{aligned} Q_\alpha &= \frac{\partial P}{\partial\alpha} = -\Big(\frac{\sigma}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big]\frac{\partial \mathbf{w}}{\partial\alpha}+\sigma[T-t]\frac{\partial \mathbf{w}}{\partial\alpha}\Big)P \\ &= -\Big(\frac{\sigma}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big]+\sigma[T-t]\Big)\frac{\partial \mathbf{w}}{\partial\alpha}P \\ &= -\Big(\frac{\sigma}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big]+\sigma[T-t]\Big)\Big(\frac{\delta\alpha}{\sqrt{\alpha^2-(\beta+1)^2}} - \frac{\delta\alpha}{\sqrt{\alpha^2-\beta^2}}\Big)P. \end{aligned}$$

Therefore, the Malliavin derivative

$$\begin{aligned} DQ_\alpha = &-\Big(\frac{\sigma}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big]+\sigma[T-t]\Big)\Big[\frac{\delta\alpha}{\sqrt{\alpha^2-(\beta+1)^2}} - \frac{\delta\alpha}{\sqrt{\alpha^2-\beta^2}}\Big] \\ &\cdot\Big(-\Big[\sigma\delta\sum_{t\le u\le T}\sum_{0\le s\le t}(\Delta\sqrt{IG(s)}e^{-a(u-s)}) + \sigma\delta\sum_{t\le u\le T}(\Delta\sqrt{IG(u)}) \\ &-\sigma^2\delta^2\Big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Big)\Big]\Big)P. \end{aligned}$$

$$= \left(\frac{\sigma}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big]+\sigma[T-t]\right)\left[\frac{\delta\alpha}{\sqrt{\alpha^2-(\beta+1)^2}}-\frac{\delta\alpha}{\sqrt{\alpha^2-\beta^2}}\right]$$

$$\cdot\Bigg[\sigma\delta\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big)+\sigma\delta\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})$$

$$-\sigma^2\delta^2\big(\sum_{0\le u\le T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\big)\Bigg]P. \qquad \square$$

Lemma 4.3.25. Let P be the price of the zero-coupon bond driven by NIG process. For its sensitivity with respect to α, the following is satisfied:

$$Q_\alpha\mathcal{M}(P)^{-1}LP = \left(\frac{\sigma}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big]+\sigma[T-t]\right)\left(\frac{\delta\alpha}{\sqrt{\alpha^2-(\beta+1)^2}}-\frac{\delta\alpha}{\sqrt{\alpha^2-\beta^2}}\right)$$

$$\cdot\left[\frac{\sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)}{\widetilde{K}^2}+1+\frac{Z}{\widetilde{K}}\right]. \tag{4.3.40}$$

Proof. From equations (4.3.38), (4.3.9) and (4.3.7), one gets

$Q_\alpha\mathcal{M}(P)^{-1}LP$

$$= -\left(\frac{\sigma}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big]+\sigma[T-t]\right)\left(\frac{\delta\alpha}{\sqrt{\alpha^2-(\beta+1)^2}}-\frac{\delta\alpha}{\sqrt{\alpha^2-\beta^2}}\right)P$$

$$\cdot\Bigg[\sigma\delta\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big)+\sigma\delta\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})$$

$$-\sigma^2\delta^2\Big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Big)\Bigg]^{-2}P^{-2}$$

$$\cdot\Big(-\Big[\sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)+\widetilde{K}^2+\widetilde{K}Z\Big]P$$

$$= \left(\frac{\sigma}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big]+\sigma[T-t]\right)\left(\frac{\delta\alpha}{\sqrt{\alpha^2-(\beta+1)^2}}-\frac{\delta\alpha}{\sqrt{\alpha^2-\beta^2}}\right)$$

$$\cdot\widetilde{K}^{-2}\Big[\sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)+\widetilde{K}^2+\widetilde{K}Z\Big]. \qquad \square$$

Lemma 4.3.26. Suppose that the price of a zero-coupon bond is driven by NIG process, then,

$\mathcal{M}(P)^{-1}\langle DP, DQ_\alpha\rangle$

$$= -\left(\frac{\sigma}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big]+\sigma[T-t]\right)\left[\frac{\delta\alpha}{\sqrt{\alpha^2-(\beta+1)^2}}-\frac{\delta\alpha}{\sqrt{\alpha^2-\beta^2}}\right]. \tag{4.3.41}$$

Proof. By equations (4.3.9), (4.3.6) and (4.3.39), we get

$$\mathcal{M}(P)^{-1}\langle DP, DQ_\alpha\rangle$$

$$= -\Big[\sigma\delta\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big) + \sigma\delta\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})$$

$$-\sigma^2\delta^2\Big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u) + \Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Big)\Big]^{-1}P^{-1}$$

$$\cdot\Big(\frac{\sigma}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big]+\sigma[T-t]\Big)\Big[\frac{\delta\alpha}{\sqrt{\alpha^2-(\beta+1)^2}}-\frac{\delta\alpha}{\sqrt{\alpha^2-\beta^2}}\Big]$$

$$\cdot\Big[\sigma\delta\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big) + \sigma\delta\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})$$

$$-\sigma^2\delta^2\big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u) + \Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\big)\Big]P$$

$$= -\Big(\frac{\sigma}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big]+\sigma[T-t]\Big)\Big[\frac{\delta\alpha}{\sqrt{\alpha^2-(\beta+1)^2}}-\frac{\delta\alpha}{\sqrt{\alpha^2-\beta^2}}\Big]. \qquad \square$$

Lemma 4.3.27. Suppose that the price of the zero-coupon bond driven by NIG process is given by P. Then, $Q_\alpha\langle DP, D\mathcal{M}(P)^{-1}\rangle$ can be written as

$$Q_\alpha\langle DP, D\mathcal{M}(P)^{-1}\rangle$$

$$= 2\Big(\frac{\sigma}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big]+\sigma[T-t]\Big)\Big[\frac{\delta\alpha}{\sqrt{\alpha^2-(\beta+1)^2}}-\frac{\delta\alpha}{\sqrt{\alpha^2-\beta^2}}\Big]$$

$$\cdot\Big[1+\frac{\sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)}{\widetilde{K}^2}\Big] \tag{4.3.42}$$

where $\widetilde{K}$ is given by equation (4.3.8).

Proof. By equations (4.3.38), (4.3.6) and (4.3.10), it follows that

$$Q_\alpha\langle DP, D\mathcal{M}(P)^{-1}\rangle$$

$$= -\Big(\frac{\sigma}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big]+\sigma[T-t]\Big)\Big[\frac{\delta\alpha}{\sqrt{\alpha^2-(\beta+1)^2}}-\frac{\delta\alpha}{\sqrt{\alpha^2-\beta^2}}\Big]P$$

$$\times -\Big(\Big[\sigma\delta\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big) + \sigma\delta\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})$$

$$-\sigma^2\delta^2\Big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u) + \Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Big)\Big]P\Big)$$

$$\cdot\frac{2}{\widetilde{K}^3P^2}\Big[\widetilde{K}^2+\sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)\Big]$$

$$= \left(\frac{\sigma}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big]+\sigma[T-t]\right)\left[\frac{\delta\alpha}{\sqrt{\alpha^2-(\beta+1)^2}}-\frac{\delta\alpha}{\sqrt{\alpha^2-\beta^2}}\right]$$

$$\cdot\frac{2}{\widetilde{K}^2}\left[\widetilde{K}^2+\sigma^2\delta^2\Big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\Big)\right]. \qquad \square$$

Theorem 4.3.6

Let P be the price of the zero-coupon bond driven by NIG process. Then, the Malliavin weight $H_{NIG}(P,Q_\alpha)$ for the sensitivity with respect to α satisfies

$$H_{NIG}(P,Q_\alpha) = \left(\frac{\sigma}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big]+\sigma[T-t]\right)\left(\frac{\delta\alpha}{\sqrt{\alpha^2-(\beta+1)^2}}-\frac{\delta\alpha}{\sqrt{\alpha^2-\beta^2}}\right)$$

$$\cdot\frac{1}{\widetilde{K}}\left[Z-\frac{\sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)}{\widetilde{K}}\right].$$

where $\widetilde{K}$ is given by equation (4.3.8).

Proof. The result holds by substituting equations (4.3.40), (4.3.41) and (4.3.42) into $H_{NIG}(P,Q_\alpha)$. Hence,

$$H_{NIG}(P,Q_\alpha) = Q_\alpha\mathcal{M}(P)^{-1}LP-\mathcal{M}(P)^{-1}\langle DP,DQ_\alpha\rangle-Q_\alpha\langle DP,D\mathcal{M}(P)^{-1}\rangle$$

$$= \left(\frac{\sigma}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big]+\sigma[T-t]\right)\left(\frac{\delta\alpha}{\sqrt{\alpha^2-(\beta+1)^2}}-\frac{\delta\alpha}{\sqrt{\alpha^2-\beta^2}}\right)$$

$$\cdot\left[\frac{\sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)}{\widetilde{K}^2}+1+\frac{Z}{\widetilde{K}}\right]$$

$$+\left(\frac{\sigma}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big]+\sigma[T-t]\right)\left[\frac{\delta\alpha}{\sqrt{\alpha^2-(\beta+1)^2}}-\frac{\delta\alpha}{\sqrt{\alpha^2-\beta^2}}\right]$$

$$-2\left(\frac{\sigma}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big]+\sigma[T-t]\right)\left[\frac{\delta\alpha}{\sqrt{\alpha^2-(\beta+1)^2}}-\frac{\delta\alpha}{\sqrt{\alpha^2-\beta^2}}\right]$$

$$\cdot\left[1+\frac{\sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)}{\widetilde{K}^2}\right]$$

$$= \left(\frac{\sigma}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big]+\sigma[T-t]\right)\left(\frac{\delta\alpha}{\sqrt{\alpha^2-(\beta+1)^2}}-\frac{\delta\alpha}{\sqrt{\alpha^2-\beta^2}}\right)$$

$$\cdot\left[\frac{Z}{\widetilde{K}}-\frac{\sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)}{\widetilde{K}^2}\right]. \qquad \square$$

4.3.10 Computation of *vega*$_4$ for NIG-driven interest rate derivatives

In this subsection, we compute $vega_4$ for interest rate derivative driven by NIG process. Recall that by equation (4.3.5), the price of the zero-coupon bond is given by

$$P(t,T) = \exp\Bigg(-\bigg(\big[\frac{-r_0}{a}(e^{-aT}-e^{-at})+b\big(T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big)+\frac{\sigma\mathbf{w}}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big]$$

$$+\mathbf{w}\sigma[T-t]+\sigma\delta\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}Z+\beta\delta\Delta IG(s)e^{-a(u-s)}\big)$$

$$+\sigma\delta\sum_{t\le u\le T}\big(\Delta\sqrt{IG(u)}Z+\beta\delta\Delta IG(u)\big)-\frac{\sigma^2\delta^2}{2}\big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)^2\big)\bigg)\Bigg)$$

where

$$\mathbf{w} = \delta((\alpha^2-(\beta+1)^2)^{0.5}-(\alpha^2-\beta^2)^{0.5})$$

which implies that

$$\frac{\partial\mathbf{w}}{\partial\beta} = \frac{-\delta(\beta+1)}{\sqrt{\alpha^2-(\beta+1)^2}}+\frac{\delta\beta}{\sqrt{\alpha^2-\beta^2}}.$$

Moreover,

$$\text{vega}_4^{NIG} = \mathcal{V}_4^{NIG} = \frac{\partial}{\partial\beta}e^{-r_0T}\mathbb{E}[\Phi(P)] = e^{-r_0T}\mathbb{E}\left[\Phi(P)'\frac{\partial P}{\partial\beta}\right]+e^{-r_0T}\mathbb{E}_{(\beta)}[\Phi(P)]$$

$$= e^{-r_0T}\mathbb{E}\left[\Phi(P)H_{NIG}\left(P,\frac{\partial P}{\partial\beta}\right)\right]+e^{-r_0T}\mathbb{E}_{(\beta)}[\Phi(P)],$$

where $\mathbb{E}_{(\beta)}[\Phi(P)]$ is given in the Appendix.

Lemma 4.3.28

Let P be a zero-coupon bond price driven by NIG process. Then,

$$Q_\beta = -\Bigg[\Big(\frac{\sigma}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big]+\sigma[T-t]\Big)\left(\frac{-\delta(\beta+1)}{\sqrt{\alpha^2-(\beta+1)^2}}+\frac{\delta\beta}{\sqrt{\alpha^2-\beta^2}}\right)$$

$$+\sigma\delta^2\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta IG(s)e^{-a(u-s)}+\sigma\delta^2\sum_{t\le u\le T}\Delta IG(u)$$

$$-\sigma^2\delta^3\Bigg(\sum_{t\le u\le T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)\Delta IG(u)\Bigg)\Bigg]P \qquad (4.3.43)$$

and

$$DQ_\beta = \Bigg(\sigma^2\delta^3\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})\Delta IG(u)\big)+\bar{\Lambda}\widetilde{K}\Bigg)P \qquad (4.3.44)$$

where

$$\bar{\Lambda} = \Bigg[\Big(\frac{\sigma}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big]+\sigma[T-t]\Big)\Bigg(\frac{-\delta(\beta+1)}{\sqrt{\alpha^2-(\beta+1)^2}}+\frac{\delta\beta}{\sqrt{\alpha^2-\beta^2}}\Bigg)$$

$$+\sigma\delta^2\sum_{t\leq u\leq T}\sum_{0\leq s\leq t}\Delta IG(s)e^{-a(u-s)}+\sigma\delta^2\sum_{t\leq u\leq T}\Delta IG(u)$$

$$-\sigma^2\delta^3\Bigg(\sum_{t\leq u\leq T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)\Delta IG(u)\Bigg)\Bigg] \qquad (4.3.45)$$

and $\widetilde{K}$ is given by equation (4.3.8).

Proof. From the price of the zero-coupon bond given by equation (4.3.5),

$$Q_\beta = \frac{\partial P}{\partial \beta} = -\Bigg[\frac{\sigma}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big]\frac{\partial \mathbf{w}}{\partial \beta}$$

$$+\sigma[T-t]\frac{\partial \mathbf{w}}{\partial \beta}+\sigma\delta^2\sum_{t\leq u\leq T}\sum_{0\leq s\leq t}\Delta IG(s)e^{-a(u-s)}$$

$$+\sigma\delta^2\sum_{t\leq u\leq T}\Delta IG(u)-\frac{2\sigma^2\delta^3}{2}\Bigg(\sum_{t\leq u\leq T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)\Delta IG(u)\Bigg)\Bigg]P$$

$$= -\Bigg[\frac{\sigma}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big]\Bigg(\frac{-\delta(\beta+1)}{\sqrt{\alpha^2-(\beta+1)^2}}+\frac{\delta\beta}{\sqrt{\alpha^2-\beta^2}}\Bigg)$$

$$+\sigma[T-t]\Bigg(\frac{-\delta(\beta+1)}{\sqrt{\alpha^2-(\beta+1)^2}}+\frac{\delta\beta}{\sqrt{\alpha^2-\beta^2}}\Bigg)+\sigma\delta^2\sum_{t\leq u\leq T}\sum_{0\leq s\leq t}\Delta IG(s)e^{-a(u-s)}$$

$$+\sigma\delta^2\sum_{t\leq u\leq T}\Delta IG(u)-\sigma^2\delta^3\Bigg(\sum_{t\leq u\leq T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)\Delta IG(u)\Bigg)\Bigg]P.$$

The Malliavin derivative

$$DQ_\beta = P\cdot\Bigg(-\Bigg[-\sigma^2\delta^3\Big(\sum_{t\leq u\leq T}(\Delta\sqrt{IG(u)})\Delta IG(u)\Big)\Bigg]\Bigg)+-\Bigg[\Big(\frac{\sigma}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big]$$

$$+\sigma[T-t]\Big)\Bigg(\frac{-\delta(\beta+1)}{\sqrt{\alpha^2-(\beta+1)^2}}+\frac{\delta\beta}{\sqrt{\alpha^2-\beta^2}}\Bigg)+\sigma\delta^2\sum_{t\leq u\leq T}\sum_{0\leq s\leq t}\Delta IG(s)e^{-a(u-s)}$$

$$+\sigma\delta^2\sum_{t\leq u\leq T}\Delta IG(u)-\sigma^2\delta^3\Bigg(\sum_{t\leq u\leq T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)\Delta IG(u)\Bigg)\Bigg]$$

$$\cdot -\Bigg[\sigma\delta\Bigg(\sum_{t\leq u\leq T}\sum_{0\leq s\leq t}(\Delta\sqrt{IG(s)}e^{-a(u-s)})\Bigg)$$

$$+\sigma\delta\Big(\sum_{t\leq u\leq T}(\Delta\sqrt{IG(u)})\Big)-\sigma^2\delta^2\Bigg(\sum_{t\leq u\leq T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Bigg)\Bigg]P$$

$$= \Big[\sigma^2\delta^3\big(\sum_{t\leq u\leq T}(\Delta\sqrt{IG(u)})\Delta IG(u)\big)\Big]P + \Big[\big(\frac{\sigma}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big]$$

$$+\sigma[T-t]\big)\Big(\frac{-\delta(\beta+1)}{\sqrt{\alpha^2-(\beta+1)^2}}+\frac{\delta\beta}{\sqrt{\alpha^2-\beta^2}}\Big)$$

$$+\sigma\delta^2\sum_{t\leq u\leq T}\sum_{0\leq s\leq t}\Delta IG(s)e^{-a(u-s)}+\sigma\delta^2\sum_{t\leq u\leq T}\Delta IG(u)$$

$$-\sigma^2\delta^3\Big(\sum_{t\leq u\leq T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)\Delta IG(u)\Big)\Big]\widetilde{K}P.$$

Hence,

$$DQ_\beta = \Big[\sigma^2\delta^3\big(\sum_{t\leq u\leq T}(\Delta\sqrt{IG(u)})\Delta IG(u)\big)\Big]P + \bar{\Lambda}\widetilde{K}P. \qquad \square$$

Lemma 4.3.29

Suppose a zero-coupon bond price driven by NIG process is given by P. Then,

$$Q_\beta\mathcal{M}(P)^{-1}LP = \frac{\bar{\Lambda}\sigma^2\delta^2\big(\sum_{t\leq u\leq T}(\Delta\sqrt{IG(u)})^2\big)}{\widetilde{K}^2}+\bar{\Lambda}+\frac{\bar{\Lambda}Z}{\widetilde{K}} \tag{4.3.46}$$

where $\bar{\Lambda}$ is given by equation (4.3.45) and $\widetilde{K}$ is given by equation (4.3.8).

Proof.

By equations (4.3.43), (4.3.9) and (4.3.7), it follows that

$$Q_\beta\mathcal{M}(P)^{-1}LP = -\Big[\big(\frac{\sigma}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big]$$

$$+\sigma[T-t]\big)\Big(\frac{-\delta(\beta+1)}{\sqrt{\alpha^2-(\beta+1)^2}}+\frac{\delta\beta}{\sqrt{\alpha^2-\beta^2}}\Big)+\sigma\delta^2\sum_{t\leq u\leq T}\sum_{0\leq s\leq t}\Delta IG(s)e^{-a(u-s)}$$

$$+\sigma\delta^2\sum_{t\leq u\leq T}\Delta IG(u)-\sigma^2\delta^3\Big(\sum_{t\leq u\leq T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)\Delta IG(u)\Big)\Big]P$$

$$\cdot\Big[\sigma\delta\Big(\sum_{t\leq u\leq T}\sum_{0\leq s\leq t}(\Delta\sqrt{IG(s)}e^{-a(u-s)})\Big)$$

$$+\sigma\delta\big(\sum_{t\leq u\leq T}(\Delta\sqrt{IG(u)})\big)-\sigma^2\delta^2\Big(\sum_{t\leq u\leq T}(\beta\delta\Delta IG(u)$$

$$+\Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Big)\Big]^{-2}P^{-2}\cdot-\Big[\sigma^2\delta^2\big(\sum_{t\leq u\leq T}(\Delta\sqrt{IG(u)})^2\big)+\widetilde{K}^2+\widetilde{K}Z\Big]P$$

$$=\frac{\bar{\Lambda}}{\widetilde{K}^2}\Big[\sigma^2\delta^2\big(\sum_{t\leq u\leq T}(\Delta\sqrt{IG(u)})^2\big)+\widetilde{K}^2+\widetilde{K}Z\Big]. \qquad \square$$

Lemma 4.3.30. Let the price of the zero-coupon bond driven by NIG process be given by P. Then,

$$\mathcal{M}(P)^{-1}\langle DP, DQ_\beta\rangle = -\frac{\sigma^2\delta^3\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})\Delta IG(u)\big)}{\widetilde{K}} - \bar{\Lambda} \quad (4.3.47)$$

where $\bar{\Lambda}$ is given by equation (4.3.45) and $\widetilde{K}$ is given by equation (4.3.8).

Proof.

From equations (4.3.9), (4.3.6) and (4.3.44), we get

$$\mathcal{M}(P)^{-1}\langle DP, DQ_\beta\rangle = -\Bigg[\sigma\delta\Big(\sum_{t\le u\le T}\sum_{0\le s\le t}(\Delta\sqrt{IG(s)}e^{-a(u-s)})\Big)$$

$$+\sigma\delta\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})\big)-\sigma^2\delta^2\Big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Big)\Bigg]^{-1}P^{-1}$$

$$\cdot\Big[\sigma^2\delta^3\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})\Delta IG(u)\big)\Big]P + \bar{\Lambda}\widetilde{K}P$$

$$= -\frac{1}{\widetilde{K}}\Big(\sigma^2\delta^3\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})\Delta IG(u)\big) + \bar{\Lambda}\widetilde{K}\Big)$$

$$= -\frac{\sigma^2\delta^3}{\widetilde{K}}\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})\Delta IG(u)\big) - \bar{\Lambda}. \qquad \square$$

Lemma 4.3.31. Let the price of the zero-coupon bond driven by NIG process be given by P. Then,

$$Q_\beta\langle DP, D\mathcal{M}(P)^{-1}\rangle = 2\bar{\Lambda} + \frac{2\bar{\Lambda}\sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)}{\widetilde{K}^2} \quad (4.3.48)$$

where $\bar{\Lambda}$ and $\widetilde{K}$ are given by equations (4.3.45) and (4.3.8), respectively.

Proof. From equations (4.3.43), (4.3.6) and (4.3.10), it follows that

$$Q_\beta\langle DP, D\mathcal{M}(P)^{-1}\rangle = -\Bigg[\Big(\frac{\sigma}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big]$$

$$+\sigma[T-t]\Big)\Big(\frac{-\delta(\beta+1)}{\sqrt{\alpha^2-(\beta+1)^2}} + \frac{\delta\beta}{\sqrt{\alpha^2-\beta^2}}\Big)$$

$$+\sigma\delta^2\big(\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta IG(s)e^{-a(u-s)}\big) + \sigma\delta^2\sum_{t\le u\le T}\Delta IG(u)$$

$$-\sigma^2\delta^3\Big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u) + \Delta\sqrt{IG(u)}Z)\Delta IG(u)\Big)\Bigg]P$$

$$\cdot\Bigg(-\Bigg[\sigma\delta\Big(\sum_{t\le u\le T}\sum_{0\le s\le t}(\Delta\sqrt{IG(s)}e^{-a(u-s)})\Big) + \sigma\delta\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})$$

$$-\sigma^2\delta^2\Big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Big)\Big]P\Big)$$

$$\cdot\frac{2}{\widetilde{K}^3P^2}\Big[\widetilde{K}^2+\sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)\Big]$$

$$=\Big[\big(\frac{\sigma}{a}[T-t+\frac{1}{a}(e^{-aT}-e^{-at})]+\sigma[T-t]\big)\Big(\frac{-\delta(\beta+1)}{\sqrt{\alpha^2-(\beta+1)^2}}+\frac{\delta\beta}{\sqrt{\alpha^2-\beta^2}}\Big)$$

$$+\sigma\delta^2\big(\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta IG(s)e^{-a(u-s)}\big)+\sigma\delta^2\sum_{t\le u\le T}\Delta IG(u)$$

$$-\sigma^2\delta^3\Big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)\Delta IG(u)\Big)\Big]$$

$$\cdot\frac{2}{\widetilde{K}^2}\Big[\widetilde{K}^2+\sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)\Big]$$

$$=\frac{2\bar{\Lambda}}{\widetilde{K}^2}\Big[\widetilde{K}^2+\sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)\Big].\qquad\square$$

Theorem 4.3.7. Let P be the price of the zero-coupon bond driven by NIG process. Then, the Malliavin weight $H_{NIG}(P,Q_\beta)$ is given by

$H_{NIG}(P,Q_\beta)$

$$=\frac{1}{\widetilde{K}}\Big(\bar{\Lambda}Z-\frac{\bar{\Lambda}\sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)}{\widetilde{K}}+\sigma^2\delta^3\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})\Delta IG(u)\big)\Big)$$

where $\bar{\Lambda}$ and $\widetilde{K}$ are given by equation (4.3.45) and (4.3.8), respectively.

Proof.

By substituting equations (4.3.46), (4.3.47) and (4.3.48) into

$$H_{NIG}(P,Q_\beta)=Q_\beta\mathcal{M}(P)^{-1}LP-\mathcal{M}(P)^{-1}\langle DP,DQ_\beta\rangle-Q_\beta\langle DP,D\mathcal{M}(P)^{-1}\rangle,$$

it follows that

$$H_{NIG}(P,Q)=\frac{\bar{\Lambda}\sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)}{\widetilde{K}^2}+\bar{\Lambda}+\frac{\bar{\Lambda}Z}{\widetilde{K}}$$

$$+\frac{\sigma^2\delta^3\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})\Delta IG(u)\big)}{\widetilde{K}}+\bar{\Lambda}$$

$$-2\bar{\Lambda}-\frac{2\bar{\Lambda}\sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)}{\widetilde{K}^2}$$

$$= \frac{\bar{\Lambda}Z}{\widetilde{K}} - \frac{\bar{\Lambda}\sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)}{\widetilde{K}^2} + \frac{\sigma^2\delta^3\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})\Delta IG(u)\big)}{\widetilde{K}}.$$

Hence,

$$\frac{\partial}{\partial\beta}\mathbb{E}[\Phi(P)] = E\left[\Phi(P)H_{NIG}\left(P, \frac{\partial P}{\partial \beta}\right)\right] + E_{(\beta)}[\Phi(P)].$$

For the computation of $E_{(\beta)}[\Phi(P)]$, see Appendix. □

Remark 4.3.2. The extended Vasicek model driven by NIG process is:

$$\begin{aligned} dr_t &= a(b - r_t)dt + \sigma dX_t = a(b - r_t)dt + \sigma d(\mathbf{w}t + \beta\delta^2 IG_t + \delta\sqrt{IG_t}Z) \\ &= a(b - r_t)dt + \sigma(\mathbf{w}dt + \beta\delta^2\Delta IG_t + \delta\Delta\sqrt{IG_t}Z). \end{aligned}$$

Substituting $\beta = 0$, $\delta = 1$, $IG = t$ and $\mathbf{w} = 0$, we obtain

$$\begin{aligned} dr_t &= a(b - r_t)dt + \sigma(\mathbf{w}dt + \beta\delta^2\Delta IG_t + \delta\Delta\sqrt{IG_t}Z) = a(b - r_t)dt + \sigma\Delta\sqrt{t}Z \\ &= a(b - r_t)dt + \sigma dW_t. \end{aligned}$$

As $W_t = \sqrt{t}Z$, we obtain $dr_t = a(b - r_t)dt + \sigma dW_t$, which is the original Vasicek model for a Brownian motion market.

4.4 Comparison of the greeks of the zero-coupon bond price driven by VG and NIG processes

We proceed in this section to compare some of the greeks obtained in sections 4.2 and 4.3.

4.4.1 *Delta* for the price of zero-coupon bond driven by VG and NIG processes

The greek *delta* is

$$\triangle^{VG} = e^{-r_0T}(-T\mathbb{E}(\Phi(P)) + \mathbb{E}[\Phi(P)H_{VG}(P, Q)])$$

$$\triangle^{NIG} = -Te^{-r_0T}\mathbb{E}(\Phi(P)) + e^{-r_0T}\mathbb{E}[\Phi(P)H_{NIG}(P, Q)].$$

The Malliavin weights for delta are given in Table 4.1.

Greek	Malliavin weight
$\triangle^{VG}$	$H_{VG}(P,Q) = \dfrac{\frac{\sigma^2}{a}(e^{-aT}-e^{-at})}{\mathcal{K}^2}\big(\widetilde{\sigma}^2 \sum_{t\le u\le T}(\Delta\sqrt{G(u)})^2\big) - \dfrac{\frac{1}{a}(e^{-aT}-e^{-at})Z}{\mathcal{K}}$
$\triangle^{NIG}$	$H_{NIG}(P,Q) = \dfrac{\frac{\sigma^2}{a}(e^{-aT}-e^{-at})}{\widetilde{K}^2}\big(\delta^2 \sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big) - \dfrac{Z\big(\frac{1}{a}(e^{-aT}-e^{-at})\big)}{\widetilde{K}}$

Table 4.1: Malliavin weight for delta ($\triangle^{VG}$ and $\triangle^{NIG}$)

$\mathcal{K}$ and $\widetilde{K}$ are given by equations (4.2.17) and (4.3.8), respectively, as

$$\mathcal{K} = \sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)} + \sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)})$$
$$-\sigma^2\widetilde{\sigma}\sum_{t\le u\le T}(\theta\Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}.$$
$$\widetilde{K} = \sigma\delta\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big) + \sigma\delta\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})$$
$$-\sigma^2\delta^2\Bigg(\sum_{t\le u\le T}(\beta\delta\Delta IG(u) + \Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Bigg).$$

4.4.2 *Gamma* for the price of the zero-coupon bond driven by VG and NIG processes

The greek *gamma* is given by

$$\Gamma^{VG} = T^2e^{-r_0T}\mathbb{E}[\Phi(P)] - 2Te^{-r_0T}\mathbb{E}[\Phi(P)H_{VG}(P,Q)]$$
$$+e^{-r_0T}\mathbb{E}[\Phi(P)H_{VG}(P,QH_{VG}(P,Q))]$$
$$\Gamma^{NIG} = T^2e^{-r_0T}\mathbb{E}[\Phi(P)] - 2Te^{-r_0T}\mathbb{E}[\Phi(P)H_{NIG}(P,Q)]$$
$$+e^{-r_0T}\mathbb{E}[\Phi(P)H_{NIG}(P,QH_{NIG}(P,Q))]$$

where

$$H_{VG}(P,Q_\Gamma) = H_{VG}(P,QH_{VG}(P,H_{VG}(P,Q_\Gamma)))$$
$$H_{NIG}(P,Q_\Gamma) = H_{VG}(P,QH_{NIG}(P,H_{NIG}(P,Q_\Gamma))).$$

$H_{VG}(P,Q)]$ and $H_{NIG}(P,Q)]$ are Malliavin weights for $\triangle^{VG}$ and $\triangle^{NIG}$, respectively. The Malliavin weights $H_{VG}(P,Q_\Gamma)$ and $H_{NIG}(P,Q_\Gamma)$ are given in Table 4.2.

$\mathcal{K}$ and $\widetilde{K}$ are given by equations (4.2.17) and (4.3.8), respectively.

Greek	Malliavin weight ($H_{VG}(P,Q_\Gamma)$ and $H_{NIG}(P,Q_\Gamma)$)
Γ^{VG}	$H_{VG}(P,Q_\Gamma) = \frac{1}{a}(e^{-aT}-e^{-at})H_{VG}(P,Q)\mathcal{K}^{-2}(\sigma^2\widetilde{\sigma}^2\sum_{t\le u\le T}(\Delta\sqrt{G(u)})^2) - \frac{Z(\frac{1}{a}(e^{-aT}-e^{-at}))}{\mathcal{K}}H_{VG}(P,Q) + 2\frac{\big(\frac{1}{a}(e^{-aT}-e^{-at})\big)^2}{\mathcal{K}^4}(\sigma^2\widetilde{\sigma}^2\sum_{t\le u\le T}(\Delta\sqrt{G(u)})^2)^2 - \frac{\big(\frac{1}{a}(e^{-aT}-e^{-at})\big)^2}{\mathcal{K}^2} + \big(\frac{1}{a}(e^{-aT}-e^{-at})\big)^2 Z\mathcal{K}^{-3}(-\sigma^2\widetilde{\sigma}^2\sum_{t\le u\le T}(\Delta\sqrt{G(u)})^2)$
Γ^{NIG}	$H_{NIG}(P,Q_\Gamma) = \frac{1}{a}(e^{-aT}-e^{-at})H_{NIG}(P,Q)\cdot\Big(\frac{\sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)}{\widetilde{K}^2} - \frac{Z}{\widetilde{K}}\Big) + \big(\frac{1}{a}(e^{-aT}-e^{-at})\big)^2\Big(\frac{\big[2\sigma^2\widetilde{\sigma}^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)\big]^2}{\widetilde{K}^4} - \frac{1}{\widetilde{K}^2} - \frac{Z}{\widetilde{K}^3}\sigma^2\widetilde{\sigma}^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)\Big)$

Table 4.2: Malliavin weight for gamma (Γ^{VG} and Γ^{NIG})

4.4.3 *Vega* for the price of the zero-coupon bond driven by VG and NIG processes

The greek *vega* is given by

$$\mathcal{V}^{VG} = e^{-r_0T}\mathbb{E}[\Phi(P)H_{VG}(P,Q_\sigma)]$$

$$\mathcal{V}^{NIG} = e^{-r_0T}\mathbb{E}[\Phi(P)H_{NIG}(P,Q_\sigma)].$$

The Malliavin weights $H_{VG}(P,Q_\sigma)$ and $H_{NIG}(P,Q_\sigma)$ are given in Table 4.3.

$\mathcal{K}$ and Λ are given by equations (4.2.17) and (4.2.28), respectively; while $\widetilde{\Lambda}$ and $\widetilde{K}$ are given by equations (4.3.23) and (4.3.8), respectively.

$$\Lambda = \Bigg[\frac{\mathbf{w}}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big]+\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\theta\Delta G(s)e^{-a(u-s)}+\widetilde{\sigma}\Delta\sqrt{G(s)}e^{-a(u-s)}Z\big)$$
$$+\mathbf{w}[T-t]+\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)-\sigma\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)^2\Bigg].$$

$$\widetilde{\Lambda} = \frac{\mathbf{w}}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big]+\mathbf{w}[T-t]+\delta\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}Z$$
$$+\beta\delta\Delta IG(s)e^{-a(u-s)}\big)+\delta\sum_{t\le u\le T}\big(\Delta\sqrt{IG(u)}Z+\beta\delta\Delta IG(u)\big)$$

Greek	Malliavin weight
$\mathcal{V}^{VG}$	$H_{VG}(P,Q_\sigma) = \frac{\Lambda Z}{\mathcal{K}} - \Big[\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}(\Delta\sqrt{G(s)}e^{-a(u-s)})$ $+\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)}) - 2\sigma\widetilde{\sigma}\Big(\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Big)\Big]$ $\cdot\mathcal{K}^{-1} - \frac{\Lambda}{\mathcal{K}^2}\Big[\sigma^2\widetilde{\sigma}^2\sum_{t\le u\le T}(\Delta\sqrt{G(u)})^2\Big]$
$\mathcal{V}^{NIG}$	$H_{NIG}(P,Q_\sigma) = \frac{\widetilde{\Lambda}Z}{\widetilde{K}} - \widetilde{\Lambda}\sigma^2\delta^2\frac{\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)}{\widetilde{K}^2}$ $-\frac{1}{\widetilde{K}}\Big[\delta\sum_{t\le u\le T}\sum_{0\le s\le t}(\Delta\sqrt{IG(s)}e^{-a(u-s)}) + \delta\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})$ $-2\sigma\delta^2\Big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Big)\Big]$

Table 4.3: Malliavin weight for vega ($\mathcal{V}^{VG}$ and $\mathcal{V}^{NIG}$)

Greek	Malliavin weight
Θ^{VG}	$H_{VG}(P,Q_T) = \big(r_0e^{-aT} + b(1-e^{-aT}) + \frac{\sigma\mathbf{w}}{a}(1-e^{-aT}) + \mathbf{w}\sigma\big)$ $\cdot\Big[\frac{Z}{\mathcal{K}} - \frac{\sigma^2\widetilde{\sigma}^2\big(\sum_{t\le u\le T}(\Delta\sqrt{G(u)})^2\big)}{\mathcal{K}^2}\Big]$
Θ^{NIG}	$H_{NIG}(P,Q_T) = \big(r_0e^{-aT} + b(1-e^{-aT}) + \frac{\sigma\mathbf{w}}{a}\big(1-e^{-aT}\big) + \mathbf{w}\sigma\big)$ $\cdot\Big[\frac{Z}{\widetilde{K}} - \sigma^2\delta^2\frac{\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)}{\widetilde{K}^2}\Big]$

Table 4.4: Malliavin weight for Theta (Θ^{VG} and Θ^{NIG})

$$-\sigma\delta^2\Big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)^2\Big).$$

4.4.4 *Theta* for the price of the zero-coupon bond driven by VG and NIG processes

The greek *Theta* is given by

$$\Theta^{VG} = -r_0e^{-r_0T}\mathbb{E}[\Phi(P)] + \mathbb{E}\big[\Phi(P)H_{VG}(P,Q_T)\big]$$

$$\Theta^{NIG} = -r_0e^{-r_0T}\mathbb{E}[\Phi(P)] + e^{-r_0T}\mathbb{E}[\Phi(P)H_{NIG}(P,Q_T)].$$

The Malliavin weights $H_{VG}(P,Q_T)$ and $H_{NIG}(P,Q_T)$ are given in Table 4.4.

$\mathcal{K}$ and $\widetilde{K}$ are given by equations (4.2.17) and (4.3.8), respectively.

The greeks measure the rate of change associated with the parameters of the interest rate derivative in a market driven by VG and NIG processes. An investor requires such information in order to manage the security risks since he will be able to measure how much the value of an option changes given a change in the value of its parameter in an interest rate derivative market driven by a Lévy process. Heavily-tailed processes have to be priced under the NIG distribution.

Chapter 5

Applications

5.1 Background

This chapter looks at the applications of the results obtained in the previous chapter; while Chapter 6 discusses its summary, conclusion as well as future research. 30-days Nigerian Interbank Offered Rate (NIBOR) data from January 2007 to December 2017, was collected from the website of Central Bank of Nigeria (CBN). There are many interest rates but NIBOR was chosen because it is a national floating rate index for financial contracts, processed from quotations submitted by reference banks, that is, some selected banks. It is the short term interbank lending rate in the Nigerian interbank market. Interbank rates are used as basis for settlement of interest rate contracts in many countries' financial markets. There are 1 month, 3 months and 6 months NIBOR. In sections 5.2-5.3, we give the dynamics of the short rate in the Nigerian market and obtain the parameter values of the VG and NIG processes. In section 5.4, we give the zero-coupon bond price dynamics in the Nigerian market and plot the graphs for the price of the zero-coupon bond driven by the two subordinated Lévy processes using 'Python 3.6' programming language.

5.2 Dynamics of the Vasicek short rate model for NIBOR

The parameters of the Vasicek short rate model are estimated from the NIBOR data using the least-square regression and maximum likelihood method (Van den Berg (2011)). The collected NIBOR data used in this work is in Appendix 2.

Using least-square regression,

$$r_i = g + hr_{i-1} + \epsilon, \qquad i = 1, ..., n. \tag{5.2.1}$$

Let

$$\mathcal{E} = \sum_i (r_i - \hat{r}_i)^2 = \sum_i [r_i - (g + hr_{i-1})]^2$$

$$\begin{aligned}
\frac{\partial \mathcal{E}}{\partial g} &= 2\sum_i [r_i - (g + hr_{i-1})](-1) = -2\sum_i r_i + 2gn + 2h\sum_i r_{i-1} = 0. \\
\therefore g = \frac{\sum_i r_i}{n} - h\frac{\sum_i r_{i-1}}{n} &= \mu_{r_i} - h\mu_{r_{i-1}} \text{ where } \mu_{r_i} \text{ is the mean of } r_i. \\
\frac{\partial \mathcal{E}}{\partial h} &= 2\sum_i [r_i - g - hr_{i-1})](-r_{i-1}) = 2h\sum_i r_{i-1}^2 + 2g\sum_i r_{i-1} - 2\sum_i r_i r_{i-1} = 0. \\
h\sum_i r_{i-1}^2 &= \sum_i r_i r_{i-1} - g\sum_i r_{i-1} = \sum_i r_i r_{i-1} - (\mu_{r_i} - h\mu_{r_{i-1}})\sum_i r_{i-1} \\
&= \sum_i r_i r_{i-1} - \mu_{r_i}\sum_i r_{i-1} + h\mu_{r_{i-1}}\sum_i r_{i-1} \\
h(\sum_i r_{i-1}^2 - \mu_{r_{i-1}}\sum_i r_{i-1}) &= \sum_i r_i r_{i-1} - \mu_{r_i}\sum_i r_{i-1}. \\
\therefore h = \frac{\sum_i r_i r_{i-1} - \mu_{r_i}\sum_i r_{i-1}}{\sum_i r_{i-1}^2 - \mu_{r_{i-1}}\sum_i r_{i-1}} &= \frac{\sum_i (r_i - \mu_{r_i})(r_{i-1} - \mu_{r_{i-1}})}{\sum_i (r_{i-1} - \mu_{r_{i-1}})^2}.
\end{aligned}$$

Using

$$r_i = r_{i-1}e^{-a\Delta t} + b(1 - e^{-a\Delta t}) + \sigma\int_0^t e^{-a(t-s)}dX_s. \tag{5.2.2}$$

From equations (5.2.1) and (5.2.2),

$$h = e^{-a\Delta t}, \;\; g = b(1 - e^{-a\Delta t}) \Rightarrow b = \frac{g}{1 - e^{-a\Delta t}} = \frac{g}{1-h}, \;\; \mathcal{E} = \sigma\int_0^t e^{-a(t-s)}dX_s.$$

Furthermore,

$$\begin{aligned}
a &= -\frac{\ln h}{\Delta t} = -\frac{1}{\Delta t}\ln\frac{\sum_{i=1}^n (r_i - \mu_{r_i})(r_{i-1} - \mu_{r_{i-1}})}{\sum_{i=1}^n (r_{i-1} - \mu_{r_{i-1}})^2} \\
&= -\frac{1}{\Delta t}\ln\left[\frac{\sum_{i=1}^n [r_i r_{i-1} - r_i\mu_{r_{i-1}} - r_{i-1}\mu_{r_i} + \mu_{r_i}\mu_{r_{i-1}}]}{\sum_{i=1}^n [r_{i-1}^2 - 2r_{i-1}\mu_{r_{i-1}} + \mu_{r_{i-1}}^2}\right] \\
&= -\frac{1}{\Delta t}\ln\left[\frac{\sum_{i=1}^n r_i r_{i-1} - \sum_{i=1}^n r_i\mu_{r_{i-1}} - \sum_{i=1}^n r_{i-1}\mu_{r_i} + n\mu_{r_i}\mu_{r_{i-1}}}{\sum_{i=1}^n r_{i-1}^2 - 2\sum_{i=1}^n r_{i-1}\mu_{r_{i-1}} + n\mu_{r_{i-1}}^2}\right] \\
&= -\frac{1}{\Delta t}\ln\left[\frac{r_{xy} - r_y\mu_x - r_x\mu_y + n\mu_x\mu_y}{r_{xx} - 2r_x\mu_x + n\mu_x^2}\right] \\
&= -\frac{1}{\Delta t}\ln\left[\frac{r_{xy} - br_y - br_x + nb^2}{r_{xx} - 2br_x + nb^2}\right].
\end{aligned}$$

Also, from equation (5.2.1) and (5.2.2),

$$
\begin{aligned}
b &= \frac{g}{1-h} = \frac{\sum_{i=1}^{n}(r_i - r_{i-1}h)}{n(1-h)} = \frac{\sum_{i=1}^{n}(r_i - r_{i-1}e^{-a\Delta t})}{n(1-e^{-a\Delta t})} \\
&= \frac{\mu_{r_i} - h\mu_{r_{i-1}}}{1-\left(\frac{\sum_i r_i r_{i-1} - \mu_{r_i}\sum_i r_{i-1}}{\sum_i r_{i-1}^2 - \mu_{r_{i-1}}\sum_i r_{i-1}}\right)} = \frac{\mu_{r_i} - \left(\frac{\sum_i r_i r_{i-1} - \mu_{r_i}\sum_i r_{i-1}}{\sum_i r_{i-1}^2 - \mu_{r_{i-1}}\sum_i r_{i-1}}\right)\mu_{r_{i-1}}}{1-\left(\frac{\sum_i r_i r_{i-1} - \mu_{r_i}\sum_i r_{i-1}}{\sum_i r_{i-1}^2 - \mu_{r_{i-1}}\sum_i r_{i-1}}\right)} \\
&= \mu_{r_i} - \left(\frac{\mu_{r_{i-1}}\sum_i r_i r_{i-1} - \mu_{r_i}\mu_{r_{i-1}}\sum_i r_{i-1}}{\sum_i r_{i-1}^2 - \mu_{r_{i-1}}\sum_i r_{i-1}}\right) \\
&\quad \div \frac{\sum_i r_{i-1}^2 - \mu_{r_{i-1}}\sum_i r_{i-1} - \sum_i r_i \sum_i r_{i-1} + \mu_{r_i}\sum_i r_{i-1}}{\sum_i r_{i-1}^2 - \mu_{r_{i-1}}\sum_i r_{i-1}} \\
&= \frac{\mu_{r_i}\sum_i r_{i-1}^2 - \mu_{r_i}\mu_{r_{i-1}}\sum_i r_{i-1} - \mu_{r_{i-1}}\sum_i r_i r_{i-1} + \mu_{r_i}\mu_{r_{i-1}}\sum_i r_{i-1}}{\sum_i r_{i-1}^2 - \mu_{r_{i-1}}\sum_i r_{i-1} - \sum_i r_i r_{i-1} + \mu_{r_i}\sum_i r_{i-1}} \\
&= \frac{\mu_{r_i}\sum_i r_{i-1}^2 - \mu_{r_{i-1}}\sum_i r_i r_{i-1}}{\sum_i r_{i-1}^2 - \mu_{r_{i-1}}\sum_i r_{i-1} - \sum_i r_i r_{i-1} + \mu_{r_i}\sum_i r_{i-1}} \\
&= \frac{\sum_i r_i \sum_i r_{i-1}^2 - \sum_i r_{i-1}\sum_i r_i r_{i-1}}{n\left(\sum_i r_{i-1}^2 - \frac{\sum_i r_{i-1}}{n}\sum_i r_{i-1} - \sum_i r_i r_{i-1} + \frac{\sum_i r_i}{n}\sum_i r_{i-1}\right)} \\
&= \frac{\sum_i r_i \sum_i r_{i-1}^2 - \sum_i r_{i-1}\sum_i r_i r_{i-1}}{n\left(\sum_i r_{i-1}^2 - \sum_i r_i r_{i-1}\right) - \left(\sum_i r_{i-1}\sum_i r_{i-1} - \sum_i r_i \sum_i r_{i-1}\right)}
\end{aligned}
$$

$$\therefore b = \frac{r_y r_{xx} - r_x r_{xy}}{n(r_{xx} - r_{xy}) - (r_x^2 - r_x r_y)}.$$

Hence, the parameter values of the model are obtained as

$$
\begin{aligned}
n &= \text{length of data} = 131 \\
r_x &= \sum_{i=1}^{n} r_{i-1} = r_0 + r_1 + \cdots + r_{130} = 18.8897 \\
r_{xx} &= \sum_{i=1}^{n} r_{i-1}^2 = r_0^2 + r_1^2 + \cdots + r_{130}^2 = 3.1001
\end{aligned}
$$

$$
\begin{aligned}
r_{xy} &= \sum_{i=1}^{n} r_{i-1}r_i = r_0 r_1 + r_1 r_2 + \cdots + r_{130}r_{131} = 2.9357 \\
r_y &= \sum_{i=1}^{n} r_i = r_1 + r_2 + \cdots r_{131} = 18.9212 \\
r_{yy} &= \sum_{i=1}^{n} r_i^2 = r_1^2 + r_2^2 + \cdots + r_{131}^2 = 3.1094.
\end{aligned}
$$

Parameter value	Interpretation
$a = 0.5959$	the interest rate reverts back to the long-term mean at a rate 59.59%
$b = 0.1447$	the interest rate reverts to the long-term mean 14.47%
$\sigma = 0.0585$	the volatility of the short rate model is 5.85%

Table 5.1: Interpretation of parameters of Vasicek model for NIBOR rates

The long-term mean becomes

$$b = \frac{-(r_x r_{xy} - r_y r_{xx})}{n(r_{xx} - r_{xy}) - (r_x^2 - r_x r_y)} = \frac{3.2031}{22.1315} = 0.1447.$$

The speed of reversion

$$a = -\frac{1}{\Delta t} \ln \left(\frac{r_{xy} - br_x - br_y + nb^2}{r_{xx} - 2br_x + nb^2} \right) = -\ln \frac{0.20735}{0.37631} = 0.5959$$

where the sample fixed time step $\Delta t = 1$.

To obtain the volatility, let $\eta = e^{-a\Delta t}$ and

$$\varsigma = \frac{1}{n}(r_{yy} - 2\eta r_{xy} + \eta^2 r_{xx} - 2b(1-\eta)(r_y - \eta r_x) + nb^2(1-\eta)^2);$$

then, the volatility

$$\sigma = \sqrt{\frac{2\varsigma a}{1-\eta^2}} = 0.0588.$$

These are already programmed into MATLAB. Hence, if we model Nigerian market using Vasicek short rate model, its dynamics becomes

$$dr_t = 0.5959(0.1447 - r_t)dt + 0.0585dX_t$$

where X_t is for the subordinated Lévy process and the parameter values are described in Table 5.1. This gives how the dynamics evolves with time.

5.3 Parameters of the VG and NIG processes

Parameters of VG and NIG distributions can be estimated from the NIBOR data using method of moment. Expression for the four central moments of the

VG distribution are given in Madan et al. (1998) over interval of length Δt as:

$$\begin{array}{rcl} m_1 = \mathbb{E}[X(t)] & = & \theta\Delta t \\ m_2 = \mathbb{E}[(X(t) - \mathbb{E}[X(t)])^2] & = & (\widetilde{\sigma}^2 + \theta^2\kappa)\Delta t \\ m_3 = \mathbb{E}[(X(t) - \mathbb{E}[X(t)])^3] & = & (2\theta^3\widetilde{\sigma}^2 + 3\theta\widetilde{\sigma}^2\kappa)\Delta t \\ m_4 = \mathbb{E}[(X(t) - \mathbb{E}[X(t)])^4] & = & 3(\widetilde{\sigma}^4\kappa + 4\widetilde{\sigma}^2\theta^2\kappa^2 + 2\theta^4\kappa^3)\Delta t \\ & & +3(\widetilde{\sigma}^4 + 2\widetilde{\sigma}^2\theta^2\kappa + \theta^4\kappa^2)(\Delta t)^2. \end{array}$$

m_1 is the mean and m_2 is the variance,

$$\text{skewness} = m_3 m_2^{-\frac{3}{2}} = \frac{3\theta\kappa}{\widetilde{\sigma}\sqrt{\Delta t}}$$

where θ^2, θ^3 and θ^4 are ignored by assuming θ to be small.

$$\text{kurtosis} = m_4 m_2^{-2} = 3\left(1 + \frac{\kappa}{\Delta t}\right).$$

Hence,

$$\theta = \frac{\text{skewness} \times \widetilde{\sigma}\sqrt{\Delta t}}{3\kappa}, \qquad \widetilde{\sigma} = \sqrt{\frac{\text{variance}}{\Delta t}},$$

$$\text{and} \quad \kappa = \left(\frac{\text{kurtosis}}{3} - 1\right)\Delta t,$$

provides the initial values of the parameters. The software takes over from here in generating the parameters of the VG process.

Furthermore, the method of moments for the estimation of the parameters of NIG process gives the following:

$$\text{Mean} = \mu + \frac{\beta\delta}{\sqrt{\alpha^2 - \beta^2}}, \qquad \text{Variance} = \frac{\alpha^2\delta}{(\alpha^2 - \beta^2)^3}$$

$$\text{Skewness} = \frac{3\beta}{\alpha(\delta\sqrt{\alpha^2 - \beta^2})}, \qquad \text{Kurtosis} = \frac{3(\alpha^2 + 4\beta^2)}{\alpha^2\delta\sqrt{\alpha^2 - \beta^2}}.$$

In this work, parameters of the VG and NIG processes were estimated from R-programming language using R-Studio version 3.2.2. An R-package called *VarianceGamma* is used to fit the VG distribution to NIBOR data, while an R-package called *GeneralizedHyperbolic* is used to fit the NIG distribution to the NIBOR data. The packages use start values from method of moments (MoM), User-supplied (US) and (SL) fitted skew-Laplace distribution (Scott and Dong, 2015).

The packages use the following optimisation methods to estimate the parameters:

(i) *BFGS*: a quasi-Newton method, also called variable matrix algorithm, which was introduced by Brodyden, Fletcher, Goldferb and Shanno in 1970.

(ii) *Nelder-Mead*: a method by Nelder and Mead (1965), very useful for non-differentiable functions.

(iii) *nlm*: non-linear minimisation function which carries out a minimisation of a function using a Newton-type algorithm.

The following values were obtained for the VG distribution based on the optimisation toolbox in R:

Parameters	$\widetilde{\sigma}$	θ	κ	iterations
Method: BFGS	0.46456	-0.01610	0.24507	72
Nelder-Mead	0.46456	-0.01633	0.24514	271
nlm	0.46456	-0.01610	0.24508	1

The following values were obtained for the NIG distribution based on the optimisation toolbox in R:

Parameters	μ	δ	α	β	criterion	iterations
Method: BFGS	0.01839	0.89963	4.16530	-0.08721	MLE	58
Nelder-Mead	0.01826	0.89962	4.16565	-0.08665	MLE	239
nlm	0.01838	0.89955	4.16493	-0.08718	MLE	1

The maximum likelihood estimation (MLE) determines the parameter values that make the data more likely to occur than any other parameter values from probability point of view (Cliff, 2009). We use the parameter values obtained from Nelder-Mead method. The interpretation of the parameter values are presented in Table 5.2 and Table 5.3, respectively.

Substituting the parameter values, the solution to the short rate dynamics

Parameter value	Interpretation
$\theta = -0.01610$	the distribution is skewed to the left
$\kappa = 0.24508$	the variance of the gamma process is 0.24508
$\widetilde{\sigma} = 0.46456$	the volatility of the arithmetic Brownian motion is 46.46%
$\mathbf{w} = -0.09286$	$\frac{1}{\kappa}\ln(1-\theta\kappa-\frac{1}{2}\widetilde{\sigma}^2\kappa)$ (cumulant generating function)

Table 5.2: Interpretation of parameters of VG process for NIBOR rates

(given by equation (4.2.2)) for Nigerian market becomes

$$\begin{aligned} r_t &= r_0e^{-at} + b(1-e^{-at}) + \sigma\big(\frac{\mathbf{w}}{a}(1-e^{-at}) \\ &\quad +\theta\sum_{0\le s\le t}\Delta G(s)e^{-a(t-s)} + \widetilde{\sigma}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(t-s)}Z\big) \\ &= 0.1317e^{-0.5959t} + 0.1447(1-e^{-0.5959t}) + 0.0585\big(\frac{-0.09286}{0.5959}(1-e^{-0.5959t}) \\ &\quad +(-0.01610)\sum_{0\le s\le t}\Delta G(s)e^{-0.5959(t-s)} + 0.46456\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-0.5959(t-s)}Z\big). \end{aligned}$$

The solution to the short rate dynamics driven by NIG process from equation (4.3.2) becomes

$$\begin{aligned} r_t &= r_0e^{-at} + b(1-e^{-at}) + \frac{\sigma\mathbf{w}}{a}(1-e^{-at}) \\ &\quad +\sigma\big(\sum_{0\le s\le t}(\delta\Delta\sqrt{IG(s)}Z + \beta\delta^2\Delta IG(s))e^{-a(t-s)}\big) \\ &= 0.1317e^{-0.5959t} + 0.1447(1-e^{-0.5959t}) + 0.0585\frac{-0.09027}{0.5959}(1-e^{-0.5959t}) \\ &\quad +0.0585\big(\sum_{0\le s\le t}(\Delta\sqrt{IG(s)}Z + (-0.08718)0.89955^2\Delta IG(s))e^{-0.5959(t-s)}\big). \end{aligned}$$

5.4 Dynamics of the zero-coupon bond price with NIBOR

The dynamics of zero-coupon bond price for the Nigerian market is given by

$$dP = r_tPdt + 0.0585PdX_t$$

Parameter value	Interpretation
$\alpha = 4.16493$	the tail-heaviness of the distribution is 4.16493
$\beta = -0.08718$	the distribution is skewed to the left
$\delta = 0.89955$	the scale of the distribution is 0.89955
$\mathbf{w} = -0.09027$	$\delta(\sqrt{\alpha^2 - (\beta+1)^2} - \sqrt{\alpha^2 - \beta^2})$ (cumulant generating function)

Table 5.3: Interpretation of parameters of NIG process for NIBOR rates

where $\sigma = 0.0585$ is the volatility and X_t is for the subordinated Lévy processes. In the end is the price of the zero-coupon bond from equation (4.2.7) driven by VG process for the Nigerian market obtained as

$$P(t,T) = \exp\Big(-\Big(\Big[-\frac{0.1317}{0.5959}(e^{-0.5959T} - e^{-0.5959t}) + 0.1447\big(T - t + \frac{1}{0.5959}(e^{-0.5959T} - e^{-0.5959t})\big) + 0.0585\frac{-0.09286}{0.5959}\Big[T - t + \frac{1}{0.5959}(e^{-0.5959T} - e^{-0.5959t})\Big] + 0.0585\sum_{t\le u\le T}\sum_{0\le s\le t}\big(-0.01610\Delta G(s)e^{-0.5959(u-s)} + 0.46456\Delta\sqrt{G(s)}e^{-0.5959(u-s)}Z\big)\Big] - 0.09286(0.0585)[T-t] + 0.0585\sum_{t\le u\le T}(-0.01610\Delta G(u) + 0.46456\Delta\sqrt{G(u)}Z) - \frac{0.0585^2}{2}\Big(\sum_{t\le u\le T}(-0.01610\Delta G(u) + 0.46456\Delta\sqrt{G(u)}Z)^2\Big)\Big)\Big).$$

The expression for the zero-coupon bond price from equation (4.3.5) driven by NIG process is obtained as

$$P(t,T) = \exp\Big(-\big(\big[\frac{-0.1317}{0.5959}(e^{-0.5959T} - e^{-0.5959t}) + 0.1447\big(T - t + \frac{1}{0.5959}(e^{-0.5959T} - e^{-0.5959t})\big) + 0.0585\frac{-0.090269}{0.5959}\big[T - t + \frac{1}{0.5959}\cdot(e^{-0.5959T} - e^{-0.5959t})\big] - 0.0903(0.0585)[T-t] + 0.0585\sum_{t\le u\le T}\sum_{0\le s\le t}\big(0.89955\Delta\sqrt{IG(s)}e^{-0.5959(u-s)}Z - 0.08718(0.89955)^2\Delta IG(s)e^{-0.5959(u-s)}\big) + 0.0585\sum_{t\le u\le T}\big(0.89955\Delta\sqrt{IG(u)}Z - 0.08718(0.89955)^2\Delta IG(u)\big) - \frac{0.0585^2}{2}\big(\sum_{t\le u\le T}((-0.08718)(0.89955)^2\Delta IG(u) + 0.89955\Delta\sqrt{IG(u)}Z)^2)\big)\Big).$$

5.5 Graphs

In this section, we present the graphs of zero-coupon bond price driven by the subordinated Lévy processes. Starting from Figure 5.1, the graphs are obtained using Spyder under Python 3.6, Anaconda custom.

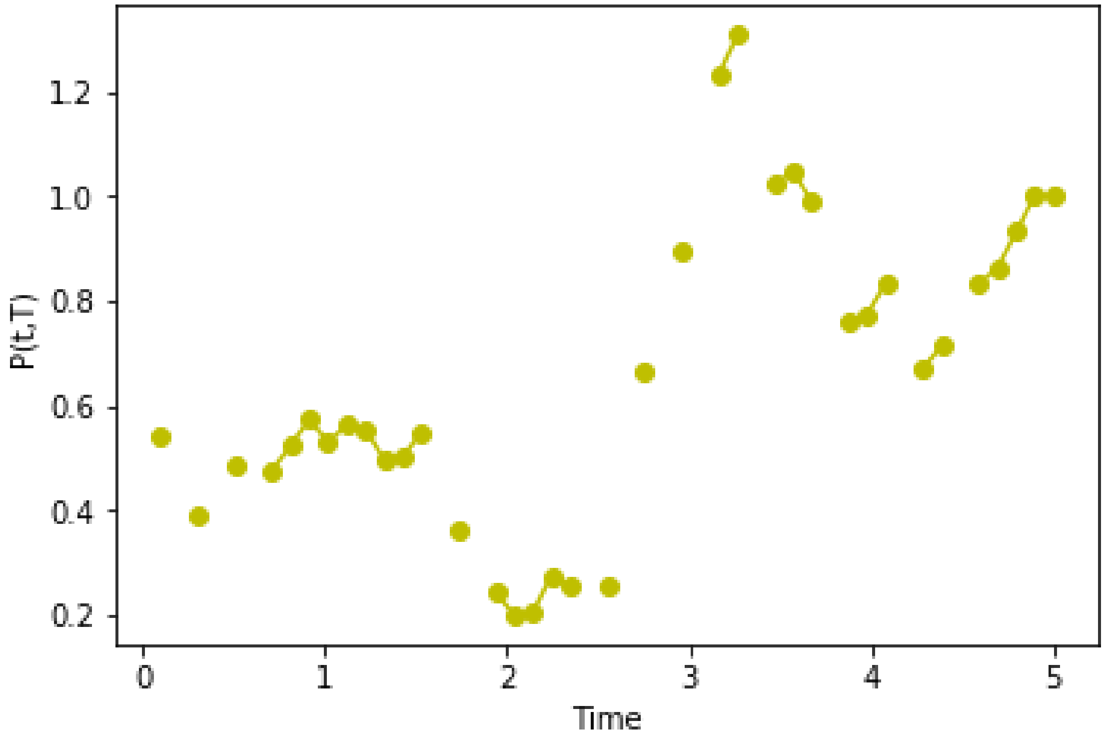

Figure 5.1: Price of zero-coupon bond driven by VG process (with jumps ± 0.08).

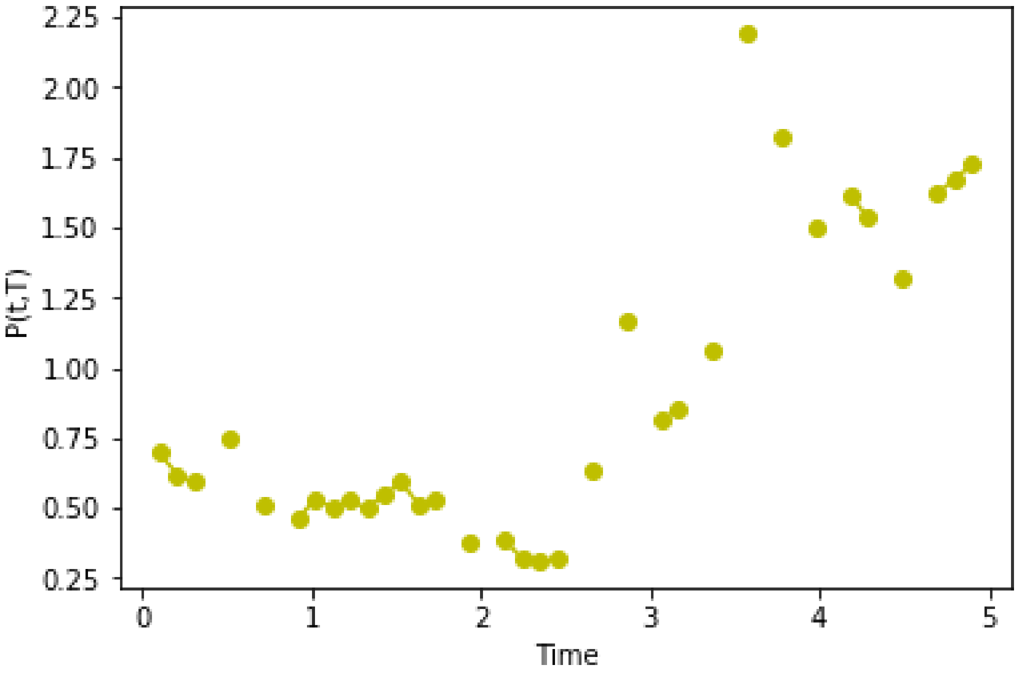

Figure 5.2: Price of zero-coupon bond driven by VG process (with jumps ± 0.1).

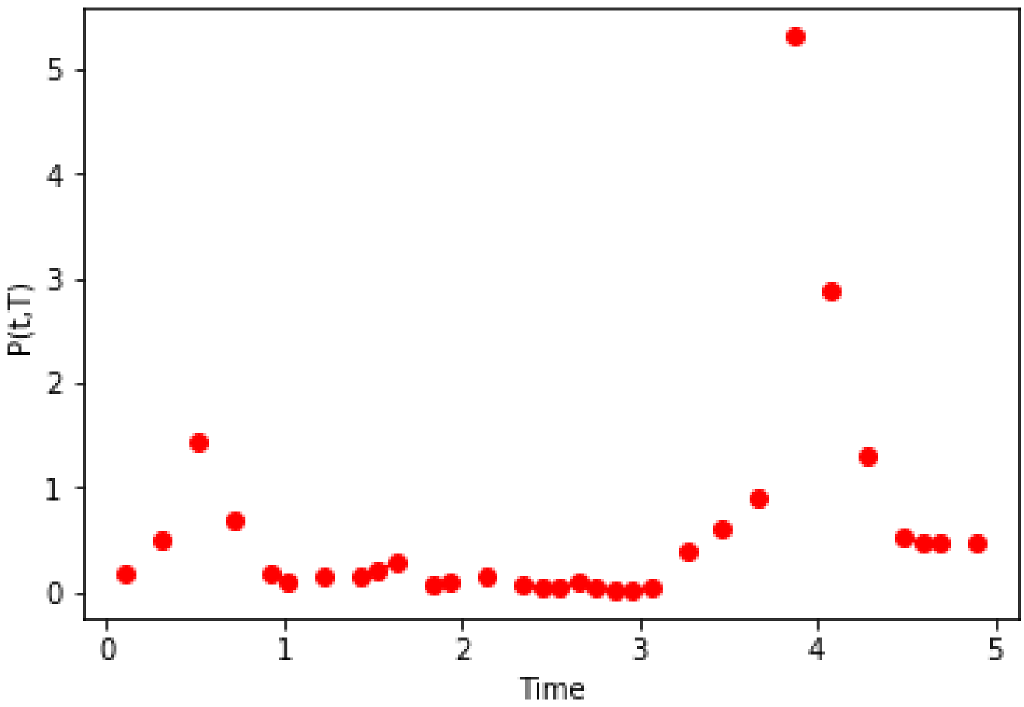

Figure 5.3: Price of zero-coupon bond driven by NIG process (with jumps ± 0.08).

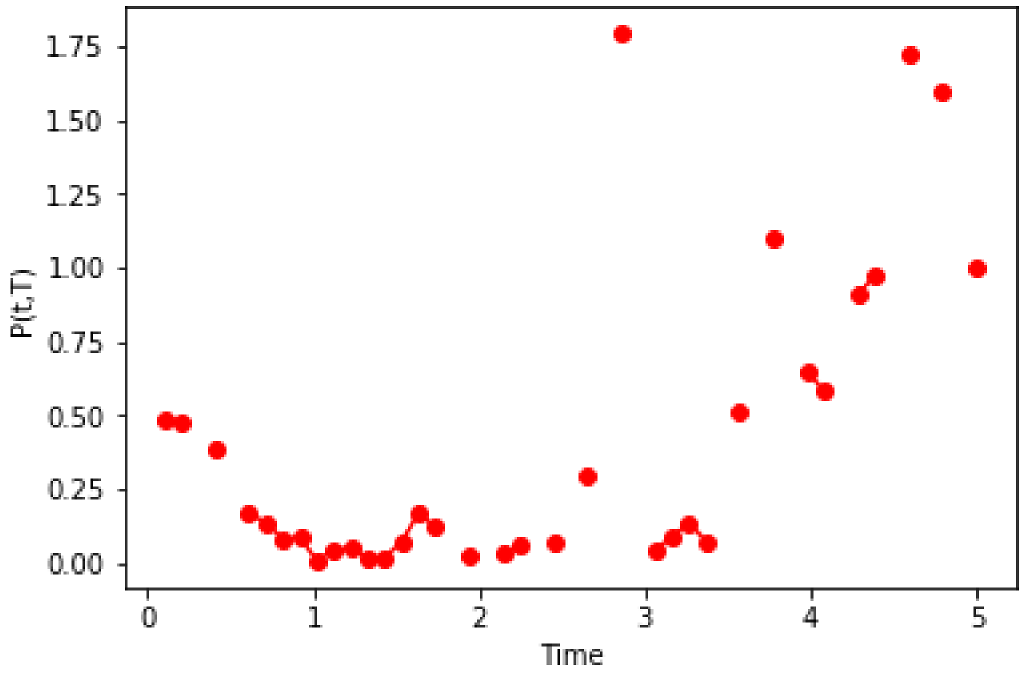

Figure 5.4: Price of zero-coupon bond driven by NIG process (with jumps ± 0.2).

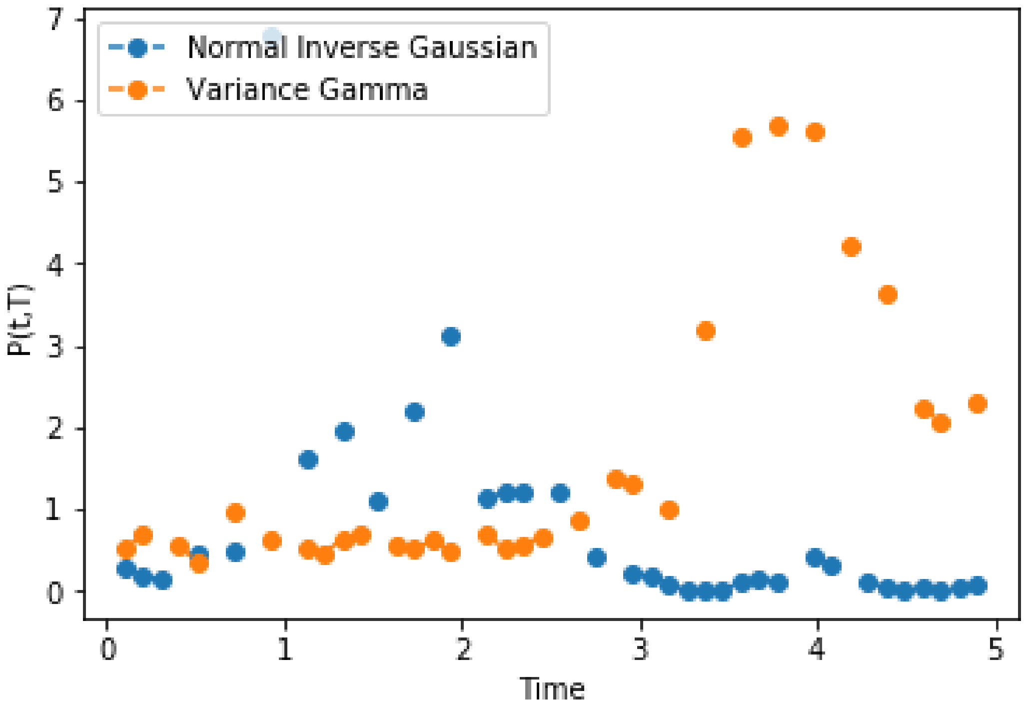

Figure 5.5: Price of zero-coupon bond driven by NIG and VG processes (with jumps ± 0.2)

Interpretation of the graphs

Figure 5.1 represents a graph of the zero-coupon bond price driven by VG process using the parameter values obtained in sections 5.2 and 5.3. The values are $a = 0.5989$, $b = 0.1447$, $\sigma = 0.0588$, $\theta = -0.01610$, $\widetilde{\sigma} = 0.46456$, $\kappa = 0.24508$ and $r_0 = 0.1317$. The maturity time $T = 5$. The values were substituted into the price of the zero-coupon bond $P(t,T)$ given by equation (4.2.7). The price is plotted on the vertical axis against time to maturity on the horizontal axis. The graph has a lot of discontinuities which indicate jumps. There are upward and downward jumps whose absolute sizes are greater than or equal to 0.08.
Figure 5.2 shows the graph of the zero-coupon bond price driven by VG process whose absolute jump sizes are greater than or equal to 0.1. Discontinuities in the graph are evidence of the occurrence of the jumps.

Figure 5.3 represents a graph of the zero-coupon bond price driven by NIG process using its parameter values obtained in sections 5.2 and 5.3. The values are $a = 0.5989$, $b = 0.1447$, $\sigma = 0.0588$, $\alpha = 4.16493$, $\beta = -0.08718$, $\delta = 0.89955$ and $r_0 = 0.1317$. The maturity time $T = 5$. The values are substituted into the zero-coupon bond price driven by the NIG process as given by equation (4.3.5). The price of the zero-coupon bond is plotted on the vertical axis against time to maturity on the horizontal axis. Discontinuities in the graph indicate presence of upward and downward jumps whose absolute sizes are greater than or equal 0.08.
Figure 5.4 shows the graph of the zero-coupon bond price driven by NIG process with upward and downward jumps whose absolute jump sizes are greater than or equal to 0.2.

Figure 5.5 shows the behaviour of the graphs of the zero-coupon bond price driven by both VG and NIG processes. The discontinuities illustrate the presence of both upward and downward jumps whose absolute jump sizes are greater than or equal to 0.2.

Chapter 6

Summary and Conclusion

6.1 Introduction

In financial markets, the prices of some assets experience jumps under certain circumstances. To take account of jumps, Lévy processes have been found to be very useful in the pricing of interest rate derivatives. The Malliavin calculus is a good tool for computing the greeks associated with the prices of zero-coupon bonds driven by subordinated Lévy processes. This helps a risk manager to understand the effects of changes in the parameters of the interest rate derivative.

6.2 Conclusion

This work has focused on the sensitivity analysis of an interest rate derivative in some Lévy markets. To this end, we extended the existing Vasicek model to incorporate jumps in sections 4.2.1 and 4.3.1 to Lévy markets driven by VG and NIG processes. We applied the improved Vasicek model to obtain the price of an interest rate derivative in a Lévy market. This was achieved by using the Itô's formula and the expressions of the extended Vasicek model to obtain the price of zero-coupon bond driven by the VG and NIG processes in sections 4.2.2 and 4.3.2. The Malliavin calculus was then applied in the sensitivity analysis of the interest rate derivatives. This was accomplished in sections 4.2.3 and 4.3.3 for the prices of zero-coupon bonds driven by VG and NIG processes. Finally, a comparison of some of the greeks obtained in sections 4.2.3 and 4.3.3 was done in section 4.4. In order to apply our results to the Nigerian market, we estimated parameter values from the dataset of 30-day NIBOR rates collected from the website (*www.cbn.com.gov.ng*) of the Central Bank of Nigeria (CBN).

We were then able to obtain the Vasicek short rate dynamics for the NIBOR rate. As a result, we determined the dynamics of zero-coupon bond prices in the Nigerian market driven by the subordinated Lévy processes VG and NIG. This work has been concerned with how to measure the risks when trading interest rate derivatives driven by certain subordinated Lévy processes. Hence, this thesis represents a contribution to the theory of interest rate derivatives in Lévy markets.

6.3 Findings

In this work, we found that

(i) the Vasicek short rate model for the Brownian motion market can be extended to a Lévy market using subordinated Lévy processes, and the short rate dynamics can be obtained for the Nigerian interest rate derivative market.

(ii) the extended Vasicek model is suitable for deriving the price of an interest rate derivative that captures jumps; and using the NIBOR rates, the price of an interest rate derivative called zero-coupon bond can be obtained. The NIG process is suitable for modelling heavily-tailed interest rate derivative market, whereas VG process is suitable for skewed market. The distribution of the Nigerian interest rate derivative market is skewed to the left and heavily-tailed, hence it is driven by an NIG process.

(iii) the Malliavin calculus provides a better way of computing the greeks in the interest rate derivative market driven by a Lévy process.

(iv) understanding the greeks will help a risk manager to monitor and minimise risks.

6.4 Future Research

In this thesis, we have extended the Vasicek short rate model to markets driven by some subordinated Lévy processes, derived expressions for the corresponding zero-coupon bond prices in the markets and obtained the associated greeks. However, there are various assumptions taken that may be adjusted to improve the model.

We have assumed a type of interest rate without coupon payments. It will be interesting to consider pricing and sensitivity analysis of bonds involving coupon payments in a Lévy market. An investor who wants to generate income without having many zero-coupon bonds will prefer bonds with coupon payments. The model will involve deriving the bond price using the improved Vasicek model in a Lévy market and applying Malliavin calculus to compute its greeks.

Furthermore, the volatility of the short rate model was assumed to be a constant, but the model can be improved by considering the possibility of stochastic volatility. This is because a zero-coupon bond has the tendency of being more volatile as there is no coupon payment throughout its life. Moreover, a stochastic volatility model will describe a more realistic market.

The foregoing will be considered for future research.

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Appendix 1

1. Computation of $\mathbb{E}_{(\kappa)}[\Phi(P)]$ for the greek $\mathcal{V}_2$

The digamma function: According to Medina and Moll (2009), the *digamma function* is given as

$$\psi(y) = \frac{d}{dy}\ln\Gamma(y) = \frac{\Gamma'(y)}{\Gamma(y)} \qquad \text{where } \Gamma(y) = \int_0^\infty t^{y-1}e^{-t}dt.$$

In what follows, the digamma function is given by

$$\psi\left(\frac{t}{\kappa}\right) = \frac{d}{d\kappa}\ln\Gamma\left(\frac{t}{\kappa}\right).$$

Let $f_{\mathcal{N}}$ and f_g be the density function for the Gaussian random variable and gamma random variable, respectively. Then, following Bayazit and Nolder (2009),

$$\begin{aligned}
\mathbb{E}_{(\kappa)}[\Phi(P)] &= \frac{\partial}{\partial\kappa}\int_{\mathbb{R}}\int_{\mathbb{R}}\Phi(P)f_{\mathcal{N}(x;0,1)}\cdot f_{g(y;\frac{t}{\kappa},\frac{1}{\kappa})}dxdy \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}}\Phi(P)f_{\mathcal{N}(x;0,1)}\cdot\frac{\partial}{\partial\kappa}\left(\frac{\kappa^{-\frac{t}{\kappa}}}{\Gamma\left(\frac{t}{\kappa}\right)}y^{\frac{t}{\kappa}-1}e^{-\frac{1}{\kappa}y}\right)dxdy \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}}\Phi(P)f_{\mathcal{N}(x;0,1)}\cdot\frac{\partial}{\partial\kappa}\exp\left(\log\left(\frac{\kappa^{-\frac{t}{\kappa}}}{\Gamma\left(\frac{t}{\kappa}\right)}y^{\frac{t}{\kappa}-1}e^{-\frac{1}{\kappa}y}\right)\right)dxdy \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}}\Phi(P)f_{\mathcal{N}(x;0,1)}f_{g(y;\frac{t}{\kappa},\frac{1}{\kappa})}\cdot\frac{\partial}{\partial\kappa}\log\left(\frac{\kappa^{-\frac{t}{\kappa}}}{\Gamma\left(\frac{t}{\kappa}\right)}y^{\frac{t}{\kappa}-1}e^{-\frac{1}{\kappa}y}\right)dxdy,
\end{aligned}$$

where

$$\begin{aligned}
\frac{\partial}{\partial\kappa}\log\left(\frac{\kappa^{-\frac{t}{\kappa}}}{\Gamma\left(\frac{t}{\kappa}\right)}y^{\frac{t}{\kappa}-1}e^{-\frac{1}{\kappa}y}\right) &= \frac{\partial}{\partial\kappa}\Big(-\frac{t}{\kappa}\log\kappa - \log\Gamma\big(\frac{t}{\kappa}\big) + \big(\frac{t}{\kappa}-1\big)\log y - \frac{1}{\kappa}y\Big) \\
&= \left(\frac{t}{\kappa^2}\log\kappa - \frac{t}{\kappa^2} + \frac{t}{\kappa^2}\frac{\Gamma'\left(\frac{t}{\kappa}\right)}{\Gamma\left(\frac{t}{\kappa}\right)} - \frac{t}{\kappa^2}\log y + \frac{1}{\kappa^2}y\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
e^{-r_0T}\mathbb{E}_{(\kappa)}[\Phi(P)] = e^{-r_0T}\mathbb{E}\Bigg[\Phi(P)\sum_{t\le u\le T}&\left(\frac{t}{\kappa^2}\log\kappa - \frac{t}{\kappa^2} + \frac{t}{\kappa^2}\frac{\Gamma'\left(\frac{t}{\kappa}\right)}{\Gamma\left(\frac{t}{\kappa}\right)}\right. \\
&\left.-\frac{t}{\kappa^2}\log(\Delta G(u)) + \frac{1}{\kappa^2}\Delta G(u)\right)\Bigg].
\end{aligned}$$

2. Computation of $\mathbb{E}_{(\beta)}[\Phi(P)]$ for the sensitivity with respect to β

$$\mathbb{E}_{(\beta)}[\Phi(P)] = \int_{\mathbb{R}}\int_{\mathbb{R}} \Phi(P)\cdot f_{\mathcal{N}(x;0,1)}\frac{\partial}{\partial\beta} f_{IG}(y;t,\delta\sqrt{\alpha^2-\beta^2})\big)dxdy.$$

The density function of the inverse Gaussian (IG) distribution is,

$$f_{IG}(y;t,\delta\sqrt{\alpha^2-\beta^2})$$
$$= \frac{ty^{-3/2}\exp(t(\delta\sqrt{\alpha^2-\beta^2}))}{\sqrt{2\pi}}\exp\bigg(-\frac{1}{2}(\frac{t^2}{y}+(\delta\sqrt{\alpha^2-\beta^2})^2y)\bigg)\cdot \mathbf{1}_{y>0}$$

while

$$\log f_{IG}(y;t,\delta\sqrt{\alpha^2-\beta^2})$$
$$= \log\left(\frac{t}{\sqrt{2\pi}}\right) + t(\delta\sqrt{\alpha^2-\beta^2}) - \frac{3}{2}\log(y) - \frac{t^2}{2y} - \frac{\delta^2(\alpha^2-\beta^2)}{2}y.$$

Thus,

$$\frac{\partial}{\partial\beta} f_{IG}(y;t,\delta\sqrt{\alpha^2-\beta^2}))$$
$$= \frac{\partial}{\partial\beta}\exp\left(\log\left(\frac{ty^{-3/2}\exp(t(\delta\sqrt{\alpha^2-\beta^2}))}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}(\frac{t^2}{y}+(\delta\sqrt{\alpha^2-\beta^2})^2y)\right)\cdot\mathbf{1}_{y>0}\right)\right)$$
$$= f_{IG}(y;t,\delta\sqrt{\alpha^2-\beta^2})\frac{\partial}{\partial\beta}\log\left(\frac{ty^{-3/2}\exp(t(\delta\sqrt{\alpha^2-\beta^2}))}{\sqrt{2\pi}}\right.$$
$$\left.\cdot\exp\bigg(-\frac{1}{2}(\frac{t^2}{y}+(\delta\sqrt{\alpha^2-\beta^2})^2y)\bigg)\cdot\mathbf{1}_{y>0}\right),$$

where

$$\frac{\partial}{\partial\beta}\log\left(\frac{ty^{-3/2}\exp(t(\delta\sqrt{\alpha^2-\beta^2}))}{\sqrt{2\pi}}\exp\bigg(-\frac{1}{2}(\frac{t^2}{y}+(\delta\sqrt{\alpha^2-\beta^2})^2y)\bigg)\cdot\mathbf{1}_{y>0}\right)$$
$$= \frac{\partial}{\partial\beta}\bigg(\log\left(\frac{t}{\sqrt{2\pi}}\right)+t(\delta\sqrt{\alpha^2-\beta^2}) - \frac{3}{2}\log(y) - \frac{t^2}{2y} - \frac{\delta^2(\alpha^2-\beta^2)}{2}y\bigg)$$
$$= \beta\delta^2 y - \frac{t\delta\beta}{\sqrt{\alpha^2-\beta^2}}.$$

Hence,

$$e^{-r_0T}\mathbb{E}_{(\beta)}[\Phi(P)] = e^{-r_0T}\mathbb{E}\bigg[\Phi(P)\sum_{t\le u\le T}\bigg(\beta\delta^2\Delta IG(u) - \frac{t\delta\beta}{\sqrt{\alpha^2-\beta^2}}\bigg)\bigg].$$

3. Computation of $\mathbb{E}_{(\delta)}[\Phi(P)]$ for the sensitivity with respect to δ

$$\mathbb{E}_{(\delta)}[\Phi(P)] = \int_{\mathbb{R}}\int_{\mathbb{R}} \Phi(P)\cdot f_{\mathcal{N}(x;0,1)}\frac{\partial}{\partial\delta}\big(f_{IG}(y;t,\delta\sqrt{\alpha^2-\beta^2})\big)dxdy.$$

Thus,

$$\frac{\partial}{\partial\delta} f_{IG}(y;t,\delta\sqrt{\alpha^2-\beta^2}) = \frac{\partial}{\partial\delta}\exp\left(\log\left(\frac{ty^{-3/2}e^{t(\delta\sqrt{\alpha^2-\beta^2})}}{\sqrt{2\pi}}e^{\left(-\frac{1}{2}\left(\frac{t^2}{y}+(\delta\sqrt{\alpha^2-\beta^2})^2y\right)\right)}\cdot\mathbf{1}_{y>0}\right)\right)$$

$$= f_{IG}(y; t, \delta\sqrt{\alpha^2-\beta^2})\frac{\partial}{\partial \delta}\log\Bigg(\frac{ty^{-3/2}\exp(t(\delta\sqrt{\alpha^2-\beta^2}))}{\sqrt{2\pi}}$$

$$\cdot\exp\Bigg(-\frac{1}{2}(\frac{t^2}{y}+(\delta\sqrt{\alpha^2-\beta^2})^2 y)\Bigg)\cdot \mathbf{1}_{y>0}\Bigg),$$

where

$$\frac{\partial}{\partial \delta}\log\Bigg(\frac{ty^{-3/2}\exp(t(\delta\sqrt{\alpha^2-\beta^2}))}{\sqrt{2\pi}}\exp\Bigg(-\frac{1}{2}(\frac{t^2}{y}+(\delta\sqrt{\alpha^2-\beta^2})^2 y)\Bigg)\cdot \mathbf{1}_{y>0}\Bigg)$$

$$= \frac{\partial}{\partial \delta}\Bigg(\log\Bigg(\frac{t}{\sqrt{2\pi}}\Bigg)+t(\delta\sqrt{\alpha^2-\beta^2})-\frac{3}{2}\log(y)-\frac{t^2}{2y}-\frac{\delta^2(\alpha^2-\beta^2)}{2}y\Bigg)$$

$$= t\sqrt{\alpha^2-\beta^2}-\delta(\alpha^2-\beta^2)y.$$

Therefore,

$$e^{-r_0T}\mathbb{E}_{(\delta)}[\Phi(P)] = e^{-r_0T}\mathbb{E}\Bigg[\Phi(P)\sum_{t\le u\le T}\Bigg(t\sqrt{\alpha^2-\beta^2}-\delta(\alpha^2-\beta^2)\Delta IG(u)\Bigg)\Bigg].$$

4. Computation of $\mathbb{E}_{(\alpha)}[\Phi(P)]$ for sensitivity with respect to α

$$\mathbb{E}_{(\alpha)}[\Phi(P)] = \int_{\mathbb{R}}\int_{\mathbb{R}}\Phi(P)\cdot f_{\mathcal{N}(x;0,1)}\frac{\partial}{\partial \alpha}f_{IG}(y;t,\delta\sqrt{\alpha^2-\beta^2})dxdy.$$

Thus,

$$\frac{\partial}{\partial \alpha}\big(f_{IG}(y;t,\delta\sqrt{\alpha^2-\beta^2})\big) = \exp\Bigg(\log\Bigg(\frac{ty^{-3/2}e^{t(\delta\sqrt{\alpha^2-\beta^2})}}{\sqrt{2\pi}}e^{\big(-\frac{1}{2}\big(\frac{t^2}{y}+(\delta\sqrt{\alpha^2-\beta^2})^2y\big)\big)}\cdot\mathbf{1}_{y>0}\Bigg)\Bigg)$$

$$= f_{IG}(y; t, \delta\sqrt{\alpha^2-\beta^2})\frac{\partial}{\partial \alpha}\log\Bigg(\frac{ty^{-3/2}\exp(t(\delta\sqrt{\alpha^2-\beta^2}))}{\sqrt{2\pi}}$$

$$\cdot\exp\Bigg(-\frac{1}{2}(\frac{t^2}{y}+(\delta\sqrt{\alpha^2-\beta^2})^2 y)\Bigg)\cdot \mathbf{1}_{y>0}\Bigg),$$

where

$$\frac{\partial}{\partial \alpha}\log\Bigg(\frac{ty^{-3/2}\exp(t(\delta\sqrt{\alpha^2-\beta^2}))}{\sqrt{2\pi}}\exp\Bigg(-\frac{1}{2}(\frac{t^2}{y}+(\delta\sqrt{\alpha^2-\beta^2})^2 y)\Bigg)\cdot \mathbf{1}_{y>0}\Bigg)$$

$$= \frac{\partial}{\partial \alpha}\Bigg(\log\Bigg(\frac{t}{\sqrt{2\pi}}\Bigg)+t(\delta\sqrt{\alpha^2-\beta^2})-\frac{3}{2}\log(y)-\frac{t^2}{2y}-\frac{\delta^2(\alpha^2-\beta^2)}{2}y\Bigg)$$

$$= \frac{t\alpha\delta}{\sqrt{\alpha^2-\beta^2}}-\delta^2\alpha y.$$

Hence,

$$e^{-r_0T}\mathbb{E}_{(\alpha)}[\Phi(P)] = e^{r_0T}\mathbb{E}\Bigg[\Phi(P)\sum_{t\le u\le T}\Bigg(\frac{t\alpha\delta}{\sqrt{\alpha^2-\beta^2}}-\delta^2\alpha\Delta IG(u)\Bigg)\Bigg].$$

Appendix II

5. NIBOR data (in percentage)

	NIBOR: 30 Days						
Jan−07	13.17	Jan−08	12.99	Jan−09	14.91	Jan−10	12.84
Feb−07	11.96	Feb−08	12.76	Feb−09	18.07	Feb−10	11.27
Mar−07	11.37	Mar−08	12.12	Mar−09	18.92	Mar−10	7.85
Apr−07	10.57	Apr−08	12.78	Apr−09	15.25	Apr−10	5.13
May−07	11.01	May−08	13.15	May−09	15.91	May−10	8.03
Jun−07	11.56	Jun−08	13.46	Jun−09	19.84	Jun−10	5.95
Jul−07	12.85	Jul−08	13.06	Jul−09	19.66	Jul−10	6.51
Aug−07	12.49	Aug−08	15.34	Aug−09	14.29	Aug−10	8.20
Sep−07	12.38	Sep−08	16.76	Sep−09	13.78	Sep−10	8.57
Oct−07	12.4	Oct−08	15.63	Oct−09	13.35	Oct−10	11.13
Nov−07	12.55	Nov−08	17.98	Nov−09	13.75	Nov−10	11.67
Dec−07	12.89	Dec−08	15.85	Dec−09	13.45	Dec−10	11.50
Jan−11	10.15	Jan−12	15.44	Jan−13	13.10	Jan−14	11.21
Feb−11	11.19	Feb−12	15.61	Feb−13	12.79	Feb−14	12.30
Mar−11	11.47	Mar−12	15.57	Mar−13	11.07	Mar−14	13.03
Apr−11	12.51	Apr−12	15.39	Apr−13	11.97	Apr−14	12.03
May−11	11.67	May−12	14.61	May−13	12.94	May−14	12.42
Jun−11	13.15	Jun−12	15.79	Jun−13	12.81	Jun−14	12.17
Jul−11	11.45	Jul−12	16.12	Jul−13	11.57	Jul−14	12.42
Aug−11	10.79	Aug−12	19.18	Aug−13	14.65	Aug−14	12.97
Sep−11	11.74	Sep−12	14.55	Sep−13	17.74	Sep−14	12.37
Oct−11	15.74	Oct−12	13.52	Oct−13	12.01	Oct−14	12.60
Nov−11	17.00	Nov−12	13.43	Nov−13	12.03	Nov−14	13.07
Dec−11	16.74	Dec−12	13.13	Dec−13	11.85	Dec−14	15.79
Jan−15	13.70	Jan−16	13.72	Jan−17	10.64		
Feb−15	15.47	Feb−16	15.19	Feb−17	26.78		
Mar−15	15.89	Mar−16	15.89	Mar−17	19.61		
Apr−15	15.17	Apr−16	15.02	Apr−17	52.74		
May−15	14.61	May−16	14.64	May−17	27.75		
Jun−15	15.45	Jun−16	15.38	Jun−17	26.03		
Jul−15	14.32	Jul−16	14.34	Jul−17	17.9		
Aug−15	17.16	Aug−16	17.08	Aug−17	26.38		
Sep−15	15.52	Sep−16	15.52	Sep−17	19.11		
Oct−15	13.05	Oct−16	13.59	Oct−17	39.24		
Nov−15	12.02	Nov−16	12.02	Nov−17	20.02		
Dec−15	9.13	Dec−16	9.13	Dec−17	16.32		

6. Python program for the price of zero-coupon bond driven by VG process

```
"""
import numpy as np
import math as m

a,b,sigma,r0,sigmatilde,theta,kappa,T,K, mu=0.5959,0.1447,
0.0585,0.1317,0.46456,-0.01610,0.24508,5,0.67,0.24508
wtilde= (1/kappa)*(np.log(1-theta*kappa-0.5*sigmatilde**2*kappa))
N=50
dt=T/N
Time=list(np.linspace(0, T, N))
gam=np.random.gamma(dt/kappa, 1/kappa)
gam5=np.random.gamma(dt/kappa, 1/kappa)

def iterat(u,gam1,data):
    W=0
    for s in data:
        gam2=np.random.gamma(dt/kappa, 1/kappa)
        E=sigmatilde*(m.sqrt(abs(gam1))-m.sqrt(abs(gam2)))
        *m.exp(-a*(u-s))*np.random.normal(0,1,1)
        F=theta*(gam1-gam2)*m.exp(-a*(u-s))
        M=E+F
        W+=M
        gam1=gam2
    return(W)

def iterat1(gam4,data):
    Y=0
    for u in data:
        gam3=np.random.gamma(dt/kappa, 1/kappa)
        E=sigmatilde*(m.sqrt(abs(gam4))-m.sqrt(abs(gam3)))
        *np.random.normal(0,1,1)
        F=theta*(gam4-gam3)
        M=E+F
        gam4=gam3
        Y+=M
    return(Y)
```

```
"""


def itera(gam4,data):
    U=0
    for u in data:
        gam3=np.random.gamma(dt/kappa, 1/kappa)
        E=sigmatilde*(m.sqrt(abs(gam4))-m.sqrt(abs(gam3)))
        *np.random.normal(0,1,1)
        F=(theta*(gam4-gam3))
        M=(E+F)**2
        gam4=gam3
        U+=M
    return(U)

def PtTgamma():
    PS=[]
    S=0
    for t in Time:
        if t == T:
            PS.append(1)
        else:
            A=(r0/a)*(m.exp(-a*T)-m.exp(-a*t))
            B=-b*(T-t +(1/a)*(m.exp(-a*T)-m.exp(-a*t)))
            C=-(sigma*wtilde/a)*(T-t+(1/a)*(m.exp(-a*T)-m.exp(-a*t)))
            D=-wtilde*sigma*(T-t)
            Times = Time[Time.index(t):N]
            Opp_Times= Time[0: Time.index(t)+1]
            gam1=np.random.gamma(dt/kappa, 1/kappa)
            gam4=np.random.gamma(dt/kappa, 1/kappa)
            for u in Times:
                S+=iterat(u, gam1,Opp_Times)
            G=-S*sigma
            I=-sigma*iterat1(gam4, Times)
            L=-((sigma**2)/2)*itera(gam4, Times)
            P= m.exp(A+B+C+D+G+I+L)
            PS.append(P)
    return(PS)

if __name__ == "__main__":


    import matplotlib.pyplot as plt
```

```
"""

    y=PtTgamma()
    for i in y:
        if abs(y[y.index(i)]-y[y.index(i)-1])>=0.04:
            y[y.index(i)]=np.nan

    #print(y)
    #print(len(y))
    plt.plot(Time, y, 'yo-')
    plt.xlabel("Time",)
    plt.ylabel("P(t,T)")
    plt.show()
```

7. **Python program for the price of zero-coupon bond driven by NIG process**

```
"""

from scipy.stats import invgauss
import numpy as np
import math as m

alpha, beta, delta,r0,a,b,sigma,T,K=4.16493, -0.08718,
0.89955,0.1317,0.5959,0.1447,0.0585,5,0.67
mu=delta*m.sqrt(alpha**2-beta**2)
w= delta*(m.sqrt(alpha**2 -(beta+1)**2)-m.sqrt(alpha**2-beta**2))
ginv=invgauss.rvs(mu)
ginv5=invgauss.rvs(mu)
N=50

Time=list(np.linspace(0, T, N))

def iterat(u,ginv1,data):
    W=0
    for s in data:
        ginv2=invgauss.rvs(mu)
        E=delta*(m.sqrt(abs(ginv1))-m.sqrt(abs(ginv2)))
        *m.exp(-a*(u-s))*np.random.normal(0,1,1)
        F=beta*delta**2*(ginv1-ginv2)*m.exp(-a*(u-s))
        M=E+F
        W+=M
        ginv1=ginv2
    return(W)

def iterat1(ginv4,data):
    Y=0
    for u in data:
        ginv3=invgauss.rvs(mu)
        E=delta*(m.sqrt(abs(ginv4))-m.sqrt(abs(ginv3)))
        *np.random.normal(0,1,1)
        F=beta*delta**2*(ginv4-ginv3)
        M=E+F
        ginv4=ginv3
        Y+=M
    return(Y)
```

```
"""
def itera(ginv4,data):
    U=0
    for u in data:
        ginv3=invgauss.rvs(mu)
        E=delta*(m.sqrt(abs(ginv4))-m.sqrt(abs(ginv3)))
        *np.random.normal(0,1,1)
        F=(beta*delta**2*(ginv4-ginv3))
        M=(E+F)**2
        ginv4=ginv3
        U+=M
    return(U)

def PtT():
        PS=[]
        S=0
        for t in Time:
            if t == T:
                PS.append(1)
            else:
                A=(r0/a)*(m.exp(-a*T)-m.exp(-a*t))
                B=-b*(T-t +(1/a)*(m.exp(-a*T)-m.exp(-a*t)))
                C=-(sigma*w/a)*(T-t+(1/a)*(m.exp(-a*T)-m.exp(-a*t)))
                D=-w*sigma*(T-t)
                Times = Time[Time.index(t):N]
                Opp_Times= Time[0: Time.index(t)+1]
                ginv1=invgauss.rvs(mu)
                ginv4=invgauss.rvs(mu)
                for u in Times:
                    S+=iterat(u, ginv1,Opp_Times)
                G=-S*sigma
                I=-sigma*iterat1(ginv4, Times)
                L=-((sigma**2)/2)*itera(ginv4, Times)
                P= m.exp(A+B+C+D+G+I+L)
                PS.append(P)
        return(PS)

if __name__ == "__main__":

    import matplotlib.pyplot as plt
```

```
"""
    y=PtT()
     for i in y:
          if abs(y[y.index(i)]-y[y.index(i)-1])>=0.2:
               y[y.index(i)]=np.nan

    # print(y)
     #print(len(y))
     plt.plot(Time, y, 'ro-')
     plt.xlabel("Time",)
     plt.ylabel("P(t,T)")
     plt.show()
```